GAME DOMINATION NUMBERS OF A DISJOINT UNION OF CHAINS AND CYCLES OF COMPLETE GRAPHS

By

MISS Nattakritta CHANTARACHADA

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

Academic Year 2019

ค่าโดมิเนชันเกมของยูเนียนของโซ่และวัฏจักรของกราฟบริบูรณ์

โดย
นางสาวณัฐกฤตา ฉันทรัชดา

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Title	Game domination numbers of a disjoint union of chains and cycles of complete graphs
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MISS NATTAKRITTA CHANTARACHADA : GAME DOMINATION NUMBERS OF A DISJOINT UNION OF CHAINS AND CYCLES OF COMPLETE GRAPHS THESIS ADVISOR : ASSISTANT PROFESSOR CHALERMPONG WORAWANNOTAI, Ph.D.

The domination game played on a graph G consists of two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen, at least one additional vertex is dominated. Playing a vertex will make all vertices in its closed neighborhood dominated. The game ends when all vertices are dominated i.e., the chosen vertices form a dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The game domination number is the total number of chosen vertices after the game ends when Dominator and Staller play the game by using optimal strategies.

In this thesis, we determine the game domination numbers of a disjoint union of chains and cycles of complete graphs together with optimal strategies for Dominator and Staller.

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Nattakritta CHANTARACHADA

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1

Introduction

A *graph* $G = (V(G), E(G))$ consists of a set $V(G)$ of vertices and a set $E(G)$ of edges where each edge is identified with an unordered pair of vertices (not necessary distinct vertices). A vertex of a graph which is not an endpoint of any edge is called *isolated*. Two vertices are *adjacent* if they are joined by an edge; they are also the *end vertices* of the edge, and the edge is said to be *incident* to each of its end vertices. *Multiple edges* are two or more edges that are incident to the same two vertices. A *loop* is an edge connecting a vertex to itself. A graph without loops or multiple edges is called a *simple graph*. From now on, we only consider simple graphs. For any graph G, the number of vertices in G is called the *order* of G and it is denoted by $|V(G)|$ or $|G|$. Consider a graph G, a graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A set S of vertices of a graph G is a *dominating set* if every vertex not in S is adjacent to some vertex of S. The *domination number* of a graph G is the number of vertices in a minimum dominating set of G, denoted by $\gamma(G)$.

In real life, domination can be used to optimize resource allocation. For example, in a school building, we want to install multiple WiFi routers so that all areas have WiFi coverage and we use as few routers as possible for saving cost.

To solve the problem we can divide the school building into smaller areas (e.g. classrooms). Each area is represented by a vertex. Two vertices are joined by an edge if a WiFi router in one area can cover the other. Then domination can be applied to solve this kind of resource allocation problem.

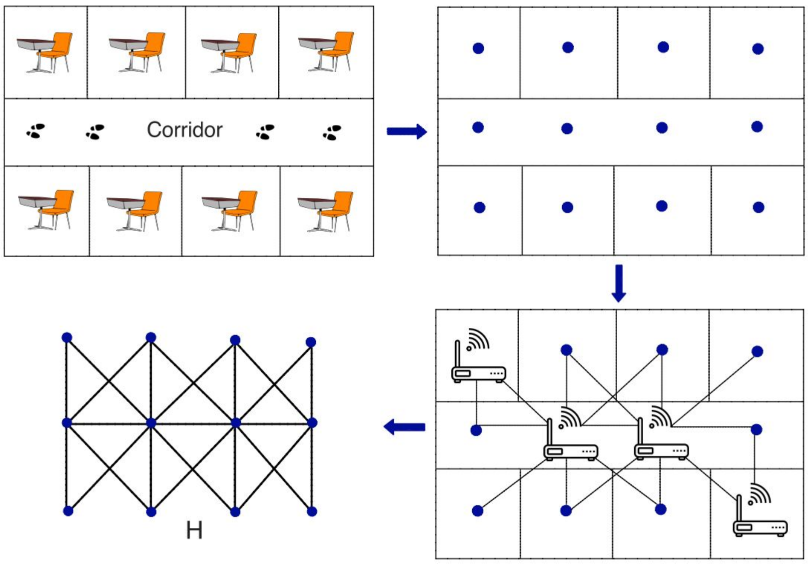


Figure 1.1: Example of a domination model for WiFi routers installation in a school building with 8 classrooms

Example 1.1. Consider Figure 1.1. If we represent the areas in the school building where we can install WiFi routers as vertices and join an edge between two vertices that WiFi signal from one area can cover the other, then we will get the graph H. Hence the domination number of H is equal to the minimum of the number of WiFi routers needed in the school building. One can see that $\gamma(H) = 2$ i.e., the minimum of the number of WiFi routers needed in this school building is 2.

There are many game variations of domination [1, 2, 3, 4, 10]. In this thesis we study the domination game introduced in 2010 by Brešar, Klavžar and

Rall [4], where the original idea of the game is attributed to Henning (2003, personal communication). The *domination game* played on a graph G consists of two players, *Dominator* and *Staller*, who alternate taking turns choosing a vertex from G such that whenever a vertex is chosen, at least one additional vertex is dominated. Playing a vertex will make all vertices in its closed neighborhood dominated. The game ends when all vertices are dominated i.e., the chosen vertices form a dominating set. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The *game domination number* is the size of the final dominating set when both players play optimally; it is denoted by $\gamma_g(G)$ when Dominator starts the game and by $\gamma_g'(G)$ when Staller starts the game.

Example 1.2. Let G be the graph in Figure 1.2.

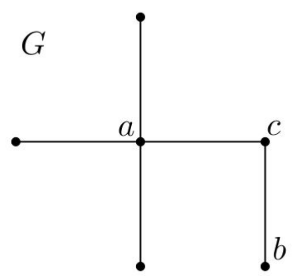


Figure 1.2: Graph G

Since every vertex not in the set $\{a, b\}$ is adjacent to a or b, the set $\{a, b\}$ is a dominating set of G. Note that the vertex a has a maximum degree but it is not adjacent to all vertices in G. Then each dominating set of G must have at least two vertices. Thus $\{a, b\}$ is a minimum dominating set, so we get that

$\gamma(G) = 2$. It implies that both games on G use at least 2 moves, i.e., $\gamma_g(G) \geq 2$ and $\gamma'_g(G) \geq 2$.

