LOGARITHMIC RATE FOR THE MODIFIED LANDWEBER METHOD OF NONLINEAR ILL-POSED PROBLEMS

By
MISS Parada SUNGCHAROEN

A Thesis Submitted in Partial Fulfillment of the Requirements
for Master of Science (MATHEMATICS)
Department of MATHEMATICS
Graduate School, Silpakorn University
Academic Year 2017

อัตราลอการิทึมสำหรับวิธีแลนด์เวเบอร์แบบปรับปรุงใหม่ของปัญหาอิลล์โพสแบบไม่เชิงเส้น

โดย
นางสาวภราดา สังข์เจริญ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรมหาบัณฑิต
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In this thesis, we consider the modified Landweber method for solving nonlinear ill-posed problems $F(x) = y$ where the noisy data $y^\delta \in Y$ with $\left\|y^\delta - y\right\| \leq \delta$ are provided and the operator $F : \mathcal{D}(F) \subseteq X \to Y$ is nonlinear operator on Hilbert spaces X and Y. We prove the convergence rate of the modified Landweber under the logarithmic source condition. We also apply this method to an inverse potential problem and show the numerical implementations.

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ในวิทยานิพนธ์นี้ เราศึกษาวิธีแลนด์เวเบอร์แบบปรับปรุงสำหรับการหาผลเฉลยของปัญหาอิลล์โพสไม่เชิงเส้น $F(x) = y$ ที่ข้อมูลมีสัญญาณรบกวน $y^\delta \in Y$ ซึ่ง $\|y^\delta - y\| \leq \delta$ และตัวดำเนินการ $F : \mathcal{D}(F) \subseteq X \to Y$ เป็นตัวดำเนินการไม่เชิงเส้นบนปริภูมิฮิลเบิร์ต X และ Y เราพิสูจน์อัตราการลู่เข้าสู่ผลเฉลยของวิธีแลนด์เวเบอร์แบบปรับปรุงใหม่ภายใต้เงื่อนไขเฉพาะที่เหมาะสม และยังแสดงตัวอย่างเชิงตัวเลขของวิธีแลนด์เวเบอร์แบบปรับปรุงใหม่กับปัญหาศักย์อีกด้วย

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Parada SUNGCHAROEN

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X, Y	:	The innite-dimensional real Hilbert spaces
F	:	The Fréchet - differentiable nonlinear operator mapping between Hilbert spaces X and Y
$\mathcal{D}(F)$	:	Domain of the nonlinear operator F
x	:	The unknown data
y	:	The exact data
δ	:	The noise level
y^δ	:	The perturbed data
x_n	:	The approximation solution via the iterative methods without the perturbed data
x_n^δ	:	The approximation solution via the iterative methods with the perturbed data
x_0	:	The initial value
x^+	:	The exact solution of nonlinear equation
F'	:	The Fréchet derivative of F
R_x	:	The family of bounded linear operators mapping between Y and Y
e	:	The Euler's number
$f(\cdot)$	:	The logarithm condition
η	:	The postitive number such as $\eta < \frac{1}{2}$
τ	:	The positive number depending on η and satises $\tau > 2\frac{1+\eta}{1-2\eta} > 2$

Chapter 1
Introduction

This chapter gives a brief introduction to nonlinear ill-posed inverse problem.

1.1 Inverse problems and ill-posed problems

Inverse problems are part of the most important mathematical problems that occur virtually anywhere in part of sciences and in the other areas where mathematical method related. Inverse problems always come paired with direct problems. However, the mathematical community has embraced what are now called direct problems with a warmth not generally extended to inverse problems. The great advances in science and technology have been made fessible by solving inverse problems. Those problems involve determining indirect measurement and observations, remote sensing, finding the nature of an inaccessible region from measurements on the boundary, and many others. Accordingly, the inverse problems are interested in many branches of mathematics.

Example 1.1. We consider the modified Helmholtz equation [12] as follows :

$$\begin{cases} \Delta u(x,y) - k^2 u(x,y) = f(x), & 0 < x \leq \pi, 0 < y < +\infty, \\ u(0,y) = u(\pi,y) = 0 & 0 \leq y < +\infty, \\ u(x,0) = 0, u(x,y)|_{y\to\infty} \quad \text{is} \quad \text{bounded}, & 0 \leq x \leq \pi \\ u(x,1) = g(x), & 0 \leq x \leq \pi \end{cases}$$

where $g(x)$ is given function in $L^2(0,\pi)$, and $f(x)$ is an unknown source. The constant k is the wave number. We use the additional condition $u(x,1) = g(x)$ to determine the unknown source $f(x)$. Physically, $g(x)$ can be measured, there will be measurement errors, and we assume the function $g^\delta(x) \in L^2(0,\pi)$ as the measurable data which satisfies

$$\left\|g - g^\delta\right\| \leq \delta,$$

where a noise level $\delta > 0$ represents a bound on the measurement error, $\|\cdot\|$ is $L^2(0,\pi)$ norm.

Example 1.2. We consider the one-dimensional steady-state diffusion equation [6]

$$-\left(a(s)u_s(s)\right)_s = f(s), \qquad s \in (0,1),$$

with the Dirichlet conditions

$$u(0) = u_0, \qquad u(1) = u_1$$

where u_0 and u_1 are real number. In the inverse problem, one gives internal measurements of the temperature u and the heat source f and tries to recover the temperature-dependent difussion coefficient a.

Example 1.3. We consider the problem of determining the shape of an unknown domain D from information of its density and of measurements of the Cauchy data of the corresponding potential on the boundary of a smooth and bounded domain.

In the direct problem, for a given domain $D \subset \Omega_R$ with a simply connected bounded domain Ω_R of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega_R$, the solution u of the boundary value problem satisfy

$$\Delta u = \chi_D \quad \text{in} \quad \Omega_R \setminus \partial D, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \Omega_R, \tag{1.2}$$

where χ_D is the characteristic function of the domain D. It is well-known [4] that there exists a unique solution $u \in C^2\left(\Omega_R \setminus \partial D\right) \cap C^1\left(\bar{\Omega}_R\right)$ for $x = \chi_D$ as follows

$$u(t, \chi_D) = \int_{\Omega_R} G(t,s)x(s)\mathrm{d}s = \int_D G(t,s)\mathrm{d}s \tag{1.3}$$

where the Green's function $G : \Omega_R \times \Omega_R \to \mathbb{R}$ is

$$G(t,s) := \frac{1}{2\pi}\ln|t-s| - \frac{1}{2\pi}\ln\left(\frac{|s|}{R}\left|t - \frac{R^2}{|s|^2}s\right|\right).$$

The inverse potential problem is to recover x from given y on $\partial\Omega_R$ where

$$\frac{\partial u}{\partial \nu} = y \quad \text{on} \quad \partial\Omega_R \tag{1.4}$$

and ν is the outer normal vector to the boundary $\partial\Omega$.

From [8], we have the uniqueness theorem of the inverse source problems as follows.

Theorem 1.4. [8]. Suppose that either (1.1), D_1 and D_2 are star-shaped with respect to their centers of gravity, or (1.2), D_1 and D_2 are convex in x. If $u(\cdot, \chi_{D_1})=$ $u(\cdot, \chi_{D_2})$ on $\Omega_R \setminus D$, then $D_1 = D_2$.
From (1.3), we can get

$$\frac{\partial u}{\partial \nu(t)}(t, \chi_D) \int_{\Omega_R} \frac{\partial G}{\partial \nu(t)}(t, s)x(s)\mathrm{d}s = \int_0^{2\pi} \int_0^{x(s)} P(r, t-s)r\mathrm{d}r\mathrm{d}s, t \in \partial\Omega_R \quad (1.5)$$

where $P(r, t)$ is the Poisson kernel.
Combined (1.4) with (1.5), we can defined an operator $F : L^2(\Omega_R) \to L^2(\Omega_R)$, satisfying

$$F(x) = y$$

where

$$[F(x)](t) = \int_{\Omega_R} \frac{\partial G}{\partial \nu(t)}(t, s)x(s)\mathrm{d}s, t \in \partial\Omega_R. \quad (1.6)$$

In early 20th century , Hadamard who worked on problems in mathematics believed that ill-posed problems do not model real-world problems. Hadamard's definition says that a problem is well-posed if it satisfied the following requirements, [3] :

1. *Existence* : The problem must have a solution.

2. *Uniqueness* : There must be only one solution to the problem.

3. *Stability* : The solution must depend continuously on the data.

A problem which is not well-posed is called ill-posed. If the data space is defined as set of solutions to the direct problem, existence of a solution to the inverse problem is clear. However, a solution may fail to exist if the data are perturbed by noise. Uniqueness of a solution is often not easy to show. However, the additional data have to be observed or the set of admissible solutions has to be restricted using a-priori information on the solution. Stability of a solution to the inverse problem is a serious numerical problem if one wants to approximate a problem whose solution does not depend continuously on the data, then one has to expect that the numerical solution becomes unstable.

The presented work concerns with a nonlinear ill-posed operator equation

$$F(x) = y, \quad (1.7)$$

where the operator $F : \mathcal{D}(F) \subseteq X \to Y$ is nonlinear operator on domain $\mathcal{D}(F) \subset X$, X and Y are Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and norms $\| \cdot \|$, respectively. They can always be identified from the context in which they appear. Due to the nonlinearity of (1.7) we assume all over that (1.7) has a solution x^+ which need not be unique. We have the approximated data y^δ with

$$\left\| y^\delta - y \right\| \leq \delta \tag{1.8}$$

where $\delta > 0$ is a noise level. In this work, we assume the ill -posed problem (1.7) in the way of the data on the right-hand is disturbed by noise. But the solution has changed a lot, that is, the solution of problem (1.7) does not depend on the change of right-hand data. Accordingly, the third well-posedness condition in the sense of Hadamard is not fulfilled.

In 1990 H.W. Engl and his researcher at Johannes Kepler University, Linz, Austria pushed the theory of the regularization methods such as Landweber iteration and Tikhonov method [3]. Then, they proved the convergence of approximated solution which obtained from Landweber iteration as follow

$$x_{n+1} = x_n + F'(x_n)^*(y - F(x_n)), n = 0, 1, 2, ... \tag{1.9}$$

where x_0 is an initial guess. The total error consists of two parts,i.e., the approximation and the data error. While the approximation error is a monotone increasing function the data error is a decreasing one. To get a good approximation one has, in general, to estimate the parameter which is called regularization parameter. Hanke et al.[5] choose the regularization parameter n according to a generalized discrepancy principle,i.e., the iteration is stopped after $N = N(y^\delta,\delta)$ steps with

$$\left\| y^\delta - F(x_N^\delta) \right\| \leq \tau\delta < \left\| y^\delta - F(x_n^\delta) \right\|, 0 \leq n < N \tag{1.10}$$

where τ is a positive number. In addition to the discrepancy principle, F satisfies the local property in an open ball $B_\rho(x_0)$ of radius ρ around x_0 :

$$\left\| F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x}) \right\| \leq \eta \left\| F(x) - F(\tilde{x}) \right\|, \eta < \frac{1}{2} \tag{1.11}$$

with $x, \tilde{x} \in B_\rho(x_0) \subset \mathcal{D}(F)$. Utilizing the triangle inequality yields

$$\frac{1}{1+\eta} \left\| F'(x)(x - \tilde{x}) \right\| \leq \left\| F(x) - F(\tilde{x}) \right\| \leq \frac{1}{1-\eta} \left\| F'(x)(x - \tilde{x}) \right\| \tag{1.12}$$

to ensure at least local convergence to a solution x^+ of (1.7) in $B_{\frac{\varrho}{2}}(x_0)$.

Next, Scherzer et al.[10] proposed other version of Landweber iteration which an additional term $\alpha_n(x_n - \zeta)$ appears. To highlight the importance of an additional term the iterative method can be written into this form

$$x_{n+1} = x_n + F'(x_n)^*(y - F(x_n)) - \alpha_n(x_n - \zeta). \tag{1.13}$$

Convergence rate results [10] for the Landweber iteration and modified Landweber method were proven essentially under the Hölder type source condition, i.e.,

$$x^+ - x_0 = (F'(x^+)^* F'(x^+))^\nu w, \qquad \nu > 0, w \in X,$$

and in additional if the Fréchet derivativeof F can be factorization in the following form

$$F'(x) = R_x F'(x^+), x \in B_\rho(x_0)$$

where $\{R_x : x \in B_\rho(x_0)\}$ is a family of bounded linear operators R_x: $Y \to Y$ with

$$\|R_x - I\| \leq C \left\|x - x^+\right\|, x \in B_\rho(x_0)$$

where C is a positive constant and I: $Y \to Y$ is an identity operator.

The Hölder type source condition above is recognized as a suitable smoothness condition only for moderately ill-posed problems but not in certain severely ill-posed problem, e.g. heat conduction and potential theory, see Hohage [7], For such problems the logarithmic source condition can be use. Böckmann et al.[1] have already shown the convergence rate of Levenberg-Marquardt method under the logarithmic source condition.

1.2 Outline of the Thesis

In this thesis we have two main parts. Firstly, we utilize logarithmic source condition instead of the Hölder type one in order to demonstrate the error analysis of the modified Landweber method. As mentioned in [1], in many important applications Hölder type source conditions are too strong. They are not fulfilled even if $x^+ - x_0$ is an element of a harmless function space. Thus, we will consider function f such that continuous and strictly increasing as follows

$$f = f_p, \quad f_p(\lambda) := \begin{cases} (\ln \frac{e}{\lambda})^{-p} & \text{for } 0 < \lambda \leq 1, \\ 0 & \text{for } \lambda = 0, \end{cases} \tag{1.14}$$

where $p > 0$, and e is Euler's number.In this work we will analyze the convergence rate of the modified Landweber method (1.13) in a Hilbert space if the logarithmic source condition is satisfied. Secondly, In numerical parts, we present the convergence rate of the modified Landweber method for an inverse potential problem.

