



VARIOUS PROBLEMS CONCERNING FACTORIALS, BINOMIAL COEFFICIENTS,
FIBONOMIAL COEFFICIENTS, AND PALINDROMES



By

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A Thesis Submitted in Partial Fulfillment of the Requirements
for Doctor of Philosophy (MATHEMATICS)

Department of MATHEMATICS

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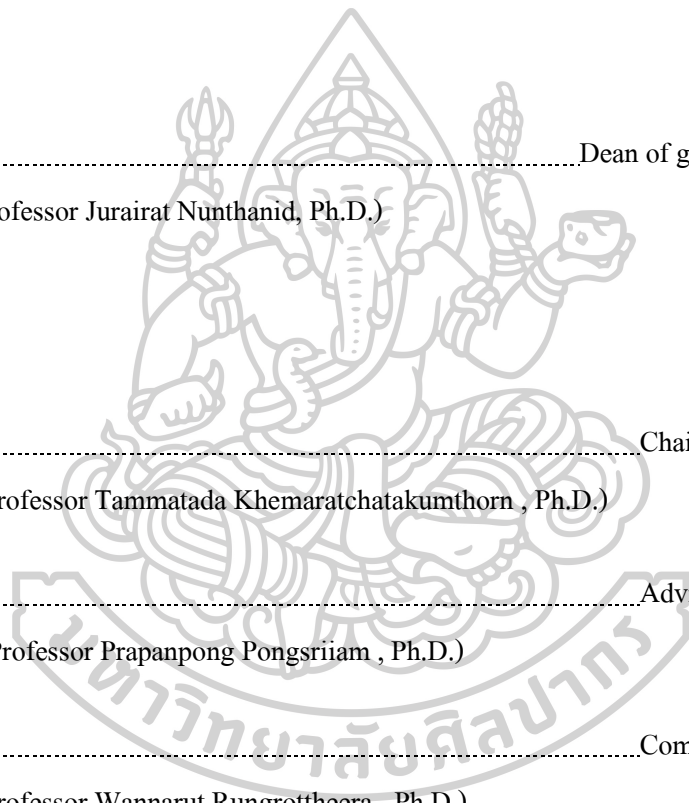
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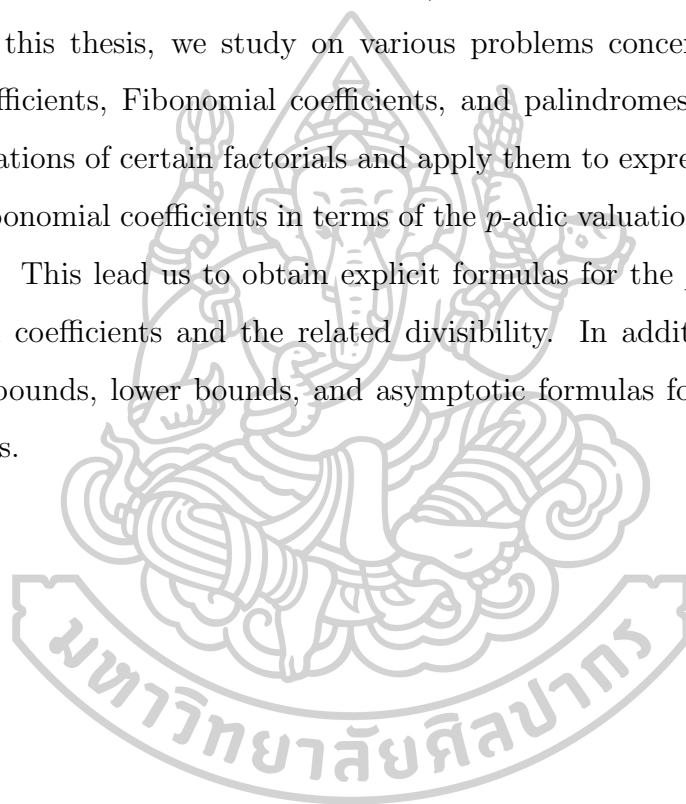


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In this thesis, we study on various problems concerning factorials, binomial coefficients, Fibonomial coefficients, and palindromes. We calculate the p -adic valuations of certain factorials and apply them to express the p -adic valuations of Fibonomial coefficients in terms of the p -adic valuations of some binomial coefficients. This lead us to obtain explicit formulas for the p -adic valuations of Fibonomial coefficients and the related divisibility. In addition, we also evaluate upper bounds, lower bounds, and asymptotic formulas for reciprocal sum of palindromes.