Now we consider the Dominator-start game. Since Dominator wants to end the game as soon as possible and he cannot finish the game in one move, the best move he can do is to play a. Then all closed neighborhoods of a are dominated. By playing a, he can force Staller to end the game by dominating b on the next move. Therefore $\gamma_g(G) \leq 2$. We can conclude that $\gamma_g(G) = 2$.

Finally let's consider the Staller-start game. Since Staller wants to prolong the game as much as possible, if Staller starts the game on a vertex not in $N(a) \setminus N(b)$, then Dominator can finish the game on the next move. So Staller starts the game on a vertex in $N(a) \setminus N(b)$ to prolong the game. Then Dominator plays a to force Staller to end the game on the next move. We can conclude that $\gamma'_g(G) = 3$.

Among many results, Brešar, Klavžar and Rall [4] gave a lower bound and the upper bound of the game domination number in terms of the domination number: for any graph G, we have $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$. They also studied the difference between the two types of the game domination numbers of a graph. Later, Kinnersley, West and Zamani [11] improved upon this result and showed that the difference is at most 1 i.e., for any graph G, $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.

The game version can be viewed as a form of negotiation. For example, in an apartment building the owner and residents want to install WiFi routers so that all areas have WiFi coverage. The owner who pays for the installation cost wishes to minimize the number of routers while the residents who want strong signal wish to maximize the number of routers. For fairness both parties can play the domination game to pick locations for router installation where the owner acts as Dominator and the residents together act as Staller.

For a graph G and a subset of vertices $S \subseteq V(G)$, we denote by $G|S$ the *partially dominated graph* where the vertices in S are already dominated initially. In particular, if $S = \{x\}$, we write $G|x$. The notion of the game domination number extends naturally to partially-dominated graphs by considering the number of moves to dominate the remaining undominated vertices. We denote the *open neighborhood* of a vertex v in a graph G by $N_G(v)$ and its *closed neighborhood* by $N_G[v]$. We simply write $N(v)$ and $N[v]$, respectively, if the graph is understood. Let v be a vertex of a graph G. The *degree* of v, denoted by $deg(v)$, is the number of edges incident with v, or equivalently, $deg(v) = |N_G(v)|$. A vertex u of a partially dominated graph G is *saturated* if every vertex in $N[u]$ is dominated. The *residual graph* of a graph G is the graph obtained from G by removing all saturated vertices and all edges joining between two dominated vertices. Since removing such vertices and edges does not affect the game, the game domination numbers of a partially dominated graph and its residual graph are the same.

Example 1.3. Let G_1 be the partially dominated graph of the graph G in Figure 1.2 where vertices b and c are already dominated. One can see that every vertex in $N[b]$ is dominated. Then vertex b is saturated. After removing vertex b and the edge joining between b and its neighbor c, we will get the residual graph G_2

of G_1 as shown in Figure 1.3. Since playing vertex b does not dominate any new vertices, removing b does not effect the game domination number of G_1. So the game domination numbers of G_1 and G_2 are equal.

To find the game domination numbers of G_1, we consider G_1 as the graph where the vertices b and c are already dominated before the game starts. In Dominator-start game, Dominator can end the game in one move by playing a vertex a and in Staller-start game, Staller can prolong the game by playing a vertex in $N(a)$. Then $\gamma_g(G_1) = 1$ and $\gamma'_g(G_1) = 2$.

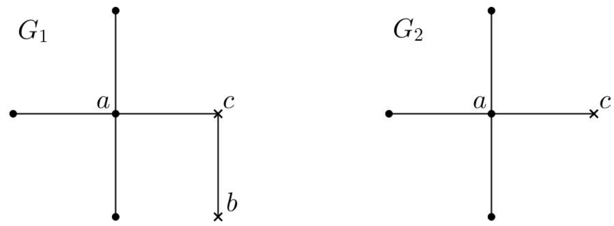


Figure 1.3: The partially dominated graph G_1 and its residual graph G_2

A *path* P_n is a graph of order n whose vertices can be listed in the order $v_1, v_2, ..., v_n$ such that v_i and v_{i+1} are adjacent and no other pairs of vertices are adjacent where $i = 1, 2, ..., n-1$. A *cycle* C_n is a connected graph of order n such that every vertex has degree 2. A graph G is said to be a *complete graph* if every pair of vertices in G are adjacent. We denote a complete graph on n vertices by K_n. In this thesis, we only consider complete graphs with at least 3 vertices. A graph G is called a *chain of complete graphs* $K_{n_1}, K_{n_2}, ..., K_{n_k}$ if G can be obtained from $K_{n_1}, K_{n_2}, ..., K_{n_k}$ by identifying a vertex in K_{n_i} and a vertex in $K_{n_{i+1}}$ for $1 \leq i \leq k-1$ (a vertex can be identified at most once). A *cycle of complete graphs*

$K_{n_1}, K_{n_2}, ..., K_{n_k}$ $(k \geq 3)$ is the graph obtained from the chain of $K_{n_1}, K_{n_2}, ..., K_{n_k}$ by identifying a vertex in $V(K_{n_1}) \setminus V(K_{n_2})$ and a vertex in $V(K_{n_k}) \setminus V(K_{n_{k-1}})$.