Chapter 2
Logarithmic convergence rate of the modified Landweber method

As introduced in Chapter 1, we begin this chapter with the modified Landweber presented by [10]

$$x_{n+1}^{\delta} = x_n^{\delta} + F'(x_n^{\delta})^*(y^{\delta} - F(x_n^{\delta})) - \alpha_n(x_n^{\delta} - \zeta) \tag{2.1}$$

where we set $\zeta = x_0$. If the iterative method is applied to exact data y, then we write x_n instead of x_n^{δ}. Moreover, we prove a convergence rates under the logarithmic condition (1.14) with $p = 1$ and the usual sourcewise representation as

$$x^+ - x_0 = f\left(F'(x^+)^* F'(x^+)\right) w, \qquad w \in X \tag{2.2}$$

where $\|w\|$ is sufficiently small and x^+ is the exact solution of $F(x^+) = y$ such that $x^+ \in \mathcal{D}(F)$. In this section, we consider a convergence rate for the modified Landweber iteration where $\{\alpha_n\}$ in (2.1) is defined as follow

$$\alpha_n = \frac{1}{2} n^{-1/2}, \qquad n \in N \qquad and \qquad \alpha_0 = \frac{1}{2}.$$

Before we state convergence rate for the modified Landweber iteration, we give some auxiliary results concerning the coefficients α_n

Lemma 2.1. *Let $l,n \in \mathbb{N}_0$, $l < n$. If $\{\alpha_s\}$ satisfies $0 < \alpha_s \leq 1$, [10] then*

$$1 - \prod_{s=l}^{n}(1 - \alpha_s) = \sum_{j=l}^{n} \alpha_j \prod_{s=j+1}^{n} (1 - \alpha_s) \leq 1.$$

If moreover

$$\sum_{n=0}^{\infty} \alpha_n < \infty$$

then $\prod_{n=0}^{\infty}(1 - \alpha_n)$ is convergent and thus

$$\prod_{n=l}^{\infty}(1 - \alpha_n) \to 1 \qquad as \qquad l \to \infty.$$

.

We define the following notations :

$$K := F'(x^+),$$
$$e_n := x^+ - x_n^\delta.$$

Lemma 2.2. *Let K be a linear operator with $\|K\| \le 1$. For $n \in N$ with $n > 1$, $e_0 := f(\lambda)w$ with f from (1.14) and $p = 1$, there exists a positive constant $C, \widehat{C}, \overline{C}$ such that the inequalities*

$$\|A_n(I - K^*K)^n e_0\| \le C \|w\| (\ln(n+e))^{-1} \tag{2.3}$$

and

$$\|A_n K(I - K^*K)^n e_0\| \le \widehat{C} \|w\| (n+1)^{-1/2} (\ln(n+e))^{-1} \tag{2.4}$$

are true where $A_n = \prod_{i=0}^{n-1}(1-\alpha_i)$, $\alpha_i = \frac{1}{2} i^{-1/2}$ and $\alpha_0 = \frac{1}{2}$.

Proof By the spectral theory and (1.14), we have

$$\begin{aligned}
\|A_n(I - K^*K)^n e_0\| &= \left\| \prod_{i=0}^{n-1}(1-\alpha_i)(I - K^*K)^n e_0 \right\| \\
&\le \prod_{i=0}^{n-1}(1-\alpha_i) \|(I - K^*K)^n f(K^*K)\| \|w\| \\
&\le \sup_{\lambda \in (0, \|K\|^2]} \{ \left|(1-\lambda)^n (\ln e - \ln\lambda)^{-1}\right| \} \|w\| \\
&\le \sup_{\lambda \in (0,1]} |h_1(\lambda)| \|w\|,
\end{aligned}$$

where $h_1(\lambda) = (1-\lambda)^n (1 - \ln\lambda)^{-1}$.

First, we are looking for critical point on (0,1). Observe that

$$h_1'(\lambda) = (1-\lambda)^n (1-\ln\lambda)^{-1} \left[\frac{1}{\lambda(1-\ln\lambda)} - \frac{n}{1-\lambda} \right].$$

We define

$$h_2(\lambda) = \frac{1}{\lambda(1-\ln\lambda)} - \frac{n}{1-\lambda}.$$

This means that a critical point of $h_1(\lambda)$ fulfills $h_2(\lambda) = 0$.

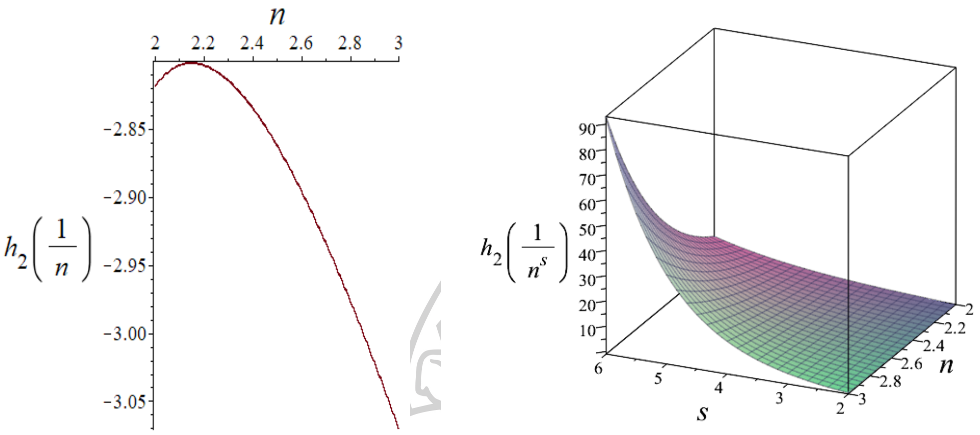
For sufficiently large $s > 1$, there exists an $n_0(s)$ such that

$$h_2(\frac{1}{n}) = \frac{(1+\ln n) - (1-\frac{1}{n})}{\frac{1}{n}(1+\ln n)(\frac{1}{n} - 1)} < 0,$$

see Figure 1(a) and

$$h_2(\frac{1}{n^s}) = \frac{(1-\frac{1}{n^s}) - \frac{n}{n^s}(1+s\ln n)}{\frac{1}{n^s}(1+s\ln n)(1-\frac{1}{n^s})} > 0$$

for all $n \geq n_0(s)$, see Figure 1(b).



(a) (b)

Figure 1 (a) shows the graph of $h_2(\frac{1}{n}) < 0$ and (b) shows the graph of $h_2(\frac{1}{n^s}) > 0$.

Thus, at least one critical point of h_1 has to lie in (n^{-s}, n^{-1}) with $n \geq \{n_0(s), 2\}$. Since $h_1'(\frac{1}{n^s}) > 0$ and $h_1'(\frac{1}{n}) < 0$, this must be a local maximum point. Then substituting $\lambda = e^{-x}$, $x \in (0, \infty)$ into h_1, we have

$$h_1(e^{-x}) = (1-e^{-x})^n(1-\ln e^{-x})^{-1} := h_3(x).$$

One has to determine that h_3 has at most two critical points on $(0, \infty)$. Therefore, we look for the roots of

$$\frac{h_3'(x)}{h_3(x)} = \frac{ne^{-x}}{1-e^{-x}} - \frac{1}{1+x} = 0$$

and get

$$e^x = nx + n + 1.$$

Thus at least one intersection point between the exponential curve $y = e^x$ and the straight line $y = nx + n + 1$ could occur. Furthermore, one concludes that if a

second critical point exists, it only be a saddle point.
For $1 \le r \le s$ we have $n^{-r} \in (n^{-s}, n^{-1})$ and

$$\begin{aligned} h_1(n^{-r}) &= (1-n^{-r})^n(1+r\ln n)^{-1} \le r(\ln n)^{-1} \\ &\le s(\frac{\ln(n+e)}{\ln n})(\ln(n+e))^{-1} \\ &\le C(\ln(n+e))^{-1} \end{aligned}$$

where C is constant and $(\frac{\ln(n+e)}{\ln n})$ bounded independently of n.
So, h_1 also attains its maxmimum in (0, 1]. Thus

$$h_1(\lambda) \le C(\ln(n+e))^{-1}$$

for any $\lambda \in (0,1]$. Note that above information imply

$$\|A_n(I-K^*K)^n e_0\| \le C\|w\|(\ln(n+e))^{-1}.$$

Similarly, in order to prove (2.4), we can proceed analogously. We have

$$\begin{aligned} \|A_nK(I-K^*K)^n e_0\| &= \left\| K\prod_{i=0}^{n-1}(1-\alpha_i)(I-K^*K)^n e_0 \right\| \\ &\le \prod_{i=0}^{n-1}(1-\alpha_i)\|K(I-K^*K)^n f(K^*K)\|\|w\| \\ &\le \sup_{\lambda\in\left(0,\|K\|^2\right]} \{\left|\lambda^{1/2}(1-\lambda)^n(\ln e-\ln\lambda)^{-1}\right|\}\|w\| \\ &\le \sup_{\lambda\in(0,1]} |v_1(\lambda)|\|w\|, \end{aligned}$$

where $v_1(\lambda) = \lambda^{1/2}(1-\lambda)^n(1-\ln\lambda)^{-1}$.
Here, we estimate the square, i.e., we introduce the function

$$v_2(\lambda) = (v_1(\lambda))^2 = \lambda(1-\lambda)^{2n}(1-\ln\lambda)^{-2}$$

with $v_2(0) = \lim_{\lambda\to 0^+} v_2(\lambda) = 0$, see the figure 2 and $v_2(1) = 0$. There exists the derivative

$$v_2{}'(\lambda) = (1-\ln\lambda)^{-2}(1-\lambda)^{2n}\left[\frac{2}{1-\ln\lambda} - \frac{2n\lambda}{1-\lambda} + 1\right].$$

Again using an auxilliary function

$$v_3(\lambda) = \frac{2}{1-\ln\lambda} - \frac{2n\lambda}{1-\lambda} + 1.$$

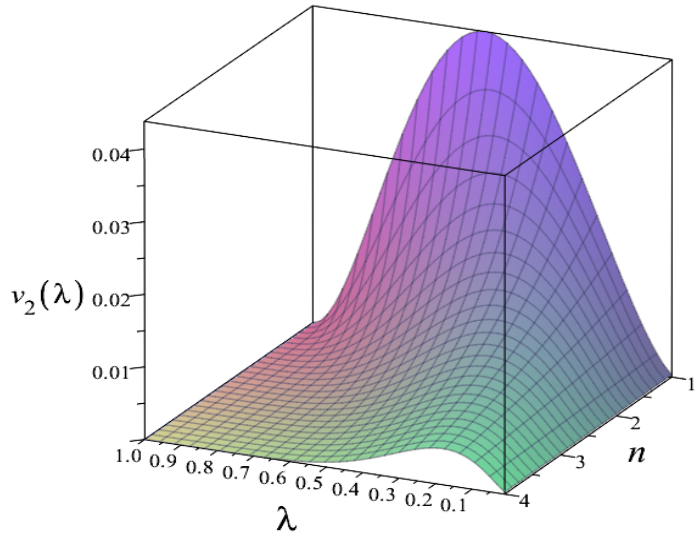


Figure 2 The plot shows the graph of $v_2(\lambda)$ for $1 \leq n \leq 4$.

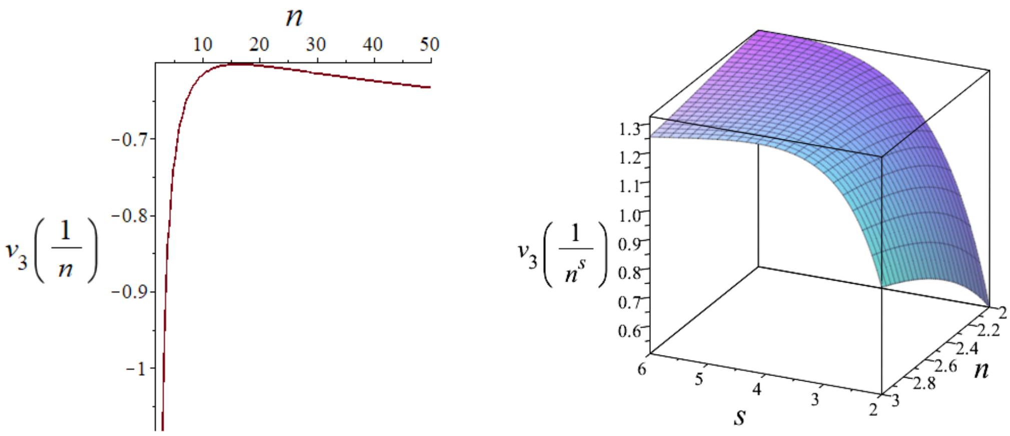
This means that a critical point of $v_1(\lambda)$ fulfills $v_3(\lambda) = 0$. For sufficiently large $s > 1$, there exists an $n_0(s)$ such that

$$v_3(\frac{1}{n}) = \frac{2}{1 + \ln n} - \frac{2n}{n-1} + 1 < 0,$$

see Figure 3(a) and

$$v_3(\frac{1}{n^s}) = \frac{2}{1 + s\ln n} - \frac{2n}{n^s - 1} + 1 > 0$$

for all $n \geq n_0(s)$, see Figure 3(b).



(a) (b)

Figure 3 The plot (a) shows the graph of $v_3(\frac{1}{n}) < 0$ and (b) shows the graph of $v_3(\frac{1}{n^s}) > 0$.

Thus, at least one critical point of v_1 has to lie in (n^{-s}, n^{-1}) with $n \geq \{n_0(s), 2\}$. Since ${v_2}'\left(\frac{1}{n^s}\right) > 0$ and ${v_2}'\left(\frac{1}{n}\right) < 0$, this must be a local maximum point. Then substituting $\lambda = e^{-x}$, $x \in (0, \infty)$ into v_2, we have

$$v_2(e^{-x}) = e^{-x}(1 - e^{-x})^{2n}(1 - \ln e^{-x})^{-2} := v_4(x).$$

One has to determine that $v_4(x)$ has at most two critical points on $(0, \infty)$. Therefore, we are looking for the roots of

$$\frac{{v_4}'(x)}{v_4(x)} = 2ne^{-x}(1 - e^{-x})^{-1} - 2(1 + x)^{-1} - 1 = 0$$

and get

$$(x + 3)e^x = (2n + 1)x + (2n + 3).$$

Thus, at least one intersection between the exponential curve $y = (x + 3)e^x$ and the straight line $y = (2n + 1)x + (2n + 3)$ could occur. Futhermore, one concludes that if a second critical point exists, it can only be a saddle point.