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Chapter 1

Introduction

Throughout this thesis, unless stated otherwise, x, y, z are real numbers, p always denotes a prime, a, b, k, ℓ, m, n, q are integers, $m, n \geq 1$, $q \geq 2$, and $\log x$ is the natural logarithm of x . Recall that for each $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x , $\{x\}$ is the *fractional part* of x given by $\{x\} = x - \lfloor x \rfloor$, and $\lceil x \rceil$ is the smallest integer larger than or equal to x . In addition, we write $a \bmod m$ to denote the least nonnegative residue of a modulo m . We also use the *Iverson notation*: if P is a mathematical statement, then

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

For example, $[5 \equiv -1 \pmod{4}] = 0$ and $[3 \equiv -1 \pmod{4}] = 1$. We define $s_q(n)$ to be the sum of digits of n when n is written in base q , that is, if $n = (a_k a_{k-1} \dots a_0)_q = a_k q^k + a_{k-1} q^{k-1} + \dots + a_0$ where $0 \leq a_i < q$ for every i , then $s_q(n) = a_k + a_{k-1} + \dots + a_0$.

In this thesis, we study on several topics of factorials, binomial coefficients, Fibonomial coefficients, and palindrome and have publications on these problems (See [30, 31, 32]). So we divide this chapter into two sections.

1.1 Explicit formulas for the p -adic valuations of Fibonomial coefficients

The *Fibonacci sequence* $(F_n)_{n \geq 1}$ is given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with the initial values $F_1 = F_2 = 1$. For each $m \geq 1$ and $1 \leq k \leq m$, the *Fibonomial coefficients* $\binom{m}{k}_F$ are defined by

$$\binom{m}{k}_F = \frac{F_1 F_2 F_3 \cdots F_m}{(F_1 F_2 F_3 \cdots F_k)(F_1 F_2 F_3 \cdots F_{m-k})} = \frac{F_{m-k+1} F_{m-k+2} \cdots F_m}{F_1 F_2 F_3 \cdots F_k},$$

where F_n is the n th Fibonacci number. If $k = 0$, we define $\binom{m}{k}_F = 1$ and if $k > m$, we define $\binom{m}{k}_F = 0$. It is well known that $\binom{m}{k}_F$ is always an integer for all integers $m \geq 1$ and $k \geq 0$. So it is natural to consider the divisibility properties and the p -adic valuation of $\binom{m}{k}_F$. As usual, the p -adic valuation (or p -adic order) of n , denoted by $\nu_p(n)$, is the exponent of p in the prime factorization of n . In addition, the order (or the rank) of appearance of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer k such that $n \mid F_k$.

In 1989, Knuth and Wilf [19] gave a short description of the p -adic valuation of $\binom{m}{k}_C$ where C is a regularly divisible sequence. However, this does not give explicit formulas for $\binom{m}{k}_F$. Then recently, there has been some interest in explicitly evaluating the p -adic valuation of Fibonomial coefficients of the form $\binom{p^b}{p^a}_F$. For example, Marques and Trojovský [26, 27] and Marques, Sellers, and Trojovský [24] deal with the case $b = a + 1$, $a \geq 1$. Then Trojovský [45] give an exact formula for $\nu_p\left(\binom{p^b}{p^a}_F\right)$ where $b > a \geq 1$. Additionally, Marques and Trojovský [25, 26, 27] and Marques, Sellers, and Trojovský [24] find all integers $n \geq 1$ such that $\binom{p^n}{n}_F$ is divisible by p in the case $p = 2, 3$ and in the case that p is any prime and $n = p^a$ for some $a \geq 1$. Ballot [6, Theorem 4.2] extends the Kummer-like theorem of Knuth and Wilf [19, Theorem 2], which gives the p -adic valuation of Fibonomials, to all Lucasnomials, and uses it to determine explicitly the p -adic valuation of Lucasnomials of the form $\binom{p^b}{p^a}_U$, for all nondegenerate fundamental Lucas sequences U and all integers $b > a \geq 0$, [6, Theorem 7.1]. In particular, Ballot [6] investigates all integers n such that $p \mid \binom{p^n}{n}_U$ for any nondegenerate fundamental Lucas sequence U and $p = 2, 3$ and for $p = 5, 7$ in the case $U = F$. Hence explicit formulas for the p -adic valuations of Fibonomial coefficients of the form $\binom{\ell_1 p^b}{\ell_2 p^a}_F$ have been investigated only in the case $\ell_1 = \ell_2 = 1$ and the relation $p \mid \binom{p^a n}{n}_F$ has been studied only in the case $p = 2, 3, 5, 7$ and $a = 1$.