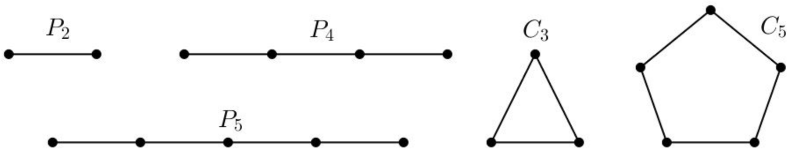


Figure 1.4: Some of paths and cycles

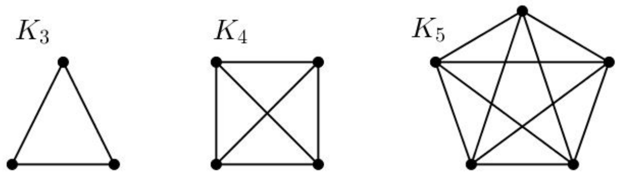


Figure 1.5: Some of complete graphs

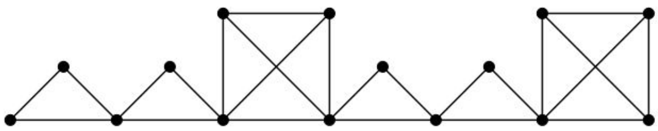


Figure 1.6: A chain of $K_3, K_3, K_4, K_3, K_3, K_4$

Domination game played on various families of graphs have been studied. In 2013, Zamani [11, 14] determined the game domination numbers of paths and cycles, and Brešar and Klavžar [5] proved a lower bound of the game domination number of a tree in terms of its order and maximum degree. In 2015, Bujtás [7] proved a lower bound of the game domination number of a certain families of forests. Dorbec, Košmrlj, and Renault [9] showed how the game domination

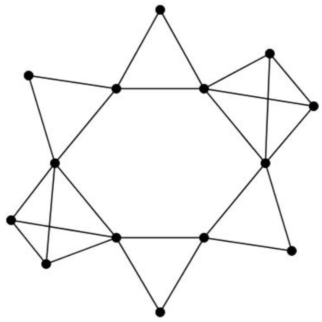


Figure 1.7: A cycle of $K_3, K_3, K_4, K_3, K_3, K_4$

number of the union of two no-minus graphs corresponds to the game domination numbers of the initial graphs. This result led to another proof of the game domination numbers of paths and cycles [12], and the game domination numbers of a graph constructed from 1-sum of paths [8]. In 2018, Raksasakcha, Onphaeng, and Worawannotai [13] determined the game domination numbers of a disjoint union of paths and cycles.

In this thesis, we determine the game domination numbers of a disjoint union of chains and cycles of complete graphs together with optimal strategies for Dominator and Staller. Chapter 2 recalls some definitions and known results of game domination numbers. Finally, Chapter 3 shows the game domination numbers of a disjoint union of chains and cycles of complete graphs together with optimal strategies for Dominator and Staller. Our proof is based on the following observation. When the domination game is played on a disjoint union of chains and cycles of complete graphs, at any stage of the game, the residual graph is a disjoint union of cycles of complete graphs and partially-dominated chains of complete graphs. In other words, the type of the residual graph does not change during the game. Therefore, if we can find an optimal first move, we have an optimal strategy for the whole game.

Chapter 2

Preliminaries

In this section, we give some definitions for describing the game domination numbers of a disjoint union of chains and cycles of complete graphs, and we also give useful lemmas for proving our results.

Definition 2.1. A graph is CC if each of its component is either a chain of complete graphs or a cycle of complete graphs.

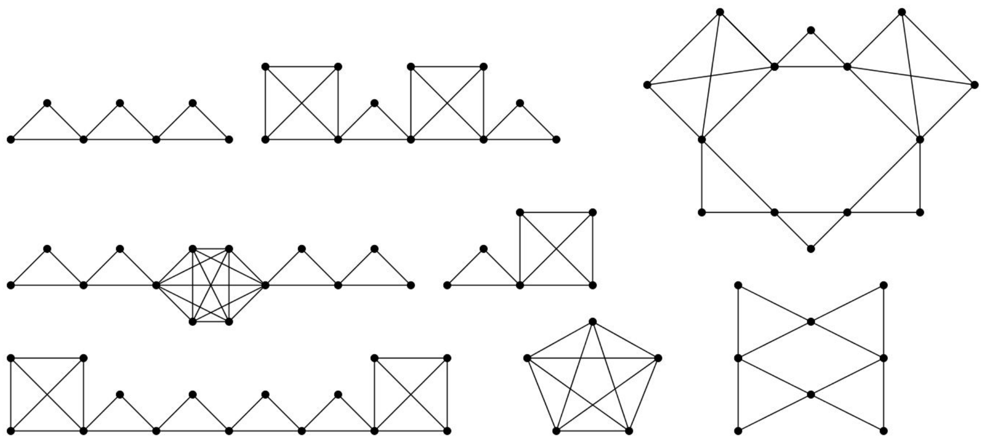


Figure 2.1: A CC graph

For convenience, we define some notations for the components of CC graphs and their residual graphs.

Definition 2.2. Let M_m denote a chain of m complete graphs. Let M'_m denote a partially-dominated chain of m complete graphs where one vertex in $V(K_{n_1}) \setminus V(K_{n_2})$ is dominated. Let M''_m denote a partially-dominated chain of m complete graphs where a vertex in $V(K_{n_1}) \setminus V(K_{n_2})$ and a vertex in $V(K_{n_m}) \setminus V(K_{n_{m-1}})$ are dominated. Let N_n denote a cycle of n complete graphs.

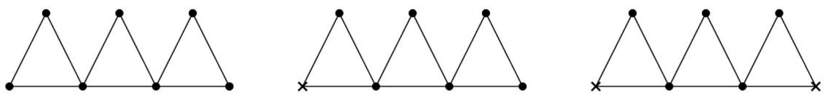


Figure 2.2: M_3, M'_3 and M''_3 of K_3's

Definition 2.3. For $i \in \{0, 1, 2, 3\}$, M_m is said to be in *class* $[i]$ if $m \equiv i \pmod 4$, where m is a positive integer, and N_n is said to be in *class* (i) if $n \equiv i \pmod 4$, where $n \geq 3$. Moreover, M_m is said to be in *class* $[m]_*$ if $m \in \{1, 2\}$ and in *class* $[i]_>$ if $m \equiv i \pmod 4$ and $m \geq 3$.