For $1 \leq r \leq s$ we have $n^{-r} \in (n^{-s}, n^{-1})$ and

$$\begin{aligned} v_2(n^{-r}) &= \left(\frac{1}{n^r}\right)\left(1-\frac{1}{n^r}\right)^{2n}\left(1+r\ln n^{-r}\right)^{-2} \\ &\leq rn^{-1}(\ln n)^{-2} \\ &\leq rn^{-1}\left(\frac{\ln(n+e)}{\ln n}\right)^{-2}(\ln(n+e))^{-2} \\ &\leq \widehat{C}(n)^{-1}(\ln(n+e))^{-2}. \end{aligned}$$

Since $(\frac{\ln(n+e)}{\ln n})$ is bounded independently of n, so

$$\begin{aligned} v_1(\lambda) &\leq \widehat{C}(n)^{-1/2}(\ln(n+e))^{-1} \\ &\leq \widehat{C}(n+1)^{-1/2}(\ln(n+e))^{-1} \end{aligned}$$

where $\widehat{C}$ is constant. Thus, we get

$$\|A_n K(I-K^*K)^n e_0\| \leq \widehat{C}\,\|w\|\,(n+1)^{-1/2}(\ln(n+e))^{-1}.$$

This completes the proof of our Lemma 2.2.

■

Proposition 2.3. *Let the linear operator K be bounded such that $\|K\| \leq 1$ and for $i = 1, 2, ..., n$, $\alpha_i = \frac{1}{2}i^{-1/2}$ and $\alpha_0 = \frac{1}{2}$ be given. The following estimates hold with positive constant $C, \widehat{C}, \overline{C}, \widetilde{C}, E$:*

$$\left\|\sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})e_0\right\| \leq \overline{C}(\ln(n+e))^{-1}\|w\|\,E \tag{2.5}$$

$$\left\|K\sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})e_0\right\| \leq \widetilde{C}(n+1)^{-1/2}(\ln(n+e))^{-1}\|w\|\,E. \tag{2.6}$$

Proof Firstly, we start with using the spectral theory that may proceed in the same way as in Lemma 2.2 where n is substituted by j to show (2.5) as follow

$$\begin{aligned} \left\|(I-K^*K)^j e_0\right\| &= \left\|(I-K^*K)^j f(K^*K)w\right\| \\ &\leq \sup_{\lambda\in\left(0,\|K\|^2\right]}\left\{\left|(1-\lambda)^j\left(\ln\left(\frac{e}{\lambda}\right)\right)^{-1}\right|\right\}\|w\| \\ &\leq \sup_{\lambda\in\left(0,\|K\|^2\right]}|h_1(\lambda)|\,\|w\| \\ &\leq C\left(\ln(j+e)\right)^{-1}\|w\|\,. \end{aligned} \tag{2.7}$$

Then, using (2.7), we obtain

$$
\begin{aligned}
&\left\| \sum_{j=0}^{n-1} \alpha_{n-j-1} (I - K^*K)^j \prod_{i=1}^{j} (1 - \alpha_{n-i}) e_0 \right\| \\
&\le \sum_{j=0}^{n-1} \alpha_{n-j-1} \prod_{i=1}^{j} (1 - \alpha_{n-i}) \left\| (I - K^*K)^j e_0 \right\| \\
&\le C \sum_{j=0}^{n-1} \alpha_{n-j-1} \prod_{i=1}^{j} (1 - \alpha_{n-i}) \left(\ln(j+e)\right)^{-1} \|w\| . \qquad (2.8)
\end{aligned}
$$

Rewritting (2.8), we have

$$
\begin{aligned}
&C \sum_{j=0}^{n-1} \alpha_{n-j-1} \prod_{i=1}^{j} (1 - \alpha_{n-i}) \left(\ln(j+e)\right)^{-1} \|w\| \\
&\le C \|w\| \sum_{j=0}^{n-2} \alpha_{n-j-1} \prod_{i=1}^{j} (1 - \alpha_{n-i}) \left(\ln(j+e)\right)^{-1} \\
&\quad + \alpha_0 \prod_{i=1}^{n-1} (1 - \alpha_{n-i}) \left(\ln(n-1+e)\right)^{-1} \|w\| . \qquad (2.9)
\end{aligned}
$$

Substituting $\alpha_i = \frac{1}{2} i^{-1/2}$ into (2.9), we have

$$
\begin{aligned}
&C \|w\| \sum_{j=0}^{n-2} \alpha_{n-j-1} \prod_{i=1}^{j} (1 - \alpha_{n-i}) \left(\ln(j+e)\right)^{-1} \\
&+ \alpha_0 \prod_{i=1}^{n-1} (1 - \alpha_{n-i}) \left(\ln(n-1+e)\right)^{-1} \|w\| \\
&= C \|w\| \sum_{j=0}^{n-2} \frac{1}{2} (n-j-1)^{-1/2} \prod_{i=1}^{j} \left(1 - \frac{1}{2}(n-i)^{-1/2}\right) \left(\ln(j+e)\right)^{-1} \\
&\quad + \frac{1}{2} \prod_{i=1}^{n-1} \left(1 - \frac{1}{2}(n-i)^{-1/2}\right) \left(\ln(n-1+e)\right)^{-1} \|w\| . \qquad (2.10)
\end{aligned}
$$

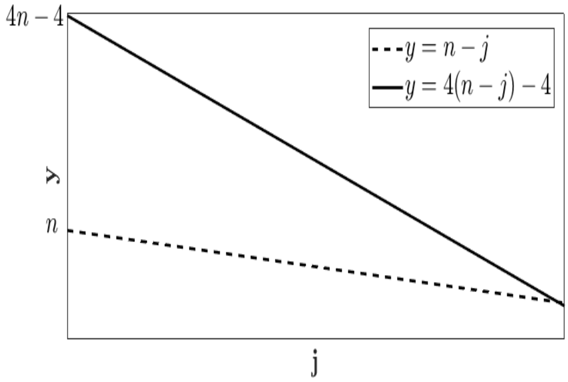


Figure 4 The plot shows that $n-j \leq 4(n-j)-4$ for $0 \leq j < n-1$.

From figure 4, we know that $n-j \leq 4(n-j)-4$ and

$$(n-j)^{1/2} \leq 2((n-j)-1)^{1/2}$$

or

$$\frac{1}{2}(n-j-1)^{-1/2} \leq (n-j)^{-1/2}.$$

Using the fact that $1-\frac{1}{2}(n-i)^{-1/2} \leq 1$ for $i=1,...,j$, we have

$$\begin{aligned}
&\prod_{i=1}^{j}\left(1-\frac{1}{2}\,(n-i)^{-1/2}\right)\\
&\leq \left(1-\frac{1}{2}\,(n-1)^{-1/2}\right)\left(1-\frac{1}{2}\,(n-2)^{-1/2}\right)...\left(1-\frac{1}{2}\,(n-j)^{-1/2}\right)\\
&\leq 1\\
&\leq (j+1)^{-1/2} \quad \text{for} \quad j \leq n.
\end{aligned}$$

Then, rewritting (2.10) by applying above information, we have

$$
\begin{aligned}
&C\,\|w\| \sum_{j=0}^{n-2} \frac{1}{2}\,(n-j-1)^{-1/2} \prod_{i=1}^{j} \left(1-\frac{1}{2}(n-i)^{-1/2}\right) (\ln(j+e))^{-1} \\
&+ \frac{1}{2} \prod_{i=1}^{n-1} \left(1-\frac{1}{2}(n-i)^{-1/2}\right) (\ln(n-1+e))^{-1}\,\|w\| \\
&\leq C\,\|w\| \sum_{j=0}^{n-2} (n-j)^{-1/2}\,(j+1)^{-1/2}\,(\ln(j+e))^{-1} \\
&+ \frac{1}{2}\,(j+1)^{-1/2}\,(\ln(n-1+e))^{-1}\,\|w\| \\
&\leq C\,\|w\| \sum_{j=0}^{n-2} \left(\frac{n-j}{n+1}\right)^{-1/2} \left(\frac{j+1}{n+1}\right)^{-1/2} (\ln(j+e))^{-1} \left(\frac{1}{n+1}\right) \\
&+ \frac{1}{2}\,(j+1)^{-1/2}\,(\ln(n-1+e))^{-1}\,\|w\|. \qquad (2.11)
\end{aligned}
$$

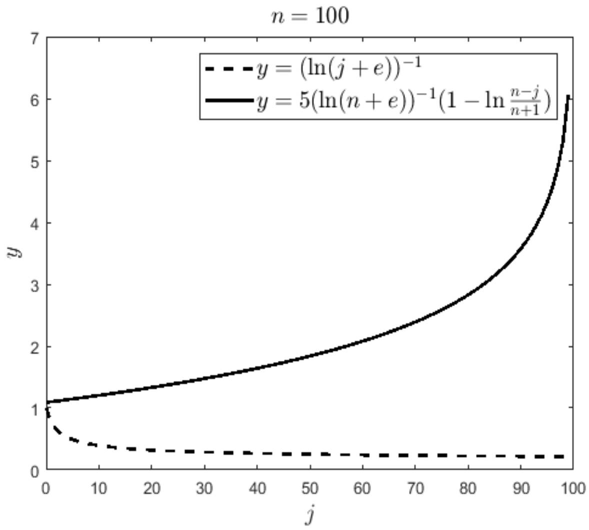


Figure 5 Graph of $(\ln(j+e))^{-1}$ and $c_0\,(\ln(n+e))^{-1}\left(1-\ln\left(\frac{n-j}{n+1}\right)\right)$ against j with $c_0 = 5$.

From Figure 5, there is a positive number c_0 such

$$
(\ln(j+e))^{-1} < c_0\,(\ln(n+e))^{-1}\left(1-\ln\left(\frac{n-j}{n+1}\right)\right).
$$

So (2.11) becomes

$$
\begin{aligned}
&C\left\|w\right\|\sum_{j=0}^{n-2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{j+1}{n+1}\right)^{-1/2}(\ln(j+e))^{-1}\left(\frac{1}{n+1}\right)\\
&+\frac{1}{2}\left(j+1\right)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\left\|w\right\|\\
&\leq c_0C\left(\ln(n+e)\right)^{-1}\left\|w\right\|\sum_{j=0}^{n-2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(1-\ln\left(\frac{n-j}{n+1}\right)\right)\left(\frac{1}{n+1}\right)\\
&+\frac{1}{2}\left(j+1\right)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\left\|w\right\|. \qquad (2.12)
\end{aligned}
$$

The first summation of (2.12) is bounded by the integral

$$
\begin{aligned}
&\sum_{j=0}^{n-2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(1-\ln\left(\frac{n-j}{n+1}\right)\right)\left(\frac{1}{n+1}\right)\\
&\leq\int_s^{1-s}x^{-1/2}(1-x)^{-1/2}(1-\ln(1-x))dx\\
&\leq E
\end{aligned}
$$

where E is positive constant and $s:=\frac{1}{2(n+1)}$. So (2.12) becomes

$$
\begin{aligned}
&C\left\|w\right\|\sum_{j=0}^{n-2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{j+1}{n+1}\right)^{-1/2}(\ln(j+e))^{-1}\left(\frac{1}{n+1}\right)\\
&+\frac{1}{2}\left(j+1\right)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\left\|w\right\|\\
&\leq\breve{C}(\ln(n+e))^{-1}\left\|w\right\|E+\frac{1}{2}\left(j+1\right)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\left\|w\right\| \qquad (2.13)
\end{aligned}
$$

with $\breve{C}=c_0C$. From (2.13) there is $c_1\in\mathbb{R}^+$ such that $\frac{1}{2}\left(j+1\right)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\leq c_1(\ln(n+e))^{-1}E$ then

$$
\begin{aligned}
&\breve{C}(\ln(n+e))^{-1}\left\|w\right\|E+\frac{1}{2}\left(j+1\right)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\left\|w\right\|\\
&\leq(\ln(n+e))^{-1}\left\|w\right\|\left[\breve{C}+c_1\right]E\\
&\leq\overline{C}(\ln(n+e))^{-1}\left\|w\right\|E \qquad (2.14)
\end{aligned}
$$

with $\overline{C}=\breve{C}+c_1$. Thus

$$
\left\|\sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})e_0\right\|\leq\overline{C}(\ln(n+e))^{-1}\left\|w\right\|E.
$$

Next, we will show the equation (2.6) in the same way of (2.5). We have

$$\begin{aligned}\left\|K(I-K^*K)^j e_0\right\| &= \left\|K\left(I-K^*K\right)^j f(K^*K)w\right\| \\ &\le \sup_{\lambda\in\left(0,\|K\|^2\right]}\left\{\left|\lambda^{1/2}\left(1-\lambda\right)^j\left(\ln\left(\frac{e}{\lambda}\right)^{-1}\right)\right|\right\}\|w\| \\ &\le \widehat{C}(j+1)^{-1/2}\left(\ln\left(j+e\right)\right)^{-1}\|w\|. \qquad (2.15)\end{aligned}$$

Using (2.15), we obtain

$$\begin{aligned}&\left\|K\sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})e_0\right\| \\ &\le \sum_{j=0}^{n-1}\alpha_{n-j-1}\prod_{i=1}^{j}(1-\alpha_{n-i})\left\|K(I-K^*K)^j e_0\right\| \\ &\le \widehat{C}\sum_{j=0}^{n-1}\alpha_{n-j-1}\prod_{i=1}^{j}(1-\alpha_{n-i})(j+1)^{-1/2}\left(\ln(j+e)\right)^{-1}\|w\|. \qquad (2.16)\end{aligned}$$

From the previous estimates we have $\prod_{i=1}^{j}(1-\alpha_{n-i})\le 1\le(n+1)^{-1/2}$, $\alpha_{n-j-1}\le(n-j)^{-1/2}$ with $\alpha_i=\frac{1}{2}i^{-1/2}$ for $i=1,...,j$, $\alpha_0=\frac{1}{2}$ and $(\ln(j+e))^{-1} < c_0\left(\ln(n+e)\right)^{-1}\left(1-\ln\left(\frac{n-j}{n+1}\right)\right)$. Then, we simplify the equation