In this thesis, we enhance Ballot's theorem [6], Theorem 7.1, in the case $U = F$ and $b \geq a > 0$ and obtain explicit formulas for $\binom{\ell_1 p^b}{\ell_2 p^a}_F$, where ℓ_1 and ℓ_2 are arbitrary positive integers such that $\ell_1 p^b > \ell_2 p^a$. This leads us to study the

p -adic valuations of integers of the factorial form

$$\left\lfloor \frac{\ell p^a}{m} \right\rfloor! \quad \text{or} \quad \left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor!,$$

where $p \equiv \pm 1 \pmod{m}$. Then we extend the examination on $\binom{p^a n}{n}_F$ to the case of any prime p and any positive integer a . Replacing n by p^a and p^a by p , this becomes the result of Marques and Trojovský [27] and Marques, Sellers, and Trojovský [24]. Substituting $a = 1$, $p \in \{2, 3, 5, 7\}$, and letting n be arbitrary, this reduces to Ballot's theorems [6]. So our results are indeed an extension of those previously mentioned. As a reward, we can easily show in Corollaries 3.11 and 3.12 that $\binom{4n}{n}_F$ is odd if and only if n is a nonnegative power of 2, and $\binom{8n}{n}_F$ is odd if and only if $n = (1 + 3 \cdot 2^k)/7$ for some $k \equiv 1 \pmod{3}$.

1.2 Reciprocal sum of Palindrome

Let $n \geq 1$ and $b \geq 2$ be integers. We call n a *palindrome in base b* (or *b -adic palindrome*) if the b -adic expansion of $n = (a_k a_{k-1} \cdots a_0)_b$ with $a_k \neq 0$ has the symmetric property $a_{k-i} = a_i$ for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$. As usual, if we write a number without specifying the base, then it is always in base 10. So, for example, $9 = (1001)_2 = (100)_3$ is a palindrome in bases 2 and 10 but not in base 3.

In recent years, there has been an increasing interest in the importance of palindromes in mathematics [1, 2, 3, 15, 23, 40], theoretical computer science [4, 11, 14], and theoretical physics [13, 17]. For example, Pongsriiam and Subwattanachai [36] obtain an exact formula for the number of b -adic palindromes not exceeding N for all $N \geq 1$. There are also some discussions on the reciprocal sum of palindromes on the internet but as far as we are aware, our observation has not appeared in the literature. Throughout this thesis, we let $b \geq 2$, s_b the reciprocal sum of all b -adic palindromes, and $s_{b,k}$ the reciprocal sum of all b -adic palindromes which have k digits in their b -adic expansion. The set of all b -adic palindromes is infinite but quite sparse, so it is not difficult to see that s_b converges. In fact, Shallit proposed the convergence of s_b as a problem proved by Klauser in the Fibonacci Quarterly [41, 42].

In this thesis, we obtain upper and lower bounds for s_b which enable us to show that $s_{b+1} > s_b$ for all $b \geq 2$ and $s_b^2 - s_{b-1}s_{b+1} > 0$ for all $b \geq 3$. That is the sequence $(s_b)_{b \geq 2}$ is strictly increasing and log-concave. We also give an asymptotic formula for s_b of the form $s_b = g(b) + O(h(b))$ where the implied constant can be taken to be 1 and the order of magnitude of $h(b)$ is $\frac{\log b}{b^3}$ as $b \rightarrow \infty$. Our result $s_{b+1} > s_b$ for all $b \geq 2$ also implies that if $b_1 > b_2 \geq 2$ and if we use the logarithmic measure, then we can say that the palindromes in base b_1 occur more often than those in base b_2 . On the other hand, if we use the usual counting measure, then we obtain from Pongsriiam and Subwattanachai's exact formula [36] that the number of palindromes in different bases which are less than or equal to N are not generally comparable. It seems that there are races between palindromes in different bases which may be similar to races between primes in different residue classes.

The reciprocal sum of an integer sequence is also of general interest in mathematics and theoretical physics as proposed by Bayless and Klyve [10], and by Roggero, Nardelli, and Di Noto [39]. See also the work of Nguyen and Pomerance [28] on the reciprocal sum of the amicable numbers, the study of Kinlaw, Kobayashi, and Pomerance [18] on the integers n satisfying $\varphi(n) = \varphi(n+1)$, and the article by Lichtman [22] on the reciprocal sum of primitive nondeficient numbers. In addition, Banks [7], Cilleruelo, Luca, and Baxter [12], and Rajasekaran, Shallit, and Smith [38] have recently investigated some additive properties of palindromes. Banks, Hart, and Sakata [8] and Banks and Shparlinski [9] show some multiplicative properties of palindromes.

We organize this thesis as follows. In Chapter 2, we give some preliminaries and useful results which are needed in the proof of the main theorems. In Chapter 3, we give an explicit formula for the p -adic of Fibonomial coefficients and some related results and apply it to obtain a characterization of the integers n such that $\binom{p^a}{n}_F$ is divisible by p where p is any prime which is congruent to $\pm 2 \pmod{5}$. Finally, in Chapter 4, we show upper and lower bounds, and asymptotic formula for s_b .