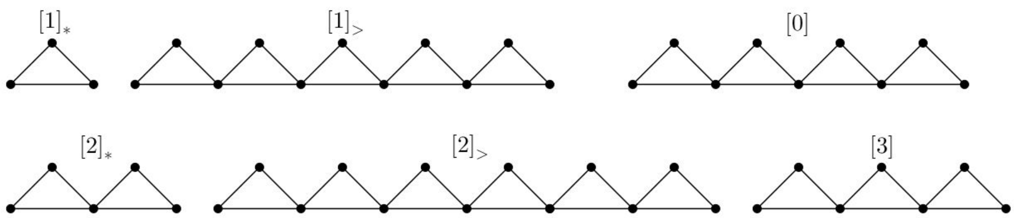


Figure 2.3: Some of chains of complete graphs in different classes

Definition 2.4. For a CC graph G, let $a(G)$ denote the numbers of components of G that are in [2] or [3] and let $b(G)$ denote the numbers of components of G that are (2).

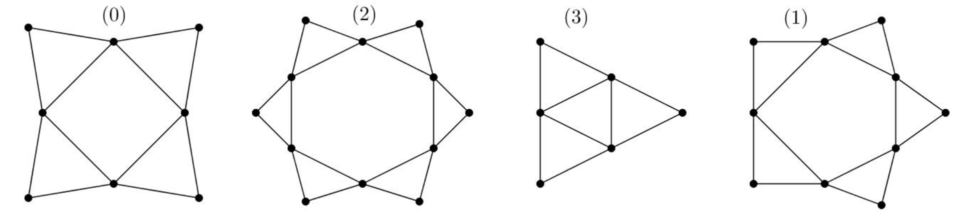


Figure 2.4: Some of cycles of complete graphs in different classes

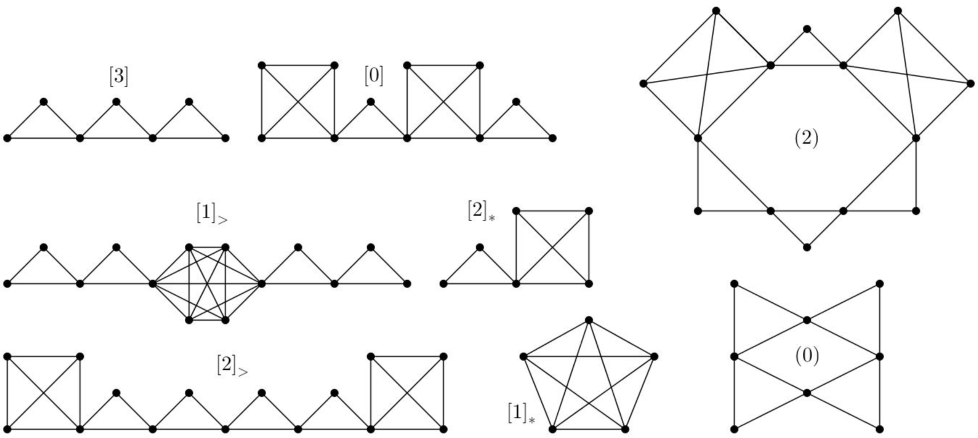


Figure 2.5: Example of a CC graph

Example 2.5. Let G be the CC graph in Figure 2.5. Since M_3, M_2 and M_6 of G are in [2] or [3], we get that $a(G) = 3$. And since there is only N_6 of G that is in (2), we get that $b(G) = 1$.

When a graph G is a disjoint union of graphs $G_1, G_2, ..., G_n$, we simply write $G = G_1 + G_2 + ... + G_n$.

Definition 2.6. Let $G = M_{m_1} + M_{m_2} + ... + M_{m_k} + N_{n_1} + N_{n_2} + ... + N_{n_l}$. Define $\theta(G) = \sum_{i=1}^{k}(m_i - \lfloor \frac{m_i}{4} \rfloor) + \sum_{j=1}^{l}(n_j - \lfloor \frac{n_j+2}{4} \rfloor)$.

When there are two vertices in a graph whose closed neighborhoods are the same, we can remove one of the vertices or mark it dominated without changing the game domination number of the graph.

Lemma 2.7. [6, Proposition 1.4] *Let G be a graph and let u, v be two distinct vertices of G. If $N[u] = N[v]$, then $\gamma_g(G \setminus \{u\}) = \gamma_g(G) = \gamma_g(G|\{u\})$ and $\gamma'_g(G \setminus \{u\}) = \gamma'_g(G) = \gamma'_g(G|\{u\})$.*

From the lemma above and since we only consider complete graphs with at least 3 vertices, the game domination numbers of a chain of m complete graphs are the same as those of a chain of m K_3's and the game domination numbers of a cycle of n complete graphs are the same as those of a cycle of n K_3's. Moreover, Lemma 2.7 yields the following important result.

Lemma 2.8. *Let G be a graph. Then for any positive integer m, $\gamma_g(G + M_m) = \gamma_g(G + M'_m) = \gamma_g(G + M''_m)$ and $\gamma'_g(G + M_m) = \gamma'_g(G + M'_m) = \gamma'_g(G + M''_m)$.*

A fundamental tool for comparing choices of moves called Continuation Principle is given below.

Theorem 2.9. [11, Lemma 2.1 (Continuation Principle)] *Let G be a (partially-dominated) graph and let A and B be subsets of $V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.*

Lemma 2.10. For each turn of Dominator, playing an identified vertex on a chain of complete graphs or a cycle of complete graphs is not worse than playing an unidentified vertex. For each turn of Staller, playing an unidentified vertex on a chain of complete graphs or a cycle of complete graphs is not worse than playing an identified vertex.

Proof. For any two consecutive K_3's on a chain or a cycle of complete graphs, let $A = \{a, b, c, d, e\}$ be a set of vertices for the two consecutive K_3's where c is an identified vertex of them. Let $A_1 = \{a, b, c\}$ and $A_2 = \{c, d, e\}$ be two sets of vertices for each K_3 i.e., $A_1 \subset A$ and $A_1 \subset A$.

Consider playing a vertex in A, the dominating set after playing c is A but the dominating set after playing a vertex in $A \setminus \{c\}$ is either A_1 or A_2 for these two consecutive K_3's. Since A_1 and A_2 are subset of A, by the continuation principle, we get that playing an identified vertex will make the game domination numbers less than or equal to playing an unidentified vertex. Thus as a Dominator, playing an identified vertex on a chain or a cycle of complete graphs is not worse than playing an unidentified vertex. On the other hand, as a Staller, playing an unidentified vertex on a chain or a cycle of complete graphs is not worse than playing an identified vertex. □

Lemma 2.10 allows us to make the following assumption which we will use throughout the paper.