(2.16) as

$$
\begin{aligned}
&\widehat{C}\sum_{j=0}^{n-1}\alpha_{n-j-1}\prod_{i=1}^{j}(1-\alpha_{n-i})(j+1)^{-1/2}\left(\ln(j+e)\right)^{-1}\|w\| \\
&\leq \widehat{C}\sum_{j=0}^{n-2}\alpha_{n-j-1}\prod_{i=1}^{j}(1-\alpha_{n-i})(j+1)^{-1/2}\left(\ln(j+e)\right)^{-1}\|w\| \\
&\quad + \alpha_0\prod_{i=1}^{j}(1-\alpha_{n-i})n^{-1/2}\left(\ln(n-1+e)\right)^{-1}\|w\| \\
&\leq \widehat{C}(n+1)^{-1/2}\|w\|\sum_{j=0}^{n-2}(n-j)^{-1/2}(j+1)^{-1/2}\left(\ln(j+e)\right)^{-1} \\
&\quad + \frac{1}{2}n^{-1/2}\left(\ln(n-1+e)\right)^{-1}\|w\| \\
&\leq \widehat{C}(n+1)^{-1/2}\|w\|\sum_{j=0}^{n-2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(\ln(j+e)\right)^{-1}\left(\frac{1}{n+1}\right) \\
&\quad + \frac{1}{2}n^{-1/2}(n+1)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\|w\| \\
&\leq c_0\widehat{C}(n+1)^{-1/2}\left(\ln(n+e)\right)^{-1}\|w\| \\
&\quad \times\left[\sum_{j=0}^{n-2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(1-\ln\left(\frac{n-j}{n+1}\right)\right)\left(\frac{1}{n+1}\right)\right] \\
&\quad + \frac{1}{2}n^{-1/2}(n+1)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\|w\| \\
&= \ddot{C}(n+1)^{-1/2}\left(\ln(n+e)\right)^{-1}\|w\|E+\frac{1}{2}n^{-1/2}(n+1)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\|w\|
\end{aligned}
\tag{2.17}
$$

with $\ddot{C}=c_0\widehat{C}$. From (2.17) there is $c_2\in\mathbb{R}^+$ such that $\frac{1}{2}n^{-1/2}\left(\ln(n-1+e)\right)^{-1}\leq c_2\left(\ln(n+e)\right)^{-1}E$ then

$$
\begin{aligned}
&\ddot{C}(n+1)^{-1/2}\left(\ln(n+e)\right)^{-1}\|w\|E+\frac{1}{2}n^{-1/2}(n+1)^{-1/2}\left(\ln(n-1+e)\right)^{-1}\|w\| \\
&\leq (n+1)^{-1/2}\left(\ln(n+e)\right)^{-1}\|w\|\left[\ddot{C}+c_2\right]E \\
&\leq \widetilde{C}(n+1)^{-1/2}\left(\ln(n+e)\right)^{-1}\|w\|E
\end{aligned}
$$

with $\widetilde{C}=\ddot{C}+c_2$.Thus

$$
\left\|K\sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})e_0\right\|\leq\widetilde{C}(n+1)^{-1/2}(\ln(n+e))^{-1}\|w\|E.
$$

This ends the proof of the Proposition 2.3. ■

Assumption 2.4. *There exist positive constants c_L, c_R, c_r and linear bounded operator $R_x : Y \to Y$ such that for $x \in B_\rho(x_0)$ the following condition hold*

$$F'(x) = R_x F'(x^+) \tag{2.18}$$

$$\|R_x - I\| \leq c_L \left\|x - x^+\right\| \tag{2.19}$$

$$|\|R_x\| - \|I\|| \geq c_R \tag{2.20}$$

$$\|R_x\| \leq c_r \tag{2.21}$$

where x^+ is exact solution of (1.7).

Lemma 2.5. *Let* **Assumption2.4** *be assumed. Then we have*

$$\left\|(1-\alpha_n)I - R^*_{x_n^\delta}\right\| \leq \frac{1}{2} K_R \|e_n\| \tag{2.22}$$

for some constant $K_R > 0$.

Proof We note that reverse triangle inequality and (2.20) guarantee the estimates

$$1 \leq \frac{\|R_x - I\|}{|\|R_x\| - \|I\||} \leq c_R^{-1} \|R_x - I\| \tag{2.23}$$

and

$$\|I + R_x^*\| \leq \frac{1}{|\|R_x\| - \|I\||} \times \|I - R_x^*\| \, \|I + R_x^*\| \leq c_R^{-1} \|I - R_x^*\| \, \|I + R_x^*\| \, . \tag{2.24}$$

Using the estimates (2.19), (2.21), (2.23), (2.24) and the triangle inequality, we now have

$$\begin{aligned}
\left\|(1-\alpha_n)I - R^*_{x_n^\delta}\right\| &= \left\|\frac{1}{2}\left[(1-(1+\alpha_n))(I + R^*_{x_n^\delta})\right] + \frac{1}{2}\left[(1+(1-\alpha_n))(I - R^*_{x_n^\delta})\right]\right\| \\
&\leq \frac{1}{2}\left\|(1-(1+\alpha_n))(I + R^*_{x_n^\delta})\right\| + \frac{1}{2}\left\|(1+(1-\alpha_n))(I - R^*_{x_n^\delta})\right\| \\
&= \frac{1}{2}\left[\alpha_n c_R^{-1}\left\|I + R^*_{x_n^\delta}\right\| + |2-\alpha_n|\right]\left\|(I - R^*_{x_n^\delta})\right\| \\
&\leq \frac{1}{2}\left[\alpha_n c_R^{-1}\left(\|I\| + \left\|R^*_{x_n^\delta}\right\|\right) + |2-\alpha_n|\right]\left\|(I - R^*_{x_n^\delta})\right\| \\
&= \frac{1}{2}\left[\frac{1}{2}n^{-1/2} c_R^{-1}\left(\|I\| + c_r\right) + \left|2 - \frac{1}{2}n^{-1/2}\right|\right] c_L \left\|x^+ - x_n^\delta\right\| \\
&\leq \frac{1}{2}\left[c_R^{-1}\left(\|I\| + c_r\right) + 2\right] c_L \left\|x^+ - x_n^\delta\right\| \\
&\leq \frac{1}{2} K_R \|e_n\|
\end{aligned}$$

with the positive constant $K_R = \frac{1}{2}\left[c_R^{-1}\left(\|I\| + c_r\right) + 2\right] c_L$. ■

Proposition 2.6. *Let the condition* (2.18) *and* (2.19) *in* **Assumption2**.4 *be hold. Then*

$$\left\|F(x_n^\delta) - F(x^+) - F'(x^+)(x_n^\delta - x^+)\right\| \leq \frac{1}{2}c_L \|e_n\| \|Ke_n\| \tag{2.25}$$

for $x \in B_\rho(x_0)$.

Proof *Define* $w_t = x^+ + t(x_n^\delta - x^+)$ *as* $0 \leq t \leq 1$. *So*

$$\int_0^1 F'(w_t)dt = F(x_n^\delta) - F(x^+). \tag{2.26}$$

Using the mean- value theorem with (2.18) *and* (2.19), *we obtain*

$$\begin{aligned}
\left\|F(x_n^\delta) - F(x^+) - F'(x^+)(x_n^\delta - x^+)\right\| &\leq \left\|\int_0^1 \left[F'(x^+ + t(x_n^\delta - x^+)) - F'(x^+)(x_n^\delta - x^+)\right] dt\right\| \\
&= \left\|\int_0^1 \left[R_{w_t}F'(x^+) - F'(x^+)(x_n^\delta - x^+)\right] dt\right\| \\
&= \left\|\int_0^1 \left[(R_{w_t} - I)F'(x^+)(x_n^\delta - x^+)\right] dt\right\| \\
&\leq \int_0^1 \|R_{w_t} - I\| \left\|F'(x^+)(x_n^\delta - x^+)\right\| dt \\
&\leq \int_0^1 c_L \left\|x^+ + t(x_n^\delta - x^+) - x^+\right\| \left\|F'(x^+)(x_n^\delta - x^+)\right\| dt \\
&= c_L \int_0^1 \left\|t(x_n^\delta - x^+)\right\| dt \left\|F'(x^+)(x_n^\delta - x^+)\right\| \\
&= \frac{1}{2}c_L \left\|x_n^\delta - x^+\right\| \left\|F'(x^+)(x_n^\delta - x^+)\right\| \\
&= \frac{1}{2}c_L \|e_n\| \|Ke_n\|.
\end{aligned} \tag{2.27}$$

■

Now we present the main theorem of this thesis.

Theorem 2.7. Assume that problem (1.7) has a solution x^+ in $B_{\frac{\varrho}{2}}(x_0)$, y^δ fulfills (1.8), F satisfies (2.18) and (2.19). Assume that the Fréchet derivative of F is scaling such that $\|F'(x)\| \leq 1$ for $x \in B_{\frac{\varrho}{2}}(x_0)$. Additionally assume that the source condition (1.14) and (2.2) is fulfilled with $p = 1$ and

$$\tau > \frac{2-\eta}{1-\eta}. \tag{2.28}$$

If $\|w\|$ is sufficiently small, then there exists a constant $\widehat{K}_2$ depending only on $p = 1$ and $\|w\|$ with

$$\left\|x^+ - x_n^\delta\right\| \leq \widehat{K}_2(\ln n)^{-1} \tag{2.29}$$

and

$$\left\|y^\delta - F(x_n^\delta)\right\| \le 4\widehat{K}_2(n+1)^{-1/2}(\ln n)^{-1} \tag{2.30}$$

for $2 < n < N$ where N denotes in termination index of the discrepancy (1.10).

Proof We give the abbreviation $e_n := x^+ - x_n^\delta$ for the error of the nth iteration x_n^δ of (2.1). We can rewrite the equation (2.1) into this form

$$x^+ - x_{n+1}^\delta = (1-\alpha_n)(x^+ - x_n^\delta) + F'(x_n^\delta)^*(F(x_n^\delta) - y^\delta) - \alpha_n(x_0 - x^+).$$

Since $e_n := x^+ - x_n^\delta$ and $K := F'(x^+)$, we present e_n as

$$\begin{aligned}
e_{n+1} &= (1-\alpha_n)e_n + F'(x_n^\delta)^*(F(x_n^\delta) - y^\delta) - \alpha_n(x_0 - x^+)\\
&= (1-\alpha_n)(I - K^*K)e_n + (1-\alpha_n)K^*Ke_n + F'(x_n^\delta)^*(F(x_n^\delta) - y^\delta) - \alpha_n(x_0 - x^+)\\
&= (1-\alpha_n)(I - K^*K)e_n + (1-\alpha_n)K^*\left[F(x_n^\delta) - F(x^+) - K(x_n^\delta - x^+)\right]\\
&\quad + \left[K^* - F'(x_n^\delta)^*\right]\left(y^\delta - F(x_n^\delta)\right) - \alpha_n K^*(y^\delta - F(x_n^\delta)) + (1-\alpha_n)K^*(y - y^\delta)\\
&\quad - \alpha_n(x_0 - x^+)\\
&= (1-\alpha_n)(I - K^*K)e_n + (1-\alpha_n)K^*\left[F(x_n^\delta) - F(x^+) - K(x_n^\delta - x^+)\right]\\
&\quad + \left[K^* - K^*R_{x_n^\delta}^*\right]\left(y^\delta - F(x_n^\delta)\right) - \alpha_n K^*(y^\delta - F(x_n^\delta)) + (1-\alpha_n)K^*(y - y^\delta)\\
&\quad - \alpha_n(x_0 - x^+)\\
&= (1-\alpha_n)(I - K^*K)e_n + (1-\alpha_n)K^*\left[F(x_n^\delta) - F(x^+) - K(x_n^\delta - x^+)\right]\\
&\quad + \left[(1-\alpha_n)I - R_{x_n^\delta}^*\right]K^*\left(y^\delta - F(x_n^\delta)\right) + (1-\alpha_n)K^*(y - y^\delta)\\
&\quad - \alpha_n(x_0 - x^+).
\end{aligned} \tag{2.31}$$

Rewritting the equation (2.31), we have

$$e_{n+1} = (1-\alpha_n)(I - K^*K)e_n + (1-\alpha_n)K^*(y - y^\delta) - \alpha_n(x_0 - x^+) + K^*z_n \tag{2.32}$$

where

$$z_n = [(1-\alpha_n)(F(x_n^\delta) - F(x^+) - K(x_n^\delta - x^+))] + [(1-\alpha_n)I - R_{x_n^\delta}^*](y^\delta - F(x_n^\delta)).$$

By recurrence and (2.32), we obtain the closed expression for the error

$$
\begin{aligned}
e_n = &\left[A_n(I-K^*K)^n + \sum_{j=0}^{n-1} \alpha_{n-j-1}(I-K^*K)^j \prod_{i=1}^{j}(1-\alpha_{n-i})\right] e_0 \\
&+ \left[\sum_{j=1}^{n}(I-K^*K)^{j-1} \prod_{i=1}^{j}(1-\alpha_{n-i})\right] K^*(y-y^\delta) \\
&+ \sum_{j=0}^{n-1} \prod_{i=n-j}^{n-1} (1-\alpha_i)(I-K^*K)^j K^* z_{n-j-1} \qquad (2.33)
\end{aligned}
$$

where $A_n = \prod_{i=1}^{n-1}(1-\alpha_i)$. Moreover, it holds

$$
\begin{aligned}
Ke_n = &\left[KA_n(I-K^*K)^n + K\sum_{j=0}^{n-1} \alpha_{n-j-1}(I-K^*K)^j \prod_{i=1}^{j}(1-\alpha_{n-i})\right] e_0 \\
&+ K\left[\sum_{j=1}^{n}(I-K^*K)^{j-1} \prod_{i=1}^{j}(1-\alpha_{n-i})\right] K^*(y-y^\delta) \\
&+ K\sum_{j=0}^{n-1} \prod_{i=n-j}^{n-1} (1-\alpha_i)(I-K^*K)^j K^* z_{n-j-1}. \qquad (2.34)
\end{aligned}
$$