Assumption 2.11. A Dominator's move always dominates two consecutive complete graphs (if available) and a Staller's move always dominates exactly one complete graph.

Next, we give an important lemma that can be used for establishing bounds for the game domination numbers. Recall that Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. If Dominator has a strategy, possibly suboptimal, that can end the game within a certain number of moves or Staller has a strategy, possibly suboptimal, that can prolong the game to at least a certain number of moves, then a bound for game domination numbers can be established as follows.

Lemma 2.12. *Let G be a graph, the following statements hold.*

(i) *For Dominator-start game, if Dominator has a strategy that can end the game within k moves, then $\gamma_g(G) \leq k$.*

(ii) *For Staller-start game, if Dominator has a strategy that can end the game within k moves, then $\gamma'_g(G) \leq k$.*

(iii) *For Dominator-start game, if Staller has a strategy that can prolong the game to at least k moves, then $\gamma_g(G) \geq k$.*

(iv) *For Staller-start game, if Staller has a strategy that can prolong the game to at least k moves, then $\gamma'_g(G) \geq k$.*

Lemma 2.12 can be used to prove a bound of a game domination number by presenting an appropriate Dominator's strategy or a Staller's strategy.

Chapter 3

Main results

In this section, we find the game domination numbers of a disjoint union of chains and cycles of complete graphs together with optimal strategies for both players.

Theorem 3.1. *Let G be a CC graph. Let $\theta = \theta(G)$, $a = a(G)$ and $b = b(G)$. Then*

$$\gamma_g(G) = \theta + \left\lfloor \frac{b-a}{2} \right\rfloor$$

and

$$\gamma_g'(G) = \theta + \left\lceil \frac{b-a}{2} \right\rceil.$$

Moreover, an optimal strategy for each player is as follows.

A Dominator's optimal strategy: For each turn, Dominator always plays on a component that is not in $[1]_*$ or (3) if possible. When Dominator plays on a chain component (except M_1 and M_2), he plays to dominate the first two complete graphs or the last two complete graphs of a chain.

A Staller's optimal strategy: For each turn, Staller always plays on a component that is not in $[1]_*$ or (1) if possible. When Staller plays on a chain component (except M_1), he plays to make the residual graph of this component contains a component from class $[1]$ or $[3]$.

Proof. Let $G = M_{m_1} + M_{m_2} + ... + M_{m_k} + N_{n_1} + N_{n_2} + ... + N_{n_l}$. By Lemma 2.7 we can assume that each component of G is either a chain or a cycle of K_3's. We prove the results by induction on $m_1 + m_2 + ... + m_k + n_1 + n_2 + ... + n_l$. One can check that the theorem holds for any CC graph with $\sum_{i=1}^{k} m_i + \sum_{j=1}^{l} n_j \leq 4$. Assume that $\sum_{i=1}^{k} m_i + \sum_{j=1}^{l} n_j \geq 5$. First, we determine the value of $\gamma_g(G)$. To do this, we find Dominator's optimal first move by comparing all his valid first moves on a Dominator-start game.

Let $\tilde{G}$ be the residual graph of G after Dominator plays his first move on G. Then $\gamma_g(G) \leq 1 + \gamma'_g(\tilde{G})$ with equality if Dominator plays his first move optimally. We divide our arguments based on the choice of Dominator's first move. In each case, we count the number of moves of the game with specified Dominator's first move and the remaining moves are played optimally by both players. After Dominator makes his first move, the component in G on which he plays will either be

1. reduced to nothing in $\tilde{G}$ if Dominator plays his first move on a component of G that is M_1 or M_2,
2. reduced to one component in $\tilde{G}$ if Dominator plays his first move on a component of G that is a cycle of complete graphs, or his first move dominates the first two complete graphs or the last two complete graphs of a chain of complete graphs, or
3. reduced to two components in $\tilde{G}$ if his first move does not dominate the first nor the last complete graph of a chain of complete graphs of G with at least

four complete graphs.

Table 3.1 shows the values of $1+\gamma'_g(\tilde{G})$ for all residual graphs $\tilde{G}$ obtained from Dominator making first move on G. The first column of the table shows the classes of components on which Dominator plays his first move. The second column shows the classes of residual graphs of the components that were played on. The third to fifth columns show the changes in values of parameters $g \in \{\theta, a, b\}$ where $\Delta g = g(\tilde{G}) - g(G)$. The last column shows the values of $1 + \gamma'_g(\tilde{G})$.

Now, we show how to obtain the entries in Table 3.1 by considering how Dominator makes his first move. Let R be the residual graph of the component that Dominator starts on. By Lemma 2.8, when computing the domination game of $\tilde{G}$, we can view $\tilde{G}$ as a CC graph whose vertices are not dominated.

Case 1 : Dominator starts on M_{m_j} where $m_j \equiv 0 \pmod 4$.

Case 1.1 : R contains exactly one component and it is in [2] or R contains exactly two components and one is in [0] and the other is in [2].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 1.2 : R contains exactly two components and both are in [1].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma'_g(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

Case 1.3 : R contains exactly two components and both are in [3].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+2$ and $b(\tilde{G}) = b$. By the induction hypothesis,

1st move	Residual	$\Delta\theta$	Δa	Δb	$1+\gamma_g'(\tilde{G})$
$[0]$	$[2]$ $[0],[2]$	-1	$+1$	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
	$[1],[1]$	-1	0	0	$\theta+\lceil\frac{b-a}{2}\rceil$
	$[3],[3]$	0	$+2$	0	$\theta+\lceil\frac{b-a}{2}\rceil$
$[1]_*$	$-$	-1	0	0	$\theta+\lceil\frac{b-a}{2}\rceil$
$[1]_>$	$[3]$ $[0],[3]$	-1	$+1$	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
	$[1],[2]$	-1	$+1$	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
$[2]_*$	$-$	-2	-1	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
$[2]_>$	$[0]$ $[0],[0]$	-2	-1	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
	$[1],[3]$	-1	0	0	$\theta+\lceil\frac{b-a}{2}\rceil$
	$[2],[2]$	-1	$+1$	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
$[3]$	$[1]$ $[0],[1]$	-2	-1	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
	$[2],[3]$	-1	$+1$	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
(0)	$[2]$	-1	$+1$	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
(1)	$[3]$	-1	$+1$	0	$\theta+\lceil\frac{b-a-1}{2}\rceil$
(2)	$[0]$	-1	0	-1	$\theta+\lceil\frac{b-a-1}{2}\rceil$
(3)	$[1]$	-1	0	0	$\theta+\lceil\frac{b-a}{2}\rceil$

Table 3.1: Effect of Dominator's first moves on a CC graph.