Next for $0 \le n < N$, using the discrepancy principle, triangle inequality , (1.12) and (2.28), we get

$$
\begin{aligned}
\left\|y^\delta - F(x_n^\delta)\right\| &= 2\left\|y^\delta - F(x_n^\delta)\right\| - \left\|y^\delta - F(x_n^\delta)\right\| \\
&\le 2\left\|y^\delta - F(x_n^\delta)\right\| - \tau\delta \\
&\le 2\left(\left\|y^\delta - F(x_n^\delta)\right\| - \delta\right) \\
&\le 2\left(\left\|y^\delta - F(x_n^\delta)\right\| - \left\|y^\delta - y\right\|\right) \\
&\le 2\left\|y - F(x_n^\delta)\right\| \\
&\le \frac{2}{1-\eta}\left\|Ke_n\right\|. \qquad (2.35)
\end{aligned}
$$

Using the Proposition 2.3, Proposition 2.6 and (2.35), we obtain

$$
\begin{aligned}
\|z_n\| &\le (1-\alpha_n)\left\|F(x_n^\delta) - F(x^+) - K(x_n^\delta - x^+)\right\| + \left\|(1-\alpha_n)I - R^*_{x_n^\delta}\right\| \left\|y^\delta - F(x_n^\delta)\right\| \\
&\le \frac{1}{2}(1-\alpha_n)\left\|e_n\right\|\left\|Ke_n\right\| c_L + \frac{1}{2}K_R\left\|e_n\right\|\left(\frac{2}{1-\eta}\right)\left\|Ke_n\right\| \\
&\le K_1\left\|Ke_n\right\|\left\|e_n\right\| \qquad (2.36)
\end{aligned}
$$

where $K_1 = \frac{c_L}{2} + \frac{K_R}{1-\eta}$ and we use the fact that $1-\alpha_n \le 1$.
It hold that $\|e_n\|$ is decreasing independently of the source condition for $0 \le n < N$,

see Proposition 2.2 in [10].
Next, we show by induction that

$$\|e_n\| \leq \widehat{K}_2(\ln(n+e))^{-1} \tag{2.37}$$

and

$$\|Ke_n\| \leq \widehat{K}_2(n+1)^{-1/2}(\ln(n+e))^{-1} \tag{2.38}$$

hold for all $0 \leq n < N$ with $\widehat{K}_2$ being a positive constant which does not dependent on n. For $l = 0$, we obtain

$$\begin{aligned}
\|e_0\| &= \|f(K^*K)w\| \\
&\leq \sup_{\lambda\in(0,1]} (\ln e - \ln\lambda)^{-1} \|w\| \\
&\leq \|w\| \\
&\leq \widehat{K}_2
\end{aligned}$$

and

$$\begin{aligned}
\|Ke_0\| &= \|Kf(K^*K)w\| \\
&\leq \sup_{\lambda\in(0,1]} \lambda^{-1/2}(\ln e - \ln\lambda)^{-1} \|w\| \\
&\leq \|w\| \\
&\leq \widehat{K}_2.
\end{aligned}$$

For $l = 1$ we have

$$\begin{aligned}
\|e_1\| &= \left\|(1-\alpha_0)(I-K^*K)e_0 + (1-\alpha_0)K^*(y-y^\delta) + \alpha_0(x_0-x^+) + K^*z_0\right\| \\
&\leq \frac{1}{2}\|(I-K^*K)e_0\| + \frac{1}{2}\left\|K^*(y-y^\delta)\right\| + \frac{1}{2}\|e_0\| + \|K^*z_0\| \\
&\leq \frac{1}{2}\|w\| + \frac{1}{2}\delta + \frac{1}{2}\|w\| + K_1\|w\|^2 \\
&\leq \|w\| + \frac{1}{2}\delta + K_1\|w\|^2 \\
&\leq \|w\| + \frac{1}{2}\left(\frac{1}{(1-\eta)(\tau-1)}\right)\|w\| + K_1\|w\|^2 \\
&\leq \left[1 + \frac{1}{2(1-\eta)(\tau-1)}\right]\|w\| + K_1\|w\|^2
\end{aligned}$$

where we choose the sufficiently small $\|w\|$ that $\left(2+\frac{1}{(1-\eta)(\tau-1)}\right)\|w\|+2K_1\|w\|^2 \leq \widehat{K}_2$. So $\|e_1\|$ becomes

$$\begin{aligned}
\|e_1\| &\leq \frac{1}{2}\left[\left(2+\frac{1}{(1-\eta)(\tau-1)}\right)\|w\|+2K_1\|w\|^2\right] \\
&\leq \frac{1}{2}\widehat{K}_2 \\
&\leq \widehat{K}_2(\ln(1+e))^{-1}.
\end{aligned}$$

Similarly, for $\|Ke_1\|$, we have

$$\begin{aligned}
\|Ke_1\| &\leq \frac{1}{2}\|K(I-K^*K)e_0\|+\frac{1}{2}\left\|KK^*(y-y^\delta)\right\|+\frac{1}{2}\|Ke_0\|+\|KK^*z_0\| \\
&\leq \frac{1}{2}\|w\|+\|w\|+K_1\|w\|^2+\frac{1}{2}\delta+\frac{1}{2}\widehat{K}_2 \\
&= \frac{3}{2}\|w\|+K_1\|w\|^2+\frac{1}{2}\delta+\frac{1}{2}\widehat{K}_2 \\
&\leq \frac{3}{2}\|w\|+K_1\|w\|^2+\frac{1}{2(1-\eta)(\tau-1)}\|w\|+\frac{1}{2}\widehat{K}_2 \\
&= \left(\frac{3}{2}+K_1\|w\|+\frac{1}{2(1-\eta)(\tau-1)}\right)\|w\|+\frac{1}{2}\widehat{K}_2
\end{aligned}$$

where we choose the sufficiently small $\|w\|$ that $\left(\frac{3}{2}+K_1\|w\|+\frac{1}{2(1-\eta)(\tau-1)}\right)\|w\| \leq 0.01\widehat{K}_2$. So $\|Ke_1\|$ becomes

$$\begin{aligned}
\|Ke_1\| &\leq 0.51\widehat{K}_2 \\
&\leq \widehat{K}_2(1+1)^{-1/2}(\ln(n+e))^{-1/2}.
\end{aligned}$$

Thus, equations (2.37) and (2.38) are fulfilled since $\|e_l\|$ and $\|Ke_l\|$ are finite. Assuming that (2.37) and (2.38) are true for all k with $0 \leq k < n < N$, we have to show that (2.37) and (2.38) are true for all $k = n$.
Using Lemma 2.2 for $n > 1$ and Proposition 2.3, we rewrite (2.33) as follow

$$\begin{aligned}
\|e_n\| \leq \|A_n(I-K^*K)^n e_0\| &+ \left\|\sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})]e_0\right\| \\
&+ \left\|\sum_{j=1}^{n}(I-K^*K)^{j-1}\prod_{i=1}^{j}(1-\alpha_{n-i})K^*(y-y^\delta)\right\| \\
&+ \left\|\sum_{j=0}^{n-1}\prod_{i=n-j}^{n-1}(1-\alpha_i)(I-K^*K)^jK^*z_{n-j-1}\right\|.
\end{aligned}$$

By assumption $\|K\| \leq 1$, see e.g. [9] or [11] cited in [5], we have

$$\left\| \sum_{k=0}^{n-1} (I - K^*K)^k K^* \right\| \leq \sqrt{n},$$

and

$$\left\| (I - K^*K)^j K^* \right\| \leq (j+1)^{-1/2}.$$

With these bounds and $\|e_n\|$, we obtain

$$\begin{aligned} \|e_n\| &\leq C(\ln(n+e))^{-1} \|w\| + \overline{C}(\ln(n+e))^{-1} \|w\| E + \sqrt{n}\delta \\ &\quad + \sum_{j=0}^{n-1} (j+1)^{-1/2} \|z_{n-j-1}\| \end{aligned} \tag{2.39}$$

where we use

$$\begin{aligned} \left\| \sum_{j=1}^{n} (I - K^*K)^{j-1} \prod_{i=1}^{j} (1 - \alpha_{n-i}) K^*(y - y^\delta) \right\| &\leq \left\| \sum_{j=1}^{n} (I - K^*K)^{j-1} K^* \right\| \|y - y^\delta\| \\ &\leq \left\| \sum_{k=0}^{n-1} (I - K^*K)^{k} K^* \right\| \|y - y^\delta\| \\ &\leq \sqrt{n}\delta \end{aligned}$$

and

$$\begin{aligned} \left\| \sum_{j=0}^{n-1} \prod_{i=n-j}^{n-1} (1 - \alpha_i)(I - K^*K)^j K^* z_{n-j-1} \right\| &\leq \sum_{j=0}^{n-1} \left\| (I - K^*K)^j K^* \right\| \|z_{n-j-1}\| \\ &\leq \sum_{j=0}^{n-1} (j+1)^{-1/2} \| \, \|z_{n-j-1}\| \,. \end{aligned}$$

Then, using (2.36) to estimate the last term of (2.39), we obtain

$$\sum_{j=0}^{n-1} (j+1)^{-1/2} \|z_{n-j-1}\| \leq K_1 \sum_{j=0}^{n-1} (j+1)^{-1/2} \|Ke_{n-j-1}\| \|e_{n-j-1}\| \tag{2.40}$$

and we apply the assumption of the induction (2.37) and (2.38) into (2.40) we have

$$\begin{aligned}
&\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|z_{n-j-1}\right\| \\
&\leq K_1\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|Ke_{n-j-1}\right\|\left\|e_{n-j-1}\right\| \\
&\leq K_1\widehat{K}_2^2\sum_{j=0}^{n-1}(j+1)^{-1/2}(n-j)^{-1/2}(\ln(n-j-1+e))^{-2} \\
&= K_1\widehat{K}_2^2\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(\frac{n-j}{n+1}\right)^{-1/2}(\ln(n-j-1+e))^{-2}\left(\frac{1}{n+1}\right). \qquad (2.41)
\end{aligned}$$

Rewritting (2.41), we have

$$\begin{aligned}
&\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|z_{n-j-1}\right\| \\
&= K_1\widehat{K}_2^2\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{1}{n+1}\right)(\ln(n-j-1+e))^{-2}\times\left[\frac{\ln(n+e)}{\ln(n+e)}\right]^{-2} \\
&= K_1\widehat{K}_2^2(\ln(n+e))^{-2}\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{1}{n+1}\right)\left[\frac{\ln(n+e)}{\ln(n-j-1+e)}\right]^{2} \\
&\leq K_1\widehat{K}_2^2(\ln(n+e))^{-1}\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{1}{n+1}\right)\left[\frac{\ln(n+e)}{\ln(n-j-1+e)}\right]^{2}.
\end{aligned}$$

We know from Figure 6 that $\frac{\ln(n+e)}{\ln(n-j-1+e)} < 1-\ln(\frac{n-j}{n+1})$.
The above equation can be estimated as follow

$$\begin{aligned}
&\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|z_{n-j-1}\right\| \\
&\leq K_1\widehat{K}_2^2(\ln(n+e))^{-1}\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1/2}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{1}{n+1}\right)\left[1-\ln(\frac{n-j}{n+1})\right]^{2}. \qquad (2.42)
\end{aligned}$$

The last summation is bounded since with $s := \frac{1}{2(n+1)}$ the integral

$$\int_s^{1-s} x^{-1/2}(1-x)^{-1/2}(1-\ln(1-x))^2dx$$

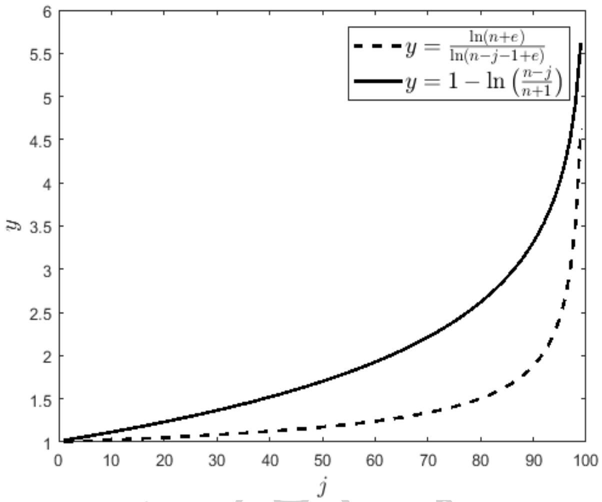


Figure 6 The plot shows the graph of $\frac{\ln(n+e)}{\ln(n-j-1+e)} < 1 - \ln(\frac{n-j}{n+1})$ where $n = 100$.

is bounded from above by a positive constant G independently of n. Substituting the above estimation into (2.39) yields

$$\begin{aligned}\|e_n\| &\leq C(\ln(n+e))^{-1}\|w\| + \overline{C}(\ln(n+e))^{-1}\|w\| E + \sqrt{n}\delta + K_1\widehat{K}_2^2(\ln(n+e))^{-1}G \\ &\leq \left[(C+\overline{C}E)\|w\| + K_1\widehat{K}_2^2 G\right](\ln(n+e))^{-1} + \sqrt{n}\delta \\ &\leq \left[(C+\overline{C}E)\|w\| + K_3\widehat{K}_2^2\right](\ln(n+e))^{-1} + \sqrt{n}\delta\end{aligned}$$

with $K_3 := K_1 G$.
Similarly, (2.34) can be rewritten as

$$\begin{aligned}\|Ke_n\| \leq \|KA_n(I-K^*K)^n e_0\| &+ \left\| K\sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})e_0\right\| \\ &+ \left\| K\sum_{j=1}^{n}(I-K^*K)^{j-1}\prod_{i=1}^{j}(1-\alpha_{n-i})K^*(y-y^\delta)\right\| \\ &+ \left\| K\sum_{j=0}^{n-1}\prod_{i=n-j}^{n-1}(1-\alpha_i)(I-K^*K)^j K^* z_{n-j-1}\right\|.\end{aligned}$$

By assumption $\|K\| \leq 1$, see e.g. [9] or [11] cited in [5], we have

$$\left\|(I-K^*K)^j KK^*\right\| \leq (j+1)^{-1}$$

and

$$\left\| K \sum_{k=0}^{n-1} (I - K^*K)^k K^* \right\| \leq \left\| I - (I - KK^*)^k \right\| \leq 1.$$

With these bounds and $\|Ke_n\|$, we obtain

$$\begin{aligned} \|Ke_n\| \leq \widehat{C}(n+1)^{-1/2}(\ln(n+e))^{-1}\|w\| + \widetilde{C}(n+1)^{-1/2}(\ln(n+e))^{-1}\|w\| E \\ + \delta + \sum_{j=0}^{n-1}(j+1)^{-1}\|z_{n-j-1}\| \end{aligned} \tag{2.43}$$

where we use

$$\begin{aligned} \left\| K \sum_{j=1}^{n} (I - K^*K)^{j-1} \prod_{i=1}^{j} (1 - \alpha_{n-i}) K^*(y - y^\delta) \right\| &\leq \left\| K \sum_{j=1}^{n} (I - K^*K)^{j-1} K^* \right\| \|y - y^*\| \\ &= \left\| K \sum_{k=0}^{n-1} (I - K^*K)^{k} K^* \right\| \| \|y - y^*\| \\ &\leq \left\| I - (I - KK^*)^k \right\| \delta \\ &\leq \delta \end{aligned}$$

and

$$\begin{aligned} \left\| K \sum_{j=0}^{n-1} \prod_{i=n-j}^{n-1} (1 - \alpha_i)(I - K^*K)^j K^* z_{n-j-1} \right\| &\leq \sum_{j=0}^{n-1} \left\| (I - K^*K)^j KK^* \right\| \|z_{n-j-1}\| \\ &\leq \sum_{j=0}^{n-1} (j+1)^{-1} \|z_{n-j-1}\| . \end{aligned}$$

We may estimate the last term of (2.43)

$$
\begin{aligned}
&\sum_{j=0}^{n-1}(j+1)^{-1}\left\|z_{n-j-1}\right\| \\
&\leq K_1 \sum_{j=0}^{n-1}(j+1)^{-1}\left\|Ke_{n-j-1}\right\|\left\|e_{n-j-1}\right\| \\
&= K_1\widehat{K}_2^2 \sum_{j=0}^{n-1}(j+1)^{-1}(n-j)^{-1/2}(\ln(n-j-1+e))^{-2} \\
&= K_1\widehat{K}_2^2 \sum_{j=0}^{n-1}(j+1)^{-1}(n-j)^{-1/2}(\ln(n-j-1+e))^{-2}\left(\frac{n+1}{n+1}\right)^{3/2} \\
&= K_1\widehat{K}_2^2 \sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1}\left(\frac{n-j}{n+1}\right)^{-1/2}(\ln(n-j-1+e))^{-2}\left(\frac{1}{n+1}\right) \\
&\quad \times (n+1)^{-1/2}\left(\frac{\ln(n+e)}{\ln(n+e)}\right)^{2} \\
&= K_1\widehat{K}_2^2 (n+1)^{-1/2}(\ln(n+e))^{-1}\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1}\left(\frac{n-j}{n+1}\right)^{-1/2} \\
&\quad \times \left(\frac{\ln(n+e)}{\ln(n-j-1+e)}\right)^{2}\left(\frac{1}{n+1}\right)(\ln(n+e))^{-1} \\
&\leq K_1\widehat{K}_2^2 (n+1)^{-1/2}(\ln(n+e))^{-1}) \\
&\quad \times \left[\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(\frac{\ln(n+e)}{\ln(n-j-1+e)}\right)^{2}\left(\frac{1}{n+1}\right)\right] \\
&\leq K_1\widehat{K}_2^2 (n+1)^{-1/2}(\ln(n+e))^{-1}) \\
&\quad \times \left[\sum_{j=0}^{n-1}\left(\frac{j+1}{n+1}\right)^{-1}\left(\frac{n-j}{n+1}\right)^{-1/2}\left(1-\ln\left(\frac{n-j}{(n+1)}\right)\right)^{2}\left(\frac{1}{n+1}\right)\right].
\end{aligned}
$$

Turning to consider the term in brackets $[\cdot]$ above. It can be estimated by

$$\int_s^{1-s} x^{-1}(1-x)^{-1/2}(1-\ln(1-x))^2 dx \leq H \qquad (2.44)$$

with a positive constant H independently of n. Substituting above information into (2.43) yields

$$\|Ke_n\| \leq \left[\left(\widehat{C}+\widetilde{C}E\right)\|w\| + K_1\widehat{K}_2^2 H\right](n+1)^{-1/2}(\ln(n+e))^{-1}+\delta.$$

With K_4= $K_1 H$, K_5:=max $\left\{C+\overline{C}E, \widehat{C}+\widetilde{C}E\right\}$, K_6:= max $\{K_3, K_4\}$, then we get

$$\|e_n\| \leq \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (\ln(n+e))^{-1} + \sqrt{n}\delta \tag{2.45}$$

and

$$\|Ke_n\| \leq \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (n+1)^{-1/2} (\ln(n+e))^{-1} + \delta. \tag{2.46}$$

Because of (1.10) and (1.12) we have

$$\begin{aligned}
\tau\delta &\leq \left\|y^\delta - F(x_n^\delta)\right\| \\
&\leq \left\|y^\delta - y\right\| + \left\|y - F(x_n^\delta)\right\| \\
&\leq \delta + \frac{1}{1-\eta} \|Ke_n\|
\end{aligned}$$

and moreover,

$$\begin{aligned}
(1-\eta)(\tau-1)\delta &\leq \|Ke_n\| \\
&\leq \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (n+1)^{-1/2} (\ln(n+e))^{-1} + \delta.
\end{aligned} \tag{2.47}$$

Due to (2.28) , $\Theta = (1-\eta)(\tau-1) - 1 > 0$. We can rewrite(2.47) as follow

$$\delta \leq \frac{1}{\Theta} \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (n+1)^{-1/2} (\ln(n+e))^{-1}. \tag{2.48}$$

Furthermore, it follows

$$\begin{aligned}
\|e_n\| &\leq \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (\ln(n+e))^{-1} + \sqrt{n}\delta \\
&\leq \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (\ln(n+e))^{-1} + \frac{\sqrt{n}}{\Theta} \left(K_5 \|w\| + K_6 \widehat{K}_2^2\right) (n+1)^{-1/2} (\ln(n+e))^{-1} \\
&\leq \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (\ln(n+e))^{-1} \\
&\quad + (n+1)^{1/2} \left[\frac{1}{\Theta} \left(K_5 \|w\| + K_6 \widehat{K}_2^2\right) (n+1)^{-1/2} (\ln(n+e))^{-1}\right] \\
&\leq \left(1 + \frac{1}{\Theta}\right) \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (\ln(n+e))^{-1} \\
&= K_7 \left[K_5 \|w\| + K_6 \widehat{K}_2^2\right] (\ln(n+e))^{-1}
\end{aligned}$$

with $K_7 = 1 + \frac{1}{\Theta}$.

In similar manner, we obtain

$$\begin{aligned}\|Ke_n\| &\le \left[K_5\|w\| + K_6\widehat{K}_2^2\right](n+1)^{-1/2}(\ln(n+e))^{-1} + \delta \\ &\le \left[K_5\|w\| + K_6\widehat{K}_2^2\right](n+1)^{-1/2}(\ln(n+e))^{-1} \\ &\quad + \frac{1}{\Theta}\left[K_5\|w\| + K_6\widehat{K}_2^2\right](n+1)^{-1/2}(\ln(n+e))^{-1} \\ &\le \left(1+\frac{1}{\Theta}\right)\left[K_5\|w\| + K_6\widehat{K}_2^2\right](n+1)^{-1/2}(\ln(n+e))^{-1} \\ &= K_7\left[K_5\|w\| + K_6\widehat{K}_2^2\right](n+1)^{-1/2}(\ln(n+e))^{-1}.\end{aligned}$$

Finally, we select $\|w\|$ such that $K_7\left[K_5\|w\| + K_6\widehat{K}_2^2\right] \le \widehat{K}_2$. This is always possible for sufficiently small $\|w\|$ and because e_0 fulfills the source condition. Therefore, the induction is finished. Thus assertion (2.29) yield due to

$$\|e_n\| \le \widehat{K}_2\left(\frac{\ln n}{\ln(n+e)}\right)(\ln n)^{-1} \le \widehat{K}_2(\ln n)^{-1}$$

and similarly, by using (2.35), we have

$$\begin{aligned}\left\|y^\delta - F(x_n^\delta)\right\| &\le \frac{2}{1-\eta}\widehat{K}_2(n+1)^{-1/2}(\ln(n+e))^{-1} \\ &\le 4\widehat{K}_2(n+1)^{-1/2}\left(\frac{\ln n}{\ln(n+e)}\right)(\ln n)^{-1} \\ &\le 4\widehat{K}_2(n+1)^{-1/2}(\ln n)^{-1}.\end{aligned}$$

Thus, the assertion (2.30) holds. ■

Theorem 2.8. Under the assumptions of the Theorem 2.7, we have

$$N^{1/2}(\ln N) \le \frac{C_1}{\delta}$$

and

$$\left\|x^+ - x_N^\delta\right\| \le C_2(-\ln\delta)^{-1}$$

with some constant C_1, $C_2 > 0$.

Proof We use the same notation as in the proof of theorem 2.7. We recall that

$$\begin{aligned}e_n =& A_n(I-K^*K)^n e_0 + \sum_{j=0}^{n-1}\alpha_{n-j-1}(I-K^*K)^j\prod_{i=1}^{j}(1-\alpha_{n-i})e_0 \\ &+ \sum_{j=1}^{n}(I-K^*K)^{j-1}\prod_{i=1}^{j}(1-\alpha_{n-i})K^*(y-y^\delta) \\ &+ \sum_{j=0}^{n-1}\prod_{i=n-j}^{n-1}(1-\alpha_i)(I-K^*K)^j K^* z_{n-j-1}\end{aligned}$$

and $e_0 = x^+ - x_0 = f(K^*K)w$ be selected from source condition (1.14). So

$$e_n = \left[A_n(I - K^*K)^n + \sum_{j=0}^{n-1} \alpha_{n-j-1}(I - K^*K)^j \prod_{i=1}^{j}(1-\alpha_{n-i})\right] f(K^*K)w$$
$$+ \sum_{j=1}^{n}(I - K^*K)^{j-1} \prod_{i=1}^{j}(1-\alpha_{n-i})K^*(y - y^\delta)$$
$$+ \sum_{j=0}^{n-1} \prod_{i=n-j}^{n-1} (1-\alpha_i)(I - K^*K)^j K^* z_{n-j-1}.$$

Then,

$$e_N = f(K^*K)w_N + \left[\sum_{j=1}^{N}(I - K^*K)^{j-1} \prod_{i=1}^{j}(1-\alpha_{N-i})K^*\right](y - y^\delta) \tag{2.49}$$

where

$$w_N = \left[A_N(I - K^*K)^N + \sum_{j=0}^{N-1} \alpha_{N-j-1}(I - K^*K)^j \prod_{i=1}^{j}(1-\alpha_{N-i})\right] w$$
$$+ \sum_{j=0}^{N-1} \prod_{i=N-j}^{N-1} (1-\alpha_i)(I - K^*K)^j \widetilde{f}(K^*K)\widetilde{z}_{N-j-1}$$

with $\|\widetilde{z}_{N-j-1}\| = \|z_{N-j-1}\|, j = 0, 1, 2, ..., N-1$
and $\widetilde{f}(K^*K) := \int_0^1 \lambda^{1/2}(\ln \frac{e}{\lambda})dE_\lambda = \int_0^1 \lambda^{1/2}(1 - \ln\lambda)dE_\lambda$. From lemma 4.1, lemma 4.3, see [2], and (2.37) − (2.38) we obtain

$$\|w_N\| \le \left\|A_N(I - K^*K)^N w\right\| + \sum_{j=0}^{N-1} \alpha_{N-j-1} \prod_{i=1}^{j}(1-\alpha_{N-i}) \left\|(I - K^*K)^j w\right\|$$
$$+ \sum_{j=0}^{N-1} \prod_{i=N-j}^{N-1} (1-\alpha_i) \left\|(I - K^*K)^j \widetilde{f}(K^*K)\right\| \|\widetilde{z}_{N-j-1}\|$$
$$\le \|w\| + \sum_{j=0}^{N-1} (N-j)^{-1/2} \|w\| + C\sum_{j=0}^{N-1}(j+1)^{-1/2}(\ln(j+1)) \|\widetilde{z}_{N-j-1}\|$$
$$\le (N+1)\|w\| + C\sum_{j=0}^{N-1}(j+1)^{-1/2}(\ln(j+1))K_1 \|Ke_{N-j-1}\| \|e_{N-j-1}\|$$
$$\le (N+1)\|w\| + C\sum_{j=0}^{N-1}(j+1)^{-1/2}(\ln(j+1))K_1\widehat{K}_2^2(N-j)^{-1/2}(\ln(N-j-1+e))^{-2}$$
$$\le (N+1)\|w\| + D \tag{2.50}$$

where D is a constant depending on w.
From (2.49), we conclude that

$$\begin{aligned}\|e_N\| &\le \|f(K^*K)w_N\| + \left\|\sum_{j=1}^{N}(I-K^*K)^{j-1}K^*\right\| \|y-y^\delta\| \\ &\le \|f(K^*K)w_N\| + \left\|\sum_{k=0}^{N-1}(I-K^*K)^{k}K^*\right\| \delta \\ &\le \|f(K^*K)w_N\| + \sqrt{N}\delta.\end{aligned}$$