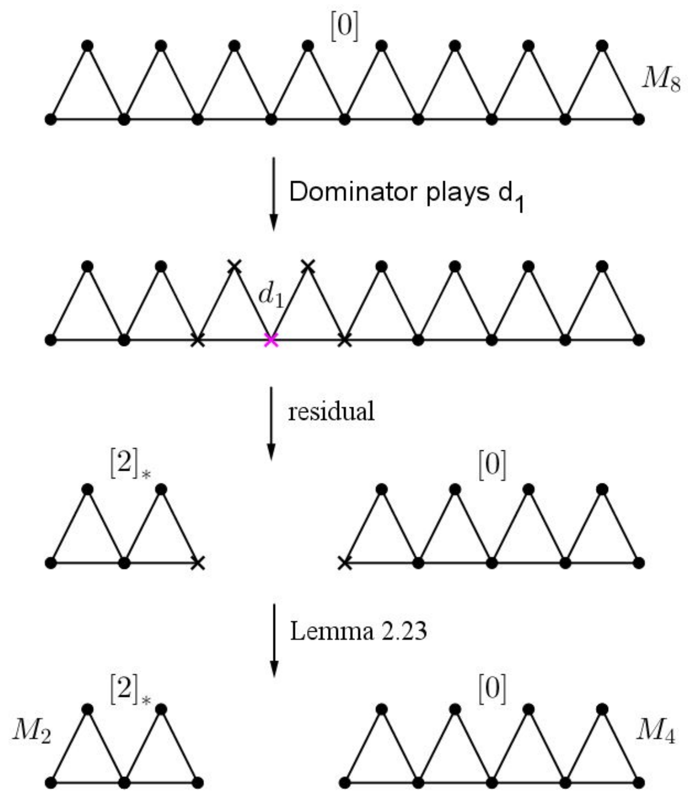


Figure 3.1: Dominator plays his first move on M_8 (in class [0]) and results in the residual graph with two new components in classes $[2]_*$ and [0], respectively

we have $\gamma_g'(\tilde{G}) = \theta + \left\lceil \frac{b-(a+2)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \lceil \frac{b-a}{2} \rceil$.

Case 2 : Dominator starts on M_1.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \lceil \frac{b-a}{2} \rceil$. Thus the number of moves in this case is $\theta + \lceil \frac{b-a}{2} \rceil$.

Case 3 : Dominator starts on M_{m_j} where $m_j \equiv 1 \pmod 4$ and $m_j \geq 5$.

Case 3.1 : R contains exactly one component and it is in [3] or R contains exactly two components and one is in [0] and the other is in [3].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 3.2 : R contains exactly two components and one is in [1] and the other is in [2].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 4 : Dominator starts on M_2.

Then $\theta(\tilde{G}) = \theta - 2$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 2 + \left\lceil \frac{b-(a-1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 5 : Dominator starts on M_{m_j} where $m_j \equiv 2 \pmod 4$ and $m_j \geq 6$.

Case 5.1 : R contains exactly one component and it is in [0] or R contains exactly two components and one is in [0] and the other is in [0].

Then $\theta(\tilde{G}) = \theta - 2$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 2 + \left\lceil \frac{b-(a-1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 5.2 : R contains exactly two components and one is in [1] and the other is in [3].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

Case 5.3 : R contains exactly two components and both are in [2].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \lceil \frac{b-a-1}{2} \rceil$.

Case 6 : Dominator starts on M_{m_j} where $m_j \equiv 3 \pmod 4$.

Case 6.1 : R contains exactly one component and it is in [1] or R contains exactly two components and one is in [0] and the other is in [1].

Then $\theta(\tilde{G}) = \theta - 2$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 2 + \left\lceil \frac{b-(a-1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \lceil \frac{b-a-1}{2} \rceil$.

Case 6.2 : R contains exactly two components and one is in [2] and the other is in [3].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \lceil \frac{b-a-1}{2} \rceil$.

Case 7 : Dominator starts on N_{n_j} where $n_j \equiv 0 \pmod 4$. Then R contains exactly one component and it is in [2].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \lceil \frac{b-a-1}{2} \rceil$.

Case 8 : Dominator starts on N_{n_j} where $n_j \equiv 1 \pmod 4$. Then R contains exactly one component and it is in [3].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a + 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-(a+1)}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 9 : Dominator starts on N_{n_j} where $n_j \equiv 2 \pmod 4$. Then R contains exactly one component and it is in [0].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b - 1$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-1-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a-1}{2} \right\rceil$.

Case 10 : Dominator starts on N_{n_j} where $n_j \equiv 3 \pmod 4$. Then R contains exactly one component and it is in [1].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g'(\tilde{G}) = \theta - 1 + \left\lceil \frac{b-a}{2} \right\rceil$. Thus the number of moves in this case is $\theta + \left\lceil \frac{b-a}{2} \right\rceil$.

From 10 cases above, we get that $\gamma_g(G) = \min(1 + \gamma_g'(\tilde{G})) = \theta + \left\lceil \frac{b-a-1}{2} \right\rceil = \theta + \left\lfloor \frac{b-a}{2} \right\rfloor$.

Next, we determine the value of $\gamma_g'(G)$. To do this, we find Staller's optimal first move by comparing all his valid first moves on a Staller-start game.