From lemma 4.2, see [2], and (2.49), we have

$$\begin{aligned}\|f(K^*K)w_N\| &\le \|f(K^*K)\| \, \|w_N\| \\ &\le \widetilde{c}(-\ln(\delta))^{-1}\left[(N+1)\|w\| + D\right] \\ &\le C_2(-\ln(\delta))^{-1}\end{aligned}$$

where $C_2 = \widetilde{c}\left[(N+1)\|w\| + D\right]$.
Thus, $\|e_N\| \le C_2(-\ln(\delta))^{-1} + \sqrt{N}\delta$. We apply (2.48), then

$$(N+1)^{1/2}(\ln(n+e)) \le \frac{1}{\delta}\left[K_5\|w\| + K_6\widehat{K}_2^2\right] = \frac{C}{\delta}$$

or

$$(N+1)(\ln(n+e))^2 \le \frac{C^2}{\delta^2}$$

for some positive C. For the fact that

$$N(\ln N)^2 \le (N+1)(\ln(n+e))^2 \le \frac{C^2}{\delta^2}, \tag{2.51}$$

we have

$$N(\ln N)^2 \le \frac{C^2}{\delta^2}.$$

We use lemma 4.4 [2] with above information, we get

$$N = \frac{\widetilde{c}(-\ln(\delta))^{-2}}{\delta^2}.$$

So$\|e_N\|$ becomes

$$\|e_N\| \le C_2(-\ln(\delta))^{-1} + \frac{\sqrt{\widetilde{c}}(-\ln(\delta))^{-1}}{\delta} = C_3(-\ln(\delta))^{-1}$$

for some positive constant C_3.
From (2.51) we have

$$N^{1/2}(\ln N) \leq \frac{C_1}{\delta}.$$

■

Chapter 3
Numerical Examples

In this section, we focus on the example 1.3 such an inverse potential problem that has already proven the numerical implementation demonstration of the convergence rate under the logarithmic with the modified Landweber in the properly condition. Firstly, we provide formulars of the operator $F(x)$ and $F'(x)$ by using (1.5) and (1.6). If the poisson kernel given by

$$P(r, t-s) = \frac{1}{2\pi R} + \frac{1}{\pi R}\sum_{i=1}^{\infty}\left(\frac{r}{R}\right)^{i}(\cos(is)\cos(it) + \sin(is)\sin(it)),$$

the nonlinear operator is expressed as

$$\begin{aligned}[F(x)](t) = {} & \frac{1}{4\pi R}\int_0^{2\pi} x^2(s)\mathrm{ds} + \sum_{i=1}^{\infty}\frac{1}{\pi R^{i+1}(i+2)}\int_0^{2\pi} x^{i+2}(s)\cos(is)\mathrm{ds}\ \cos(it) \\ & + \sum_{i=1}^{\infty}\frac{1}{\pi R^{i+1}(i+2)}\int_0^{2\pi} x^{i+2}(s)\sin(is)\mathrm{ds}\ \sin(it) \qquad (3.1)\end{aligned}$$

where $F : L^2[0, 2\pi] \to L^2[0, 2\pi]$. Moreover, the $Fr\acute{e}$chet derivative of the operator F is

$$\begin{aligned}[F'(x)h](t) = {} & \frac{1}{2\pi R}\int_0^{2\pi} x(s)h(s)\mathrm{ds} \\ & + \sum_{i=1}^{\infty}\frac{1}{\pi R^{i+1}}\left(\int_0^{2\pi} x^{i+1}(s)h(s)\cos(is)\mathrm{ds}\ \cos(it)\right. \\ & \left. + \int_0^{2\pi} x^{i+1}(s)h(s)\sin(is)\mathrm{ds}\ \sin(it)\right) \qquad (3.2)\end{aligned}$$

see [1], for more detail. Recall the equation (2.1)

$$x_{n+1}^{\delta} = x_n^{\delta} + F'(x_n^{\delta})^*(y^{\delta} - F(x_n^{\delta})) - \alpha_n(x_n^{\delta} - \zeta) \qquad (3.3)$$

we set $\zeta = \delta x_*(s)$. Since $X = L^2[0, 2\pi]$ and $Y = L^2[0, 2\pi]$ we discretized $[0, 2\pi]$ to m interval with the grid points $0 = t_0, t_1, ..., t_m = 2\pi$ and $0 = s_0, s_1, ..., s_m = 2\pi$. The sets $\left\{\varphi_1^{(m)}...\varphi_m^{(m)}\right\}$ and $\left\{\psi_1^{(m)}...\psi_m^{(m)}\right\}$ are the orthogonal bases in the space $L^2[0, 2\pi]$. The orthogonal base are dened with respect to the step length

$h^{(m)} := 2\pi/m, m \in \mathbb{N}$ by the piecewise continuous function with $\varphi_j^{(m)}(s) = 1$ for $s \in [s_{j-1}, s_j]$, $\psi_j^{(m)}(t) = 1$ for $t \in [t_{j-1}, t_j]$ and $\varphi_j^{(m)}(s) = 0$, $\psi_j^{(m)}(t) = 0$ otherwise. Note that $X_m = \text{span } \left\{\varphi_j^{(m)}\right\}_{j=1,\dots,m}$ and $Y_m = \text{span } \left\{\psi_j^{(m)}\right\}_{j=1,\dots,m}$.Therefore

$$x_n^\delta(s) = \sum_{j=1}^m u_j^{(m)} \varphi_j^{(m)}(s)$$

$$x_{n+1}^\delta(s) = \sum_{j=1}^m v_j^{(m)} \varphi_j^{(m)}(s)$$

$$\zeta = \sum_{j=1}^m z_j^{(m)} \varphi_j^{(m)}(s) = \delta x_*(s)$$

for some vectors $U^{(m)} = (u_1^{(m)} \dots u_m^{(m)})^T$, $V^{(m)} = (v_1^{(m)} \dots v_m^{(m)})^T$ and $Z^{(m)} = (z_1^{(m)} \dots z_m^{(m)})^T$. Applying the above information to (3.3) we have

$$\sum_{j=1}^m v_j^{(m)} \varphi_j^{(m)}(s) = \sum_{j=1}^m u_j^{(m)} \varphi_j^{(m)}(s) + F'(x_n^\delta(s))^*[y^\delta - F(x_n^\delta(s))] - \alpha_n(\sum_{j=1}^m u_j^{(m)} \varphi_j^{(m)}(s) - \delta x_*(s)). \tag{3.4}$$

For $j = 1$ we will consider the inner product of (3.4) and $\varphi_1^{(m)}$. We know that an orthogonal bases is defined by the piecewise continuous function such that $\varphi_j^{(m)}(s) = 1$ for $s \in [s_{j-1}, s_j]$ and $\varphi_j^{(m)}(s) = 0$ otherwise. So for $j = 1$ we have

$$\left\langle \sum_{j=1}^m v_j^{(m)} \varphi_j^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle = \left\langle \sum_{j=1}^m u_j^{(m)} \varphi_j^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle + \left\langle F'(x_n^\delta(s))^*[y^\delta(t) - F(x_n^\delta(s))], \varphi_1^{(m)}(s) \right\rangle - \alpha_n \left\langle \sum_{j=1}^m u_j^{(m)} \varphi_j^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle + \alpha_n \left\langle \delta x_*(s), \varphi_1^{(m)}(s) \right\rangle \tag{3.5}$$

and

$$\sum_{j=1}^m v_j^{(m)} \left\langle \varphi_j^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle = \sum_{j=1}^m u_j^{(m)} \left\langle \varphi_j^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle + \left\langle F'(x_n^\delta(s))^*[y^\delta(t) - F(x_n^\delta(s))], \varphi_1^{(m)}(s) \right\rangle - \alpha_n \sum_{j=1}^m u_j^{(m)} \left\langle \varphi_j^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle + \alpha_n \left\langle \delta x_*(s), \varphi_1^{(m)}(s) \right\rangle . \tag{3.6}$$

Since $\varphi_j^{(m)}(s) = 0$ for $j \neq 1$ and (3.6) we have

$$v_1^{(m)} \left\langle \varphi_1^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle = u_1^{(m)} \left\langle \varphi_1^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle + \left\langle F'(x_n^\delta(s))^*[y^\delta(t) - F(x_n^\delta(s))], \varphi_1^{(m)}(s) \right\rangle$$
$$- \alpha_n u_1^{(m)} \left\langle \varphi_1^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle + \alpha_n \left\langle \delta x_*(s), \varphi_1^{(m)}(s) \right\rangle . \tag{3.7}$$

Next, considering $\left\langle \varphi_1^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle$ we get

$$\begin{aligned}
\left\langle \varphi_1^{(m)}(s), \varphi_1^{(m)}(s) \right\rangle_{L^2[0,2\pi]} &= \int_0^{2\pi} \left| \varphi_1^{(m)}(s) \right|^2 \mathrm{d}s \\
&= \int_0^{s_1} (\varphi_1^{(m)}(s))^2 \mathrm{d}s + \int_{s_1}^{s_2} (\varphi_1^{(m)}(s))^2 \mathrm{d}s + ... + \int_{s_{m-1}}^{s_m} (\varphi_1^{(m)}(s))^2 \mathrm{d}s \\
&= (s_1 - 0) + 0 + ... + 0 \\
&= h^{(m)}.
\end{aligned}$$

Using above estimation, then (3.7) becomes

$$v_1^{(m)} h^{(m)} = u_1^{(m)} h^{(m)} + \left\langle F'(x_n^\delta(s))^* \left(y^\delta(t) - [F(x_n^\delta(s))](t)\right), \varphi_1^{(m)}(s) \right\rangle$$
$$- \alpha_n u_1^{(m)} h^{(m)} + \alpha_n \left\langle \delta x_*(s), \varphi_1^{(m)}(s) \right\rangle. \tag{3.8}$$

Dividing (3.8) by $h^{(m)}$ we have

$$v_1^{(m)} = u_1^{(m)} + \frac{1}{h^{(m)}} \left\langle F'(x_n^\delta(s))^* \left(y^\delta(t) - [F(x_n^\delta(s))](t)\right), \varphi_1^{(m)}(s) \right\rangle$$
$$- \alpha_n u_1^{(m)} + \frac{\alpha_n}{h^{(m)}} \left\langle \delta x_*(s), \varphi_1^{(m)}(s) \right\rangle. \tag{3.9}$$

Furthermore, we estimate the second term of the right - hand side of above equation as follow

$$\begin{aligned}
&\left\langle F'(x_n^\delta(s))^*(y^\delta(t) - \left[F(x_n^\delta(s))\right](t)), \varphi_1^{(m)}(s)\right\rangle \\
&= \left\langle y^\delta(t) - \left[F(x_n^\delta(s))\right](t), F'(x_n^\delta(s))\varphi_1^{(m)}(s)\right\rangle \\
&= \int_0^{2\pi} \left[y^\delta(t) - \left[F(x_n^\delta(s))\right](t)\right] \left[F'(x_n^\delta(s))\varphi_1^{(m)}(s)\right] dt \\
&= \int_0^{2\pi} \left[y^\delta(t) - \left[F(x_n^\delta(s))\right](t)\right] \left[\frac{1}{2\pi R}\int_0^{2\pi} x_n^\delta(s)\varphi_1^{(m)}(s)\mathrm{d}s \right.\\
&\quad + \sum_{j=1}^{\infty} \frac{1}{\pi R^{i+1}} \left(\int_0^{2\pi} x_n^{\delta(i+1)}(s)\varphi_1^{(m)}(s)\cos(is)\mathrm{d}s\,\cos(it)\right.\\
&\quad \left.\left. + \int_0^{2\pi} x_n^{\delta(i+1)}(s)\varphi_1^{(m)}(s)\sin(is)\mathrm{d}s\,\sin(it)\right)\right] \mathrm{d}t \\
&= \int_0^{2\pi} \left[y^\delta(t) - \left[F(x_n^\delta(s))(t)\right]\right] \left[\frac{1}{2\pi}\int_0^{s_1} x_n^\delta(s)\mathrm{d}s \right.\\
&\quad + \sum_{j=1}^{\infty} \frac{1}{\pi} \left(\int_0^{s_1} x_n^{\delta(i+1)}(s)\cos(is)\mathrm{d}s\,\cos(it)\right.\\
&\quad \left.\left. + \int_0^{s_1} x_n^{\delta(i+1)}(s)\sin(is)\mathrm{d}s\,\sin(it)\right)\right] \mathrm{d}t
\end{aligned}$$

where we use $R = 1$ and (3.2). Using a technique for approximating the integral with the trapezoidal rule and $h^{(m)} = \frac{2\pi}{m}$, we obtain

$$\int_0^{s_1} x_n^\delta(s)\mathrm{d}s = \frac{h^{(m)}}{2}\left[x_n^\delta(0) + x_n^\delta(s_1)\right],$$

$$\int_0^{s_1} x_n^{\delta(i+1)}(s)\cos(is)\mathrm{d}s\,\cos(it) = \frac{h^{(m)}}{2}\left[x_n^{\delta(i+1)}(0)\cos(0) + x_n^{\delta(i+1)}(s_1)\cos(is_1)\right]\cos(it),$$

and

$$\int_0^{s_1} x_n^{\delta(i+1)}(s)\sin(is)\mathrm{d}s\,\sin(it) = \frac{h^{(m)}}{2}\left[x_n^{\delta(i+1)}(0)\sin(0) + x_n^{\delta(i+1)}(s_1)\sin(is_1)\right]\sin(it).$$

So,

$$\begin{aligned}
&\int_0^{2\pi} \left[y^\delta(t) - \left[F(x_n^\delta(s))(t)\right]\right] \\
&\quad \times \left[\frac{1}{2\pi}\left(\frac{h^{(m)}}{2}(x_n^\delta(0) + x_n^\delta(s_1))\right) + \sum_{j=1}^{\infty}\frac{1}{\pi}\left(\frac{h^{(m)}}{2}\left(x_n^{\delta(i+1)}(0)\cos(0) + x_n^{\delta(i+1)}(s_1)\cos(is_1)\right)\cos(it)\right.\right.\\
&\quad \left.\left. + \frac{h^{(m)}}{2}\left(x_n^{\delta(i+1)}(0)\sin(0) + x_n^{\delta(i+1)}(s_1)\sin(is_1)\right)\sin(it)\right)\right] \mathrm{d}t
\end{aligned}$$

Let

$$A_{i1} = x_n^{\delta(i+1)}(0)\cos(0) + x_n^{\delta(i+1)}(s_1)\cos(is_1)$$

$$B_{i1} = x_n^{\delta(i+1)}(0)\sin(0) + x_n^{\delta(i+1)}(s_1)\sin(is_1).$$

Then,

$$q_1 := \int_0^{2\pi} \left[y^\delta(t) - \left[F(x_n^\delta(s))(t)\right]\right] \times \left[\frac{1}{2\pi}\left(\frac{h^{(m)}}{2}(x_n^\delta(0) + x_n^\delta(s_1))\right) + \sum_{j=1}^{\infty}\left(\frac{h^{(m)}}{2\pi}A_{i1}\cos(it) + \frac{h^{(m)}}{2\pi}B_{i1}\sin(it)\right)\right]\mathrm{d}t. \tag{3.10}$$