Let $\tilde{G}$ be the residual graph of G after Staller plays his first move on G. Then $\gamma_g'(G) \geq 1 + \gamma_g(\tilde{G})$ with equality if Staller plays his first move optimally. We divide our arguments based on the choice of Staller's first move. In each case, we count the number of moves of the game with specified Staller's first move and the remaining moves are played optimally by both players. After Staller makes his first move, the component in G on which he plays will either be

1. reduced to nothing in $\tilde{G}$ if Staller plays his first move on a component of G that is M_1,

2. reduced to one component in $\tilde{G}$ if Staller plays his first move on a component of G that is a cycle of complete graphs, or his first move dominates the first complete graph or the last complete graph of a chain of complete graphs, or

3. reduced to two components in $\tilde{G}$ if his first move does not dominate the first complete graph nor the last complete graph of a chain of complete graphs of G with at least three complete graphs.

Table 3.2 shows the values of $1+\gamma_g(\tilde{G})$ for all residual graphs $\tilde{G}$ obtained from Staller making first move on G. The first column of the table shows the classes of components on which Staller plays his first move. The second column shows the classes of residual graphs of the components that were played on. The third to fifth columns show the changes in values of parameters $g \in \{\theta, a, b\}$ where $\Delta g = g(\tilde{G}) - g(G)$. The last column shows the values of $1 + \gamma_g(\tilde{G})$.

Now, we show how to obtain the entries in Table 3.2 by considering how Staller makes his first move. Let R be the residual graph of the component that Staller starts on. By Lemma 2.8, when computing the domination game of $\tilde{G}$, we can view $\tilde{G}$ as a CC graph whose vertices are not dominated.

Case 1 : Staller starts on M_{m_j} where $m_j \equiv 0 \pmod 4$.

Case 1.1 : R contains exactly one component and it is in [3] or R contains exactly two components and one is in [0] and the other is in [3].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

Case 1.2 : R contains exactly two components and one is in [1] and the

1st move	Residual	$\Delta\theta$	Δa	Δb	$1+\gamma_g(\tilde{G})$
$[0]$	$[3]$ $[0],[3]$	0	$+1$	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
	$[1],[2]$	0	$+1$	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
$[1]_*$	$-$	-1	0	0	$\theta+\left\lfloor\frac{b-a}{2}\right\rfloor$
$[1]_>$	$[0]$ $[0],[0]$	-1	0	0	$\theta+\left\lfloor\frac{b-a}{2}\right\rfloor$
	$[1],[3]$	0	$+1$	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
	$[2],[2]$	0	$+2$	0	$\theta+\left\lfloor\frac{b-a}{2}\right\rfloor$
$[2]_*$	$[1]_*$	-1	-1	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
$[2]_>$	$[1]$ $[0],[1]$	-1	-1	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
	$[2],[3]$	0	$+1$	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
$[3]$	$[2]$ $[0],[2]$	-1	0	0	$\theta+\left\lfloor\frac{b-a}{2}\right\rfloor$
	$[1],[1]$	-1	-1	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
	$[3],[3]$	0	$+1$	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
(0)	$[3]$	0	$+1$	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
(1)	$[0]$	-1	0	0	$\theta+\left\lfloor\frac{b-a}{2}\right\rfloor$
(2)	$[1]$	0	0	-1	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$
(3)	$[2]$	0	$+1$	0	$\theta+\left\lfloor\frac{b-a+1}{2}\right\rfloor$

Table 3.2: Effect of Staller's first moves on a *CC* graph.

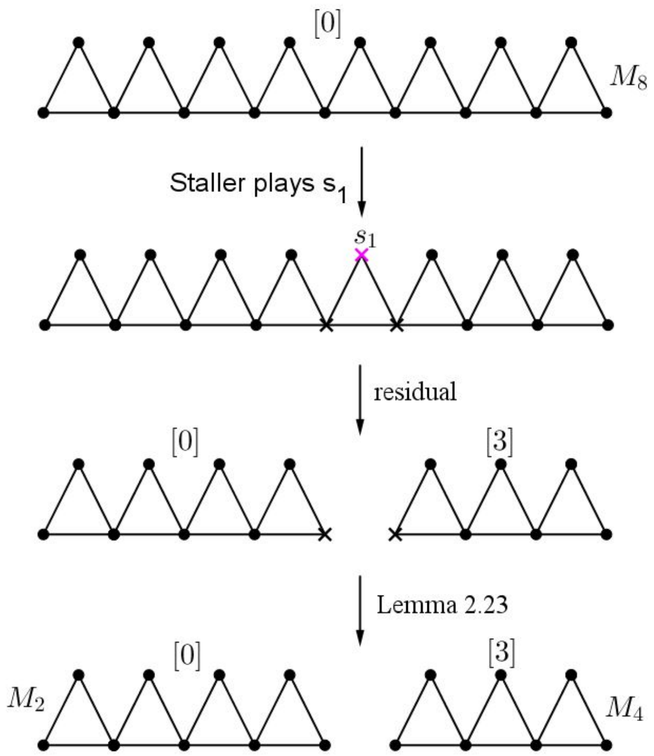


Figure 3.2: Staller plays his first move on M_8 (in class [0]) and results in the residual graph with two new components in classes [0] and [3], respectively

other is in [2].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 2 : Staller starts on M_1.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \lfloor \frac{b-a}{2} \rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a}{2} \rfloor$.

Case 3 : Staller starts on M_{m_j} where $m_j \equiv 1 \pmod 4$ and $m_j \geq 5$.

Case 3.1 : R contains exactly one component and it is in [0] or R contains exactly two components and both are in [0].

Then $\theta(\tilde{G}) = \theta-1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-a}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a}{2} \right\rfloor$.

Case 3.2 : R contains exactly two components and one is in [1] and the other is in [3].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

Case 3.3 : R contains exactly two components and both are in [2].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+2$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+2)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a}{2} \right\rfloor$.

Case 4 : Staller starts on M_2.

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-(a-1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

Case 5 : Staller starts on M_{m_j} where $m_j \equiv 2 \pmod 4$.