We use the right endpoint appoximation for above integral term as follow

$$q_1 = h^{(m)}\sum_{k=1}^{m}\left[y^\delta(t_k) - \left[F(x_n^\delta(s))(t_k)\right]\right] \times \left[\frac{1}{2\pi}\left(\frac{h^{(m)}}{2}(x_n^\delta(0) + x_n^\delta(s_1))\right) + \sum_{j=1}^{\infty}\left(\frac{h^{(m)}}{2\pi}A_{i1}\cos(it_k) + \frac{h}{2\pi}B_{i1}\sin(it_k)\right)\right]. \tag{3.11}$$

Applying (3.1) into term $\left[F(x_n^\delta(s))(t_k)\right]$ in (3.10) we have

$$\left[F(x_n^\delta(s))(t_k)\right] = \frac{1}{4\pi}\int_0^{2\pi}(x_n^\delta(s))^2\mathrm{d}s + \sum_{i=1}^{\infty}\frac{1}{\pi(i+2)}\int_0^{2\pi} x_n^{\delta(i+2)}(s)\cos(is)\mathrm{d}s\cos(it_k) + \sum_{i=1}^{\infty}\frac{1}{\pi(i+2)}\int_0^{2\pi} x_n^{\delta(i+2)}(s)\sin(is)\mathrm{d}s\,\sin(it_k).$$

We use the right endpoint appoximation for above integral term as follow

$$\left[F(x_n^\delta(s))(t_k)\right] = \frac{h^{(m)}}{4\pi}\sum_{l=1}^{n}(x_n^\delta(s_l))^2 + \sum_{i=1}^{\infty}\frac{1}{\pi(i+2)}\left[h^{(m)}\sum_{l=1}^{m}x_n^{\delta(i+2)}(s_l)\cos(is_l)\cos(it_k)\right] + \sum_{i=1}^{\infty}\frac{1}{\pi(i+2)}\left[h^{(m)}\sum_{l=1}^{m}x_m^{\delta(i+2)}(s_l)\sin(is_l)\sin(it_k)\right].$$

Define

$$C = \frac{h^{(m)}}{4\pi}\sum_{l=1}^{m}(x_n^\delta(s_l))^2$$

$$D_{ki} = \frac{h^{(m)}}{\pi(i+2)}\sum_{l=1}^{m}x_n^{\delta(i+2)}(s_l)\cos(is_l)\cos(it_k)$$

$$E_{ki} = \frac{h^{(m)}}{\pi(i+2)}\sum_{l=1}^{m}x_n^{\delta(i+2)}(s_l)\sin(is_l)\sin(it_k).$$

Using the fact that $x_n^\delta(s) = \sum_{j=1}^m u_j^{(m}\varphi_j^{(m)}(s)$ for C, D_{ki} and E_{ki} we can rewrite C, D_{ki} and E_{ki} as follow

$$C = \frac{h^{(m)}}{4\pi}\sum_{l=1}^{m}\left(x_n^\delta(s_k)\right)^2 = \frac{h}{4\pi}\sum_{l=1}^{m}\left(u_l^{(m)}\right)^2,$$

$$D_{ki} = \frac{h^{(m)}}{\pi(i+2)}\sum_{l=1}^{m}\left(u_l^{(m)}\right)^{i+2}\cos(is_l)\cos(it_k)$$

and

$$E_{ki} = \frac{h^{(m)}}{\pi(i+2)}\sum_{l=1}^{m}\left(u_l^{(m)}\right)^{i+2}\sin(is_l)\sin(it_k).$$

So,

$$\left[F(x_n^\delta(s))(t_k)\right] = C + \sum_{i=1}^{\infty}(D_{ki} + E_{ki}). \tag{3.12}$$

Substituting(3.12) into (3.10), we get

$$q_1 = h^{(m)}\sum_{k=1}^{m}\left[y_k^\delta(t_k) - C - \sum_{i=1}^{\infty}(D_{ki} + E_{ki})\right]\left[\frac{h^{(m)}}{4\pi}(x_n^\delta(0) + x_n^\delta(s_1)) + \sum_{j=1}^{\infty}\left(\frac{h^{(m)}}{2\pi}A_{i1}\cos(it_k) + \frac{h^{(m)}}{2\pi}B_{i1}\sin(it_k)\right)\right]. \tag{3.13}$$

In the same manner of (3.13), we can define

$$q_r = h^{(m)}\sum_{k=1}^{m}\left[y_k^\delta(t_k) - C - \sum_{i=1}^{\infty}(D_{ki} + E_{ki})\right]\left[\frac{h^{(m)}}{4\pi}(x_n^\delta(s_{r-1}) + x_n^\delta(s_r)) + \sum_{j=1}^{\infty}\left(\frac{h^{(m)}}{2\pi}A_{ir}\cos(it_k) + \frac{h^{(m)}}{2\pi}B_{ir}\sin(it_k)\right)\right],$$

for $r = 1, ..., m$ where

$$\begin{aligned} A_{ir} &= x_n^{\delta(i+1)}(s_{r-1})\cos(is_{r-1}) + x_n^{\delta(i+1)}(s_r)\cos(is_r) \\ &= \left(u_{r-1}^{(m)}\right)^{i+1}\cos(is_{r-1}) + \left(u_r^{(m)}\right)^{i+1}\cos(is_r) \end{aligned}$$

and

$$\begin{aligned} B_{ir} &= x_n^{\delta(i+1)}(s_{r-1})\sin(is_{r-1}) + x_n^{\delta(i+1)}(s_r)\sin(is_r) \\ &= \left(u_{r-1}^{(m)}\right)^{i+1}\sin(is_{r-1}) + \left(u_r^{(m)}\right)^{i+1}\sin(is_r). \end{aligned}$$

Here we use $x_n^\delta(s) = \sum_{j=1}^m u_j^{(m)} \varphi_j^{(m)}(s)$.
Next, we consider $\left\langle \delta x_*(s), \varphi_1^{(m)}(s) \right\rangle$, where $x_*(s) = \frac{5+\sin(3s)}{6}$, then

$$\begin{aligned}
\left\langle \delta x_*(s), \varphi_1^{(m)}(s) \right\rangle &= \left\langle \delta(\frac{5+sin(3s)}{6}), \varphi_1^{(m)}(s) \right\rangle \\
&= \delta \int_0^{2\pi} \frac{5+\sin(3s)}{6} \varphi_1^{(m)}(s) \mathrm{d}s \\
&= \delta \int_{s_0}^{s_1} \frac{5+\sin(3s)}{6} \mathrm{d}s \\
&= \frac{\delta}{6} \int_{s_0}^{s_1} 5+\sin(3s) \mathrm{d}s \\
&= \frac{\delta}{6} \left[5s - \frac{1}{3}\cos(3s) \right]_{s_0}^{s_1} \\
&= \frac{\delta}{6} \left[5(s_1 - s_0) - (\frac{1}{3}\cos(3s_1) - \frac{1}{3}\cos(3s_0)) \right] \\
&= \frac{\delta}{18} \left[15h^{(m)} - (\cos(3s_1) - \cos(3s_0)) \right].
\end{aligned}$$

It becomes

$$z_1^{(m)} = \frac{\delta}{18} \left[15h^{(m)} + \cos(3s_0) - \cos(3s_1) \right].$$

We define

$$z_r^{(m)} = \frac{\delta}{18} \left[15h^{(m)} + \cos(3s_{r-1}) - \cos(3s_r) \right].$$

From (3.9), for $r = 1, ..., m$, we have

$$v_r^{(m)} = u_r^{(m)} + \frac{1}{h^{(m)}} q_r - \alpha_n u_r^{(m)} + \frac{\alpha_n}{h^{(m)}} z_r^{(m)}. \tag{3.14}$$

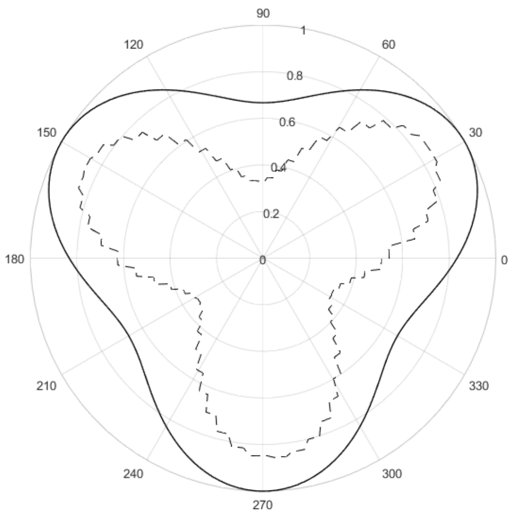


Figure 7 The polar plot shows the exact data (solid line) and the approximated data (dashed line).

The plot demonstrate the numerical example for recovering $x^{+} = \frac{5+sin(3s)}{6}$. We obtain data y by solving the direct problem for a test curve. For example, we evaluate (3.1) for the test curve. The values for the Nauman boundary conditions are provided by $F(x)$ and $F'(x)$. The result are demonstrated in Figure 7. The program was written in MATLAB. Moreover, the number of iteration is 3 (n=3) and the number of the grid points M is 200. We set the initial value $x_0 = 0.4 + \frac{sin(3s)}{6}$ and $\delta = 10^{-6}$ to be a noisy level. From the error estimation we have $\left\| x^{+} - x_N^{\delta} \right\| \leq 3.6258$. Unfortuately we cannot show the iterates that are stopped by the discrepancy principle (1.10) such that will give the better solution.

Chapter 4
Conclusions

We solve the nonlinear ill-posed problems $F(x) = y$ where the noisy data $y^\delta \in Y$ with $\|y^\delta - y\| \leq \delta$ are provided. We use the modified Landweber method which proposed by Scherzer in 1998. We also include the additional term $\alpha_n = \frac{1}{2}n^{-1/2}$ into the Landweber method. The convergence rate is provided under the specific source condition. In general, the source condition is the Hölder type. However, for severely ill-posed problems Hölder type source condition cannot be applied. Therefore in this thesis, we consider the logarithmic source condition for the modified Landeweber method. We found that

$$\|x^+ - x_N^\delta\| \leq C_2(-\ln\delta)^{-1},$$

if N is chosen according to the discrepancy principle.

Finally, we have employed the modified Landweber method to an inverse potential problem which is to recover the characteristic function of the domain D from information of its density and of measurements of the Cauchy data of the corresponding potential on the boundary of a smooth and bounded domain. We find the shape of an unknown domain D from given data $y := \frac{\partial u}{\partial \nu}$. We demonstrate the numerical example for recovering $x^+ = \frac{5+sin(3s)}{6}$. We simulate data y by solving the direct problem for a test curve, i.e. we evaluate (3.1) for the test curve x^+. The values for the Nauman boundary conditions are provided by $F(x)$ and $F'(x)$. The result are demonstrated in Figure 7. The computer programing is written in MATLAB. Moreover, we try to illustrate a plot of the error $\left\|x^+ - x_N^\delta\right\| \leq C_2(-\ln\delta)^{-1}$. Unfortunately, we cannot show the iterates that are stopped by the discrepancy principle (1.10) such that the better solution can be given.

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Appendix A

Lemma 4.1. *[2]*
Let $p > 0$ and $k \in \mathbb{N}_0$. The real-valued function $\hat{f}(\lambda) = (1-\lambda)^k(\ln \frac{\exp(1)}{\lambda})^{-p}$ defined on $[0,1]$, satisfies

$$\hat{f}(\lambda) \leq C(\ln k)^{-p}$$

with C independent of k.
Moreover, for each $p \in \mathbb{R}$ the real-valued function

$$\hat{g}(\lambda) = (1-\lambda)^k \lambda^{1/2} \left(\ln \frac{\exp(1)}{\lambda} \right)^{-p}$$

defined on $[0,1]$, satisfies

$$\hat{g}(\lambda) \leq Ck^{-1/2}(\ln k)^{-p}$$

with C independent of k.

Lemma 4.2. *[2]*
Let $p \geq 1, C > 0$, and $\delta > 0$ sufficiently small such that $1 \geq (-\ln(\delta C))^{-2p} \geq \delta$. Let

$$\int_0^1 \exp(-((1-\ln(\lambda))^{-2p})^{-1/(2p)})(1-\ln(\lambda))^{-2p} \|\mathrm{d}E_\lambda w\|^2 = C\delta^2.$$

Then

$$\int_0^1 (1-\ln(\lambda))^{-2p} \|\mathrm{d}E_\lambda w\|^2 \leq C(-\ln \delta)^{-2p}$$

with a generic constant C.

Lemma 4.3. *[2]*
Let $p \geq 1, k \in \mathbb{N}, k \geq 2$. Then there exists a constant D, which is independent of k, such that

$$\sum_{j=0}^{k-1} \left(\frac{j+1}{k+1}\right)^{-1/2} \left(\frac{k-j}{k+1}\right)^{-1/2} \frac{1}{k+1} \left(\frac{\ln(k+2)}{\ln(k-j+1)}^{2p}\right) \leq D \tag{4.1}$$

$$(\ln(k+2))^{-p} \sum j = 0k - 1 \left(\frac{j+1}{k+1}\right)^{-1} \left(\frac{k-j}{k+1}\right)^{-1/2} \frac{1}{k+1} \left(\frac{\ln(k+2)}{\ln(k-j+1)}\right)^{2p} \leq D.$$

Moreover, there exists a constant D (independent of k) such that

$$\sum_{j=0}^{k-1} (j+1)^{-1/2} (\ln(j+1))^{p} (k-j+1)^{-1/2} (\ln(k-j+1))^{-2p} \leq D. \tag{4.2}$$

Lemma 4.4. *[2]*
Let $\hat{k}$ be a solution of

$$k(\ln k)^{2p} = \frac{C}{\delta^2}. \tag{4.3}$$

Then $\hat{k}$ satisfies

$$\hat{k} = \mathrm{O}\left(\frac{(-\ln\delta)^{-2p}}{\delta^2}\right).$$

Appendix B
Publications

pom1.PNG

pa1.PNG

pam1.PNG

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