Case 5.1 : R contains exactly one component and it is in [1] or R contains exactly two components and one is in [0] and the other is in [1].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-(a-1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

Case 5.2 : R contains exactly two components and one is in [2] and the other is in [3].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 6 : Staller starts on M_{m_j} where $m_j \equiv 3 \pmod 4$.

Case 6.1 : R contains exactly one component and it is in [2] or R contains exactly two components and one is in [0] and the other is in [2].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \lfloor \frac{b-a}{2} \rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a}{2} \rfloor$.

Case 6.2 : R contains exactly two components and both are in [1].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a - 1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \left\lfloor \frac{b-(a-1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 6.3 : R contains exactly two components and both are in [3].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 7 : Staller starts on N_{n_j} where $n_j \equiv 0 \pmod 4$. Then R contains exactly one component and it is in [3].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a+1}{2} \rfloor$.

Case 8 : Staller starts on N_{n_j} where $n_j \equiv 1 \pmod 4$. Then R contains exactly one component and it is in [0].

Then $\theta(\tilde{G}) = \theta - 1$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta - 1 + \lfloor \frac{b-a}{2} \rfloor$. Thus the number of moves in this case is $\theta + \lfloor \frac{b-a}{2} \rfloor$.

Case 9 : Staller starts on N_{n_j} where $n_j \equiv 2 \pmod 4$. Then R contains exactly one component and it is in [1].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a$ and $b(\tilde{G}) = b-1$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-1-a}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

Case 10 : Staller starts on N_{n_j} where $n_j \equiv 3 \pmod 4$. Then R contains exactly one component and it is in [2].

Then $\theta(\tilde{G}) = \theta$, $a(\tilde{G}) = a+1$ and $b(\tilde{G}) = b$. By the induction hypothesis, we have $\gamma_g(\tilde{G}) = \theta + \left\lfloor \frac{b-(a+1)}{2} \right\rfloor$. Thus the number of moves in this case is $\theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor$.

From 10 cases above, we get that $\gamma_g'(G) = \max(1 + \gamma_g(\tilde{G})) = \theta + \left\lfloor \frac{b-a+1}{2} \right\rfloor = \theta + \left\lceil \frac{b-a}{2} \right\rceil$.

□

For a predicate $\mathcal{P}$, let $[\mathcal{P}]$ equals 1 if $\mathcal{P}$ is true; otherwise $[\mathcal{P}] = 0$.

Corollary 3.2. *Let G be a chain of m complete graphs. Then* $\gamma_g(G) = m - \left\lfloor \frac{m}{4} \right\rfloor - [m \equiv 2, 3 \pmod 4]$ *and* $\gamma_g'(G) = m - \left\lfloor \frac{m}{4} \right\rfloor$.

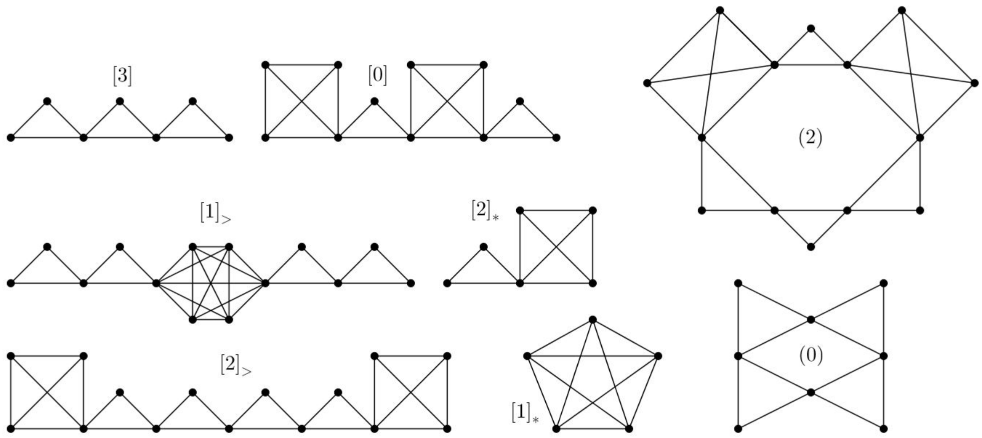
Corollary 3.3. *Let G be a cycle of n complete graphs. Then* $\gamma_g(G) = n - \left\lfloor \frac{n+2}{4} \right\rfloor$ *and* $\gamma_g'(G) = n - \left\lfloor \frac{n+2}{4} \right\rfloor + [n \equiv 2 \pmod 4]$.

Example 3.4. Recall the CC graph in Figure 2.5. We calculate the game domination numbers of this CC graph by using Theorem 3.1 as follows.

Let $G = M_3 + M_4 + M_5 + M_2 + M_6 + M_1 + N_6 + N_4$.

Then $\theta(G) = \sum_{i=1}^{6}(m_i - \lfloor \frac{m_i}{4} \rfloor) + \sum_{j=1}^{2}(n_j - \lfloor \frac{n_j+2}{4} \rfloor) = (3 - \lfloor \frac{3}{4} \rfloor) + (4 - \lfloor \frac{4}{4} \rfloor) + (5 - \lfloor \frac{5}{4} \rfloor) + (2 - \lfloor \frac{2}{4} \rfloor) + (6 - \lfloor \frac{6}{4} \rfloor) + (1 - \lfloor \frac{1}{4} \rfloor) + (6 - \lfloor \frac{6+2}{4} \rfloor) + (4 - \lfloor \frac{4+2}{4} \rfloor) = 25$.

Recall that $a(G)$ and $b(G)$ denote the numbers of components of G that are in $[2] \cup [3]$ and (2), respectively. So $a(G) = 3$ and $b(G) = 1$.



By theorem 3.1, we get that $\gamma_g(G) = \theta + \left\lfloor \frac{b-a}{2} \right\rfloor = 25 + \left\lfloor \frac{1-3}{2} \right\rfloor = 24$ and $\gamma_g'(G) = \theta + \left\lceil \frac{b-a}{2} \right\rceil = \gamma_g'(G) = 25 + \left\lceil \frac{1-3}{2} \right\rceil = 24.$

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DISSEMINATIONS

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