Chapter 2

Preliminaries and Lemmas

Recall that for each odd prime p and $a \in \mathbb{Z}$, the *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \mid a; \\ 1, & \text{if } a \text{ is a quadratic residue of } p; \\ -1, & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Then we have the following result.

Lemma 2.1. *Let $p \neq 5$ be a prime and let m and n be positive integers. Then the following statements hold.*

- (i) *If $p > 2$, then $F_{p-\left(\frac{5}{p}\right)} \equiv 0 \pmod{p}$.*
- (ii) *$n \mid F_m$ if and only if $z(n) \mid m$.*
- (iii) *$z(p) \mid p + 1$ if and only if $p \equiv 2$ or $-2 \pmod{5}$, and $z(p) \mid p - 1$ otherwise.*
- (iv) *$\gcd(z(p), p) = 1$.*

Proof. These are well known results. For example, (i) and (ii) can be found in [20, p. 410] and [46], respectively. Then (iii) follows from (i) and (ii). By (iii), $z(p) \mid p \pm 1$. Since $\gcd(p, p \pm 1) = 1$, we obtain $\gcd(z(p), p) = 1$. This proves (iv). □

Lengyel's result and Legendre's formula given in the following lemmas are important tools in evaluating the p -adic valuation of Fibonomial coefficients. We also refer the reader to [26, 27, 24, 33] for other similar applications of Lengyel's result.

Lemma 2.2. (Lengyel [21]) *For $n \geq 1$, we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, and if p is a prime distinct from 2 and 5, then

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}, \end{cases}$$

Lemma 2.3. (Legendre's formula) *Let n be a positive integer and let p be a prime.*

Then

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n - s_p(n)}{p - 1}.$$

In the proof of main results, we will deal with a lot of calculation involving the floor function. So it is useful to recall the following results.

Lemma 2.4. [35, Theorem 3.3] *For $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, the following holds*

- (i) $\lfloor n + x \rfloor = n + \lfloor x \rfloor$,
- (ii) $\{n + x\} = \{x\}$,
- (iii) $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$
- (iv) $\{-x\} = \begin{cases} 1 - \{x\}, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$
- (v) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\} + \{y\} \geq 1, \end{cases}$
- (vi) $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$ for $n \geq 1$.

The next lemma is used often in counting the number of positive integers $n \leq x$ lying in a residue class $a \pmod q$, see for instance in [37, Proof of Lemma 2.6].

Lemma 2.5. *For $x \in [1, \infty)$, $a, q \in \mathbb{Z}$ and $q \geq 1$, we have*

$$\sum_{\substack{1 \leq n \leq x \\ n \equiv a \pmod q}} 1 = \left\lfloor \frac{x-a}{q} \right\rfloor - \left\lfloor -\frac{a}{q} \right\rfloor. \quad (2.1)$$

Proof. Replacing a by $a+q$ and applying Lemma 2.4, we see that the value on the right-hand side of (2.1) is not changed. Obviously, the left-hand side is also invariant when we replace a by $a+q$. So it is enough to consider only the case $1 \leq a \leq q$. Since $n \equiv a \pmod q$, we write $n = a+kq$ where $k \geq 0$ and $a+kq \leq x$. So $k \leq \frac{x-a}{q}$. Therefore

$$\sum_{\substack{1 \leq n \leq x \\ n \equiv a \pmod q}} 1 = \sum_{0 \leq k \leq \frac{x-a}{q}} 1 = \left\lfloor \frac{x-a}{q} \right\rfloor + 1 = \left\lfloor \frac{x-a}{q} \right\rfloor - \left\lfloor -\frac{a}{q} \right\rfloor. \quad \square$$

It is convenient to use the Iverson notation and to denote the least non-negative residue of a modulo m by $a \pmod m$. Therefore we will do so from this point on.

Lemma 2.6. *Let n and k be integers, m a positive integer, $r = n \pmod m$, and $s = k \pmod m$. Then*

$$\left\lfloor \frac{n-k}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor - [r < s].$$

Proof. By Lemma 2.4(i) and the fact that $0 \leq r < m$, we obtain

$$\left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{n-r}{m} + \frac{r}{m} \right\rfloor = \frac{n-r}{m} + \left\lfloor \frac{r}{m} \right\rfloor = \frac{n-r}{m}.$$

Similarly, $\left\lfloor \frac{k}{m} \right\rfloor = \frac{k-s}{m}$. Therefore $\left\lfloor \frac{n-k}{m} \right\rfloor$ is equal to

$$\left\lfloor \frac{n-r}{m} - \frac{k-s}{m} + \frac{r-s}{m} \right\rfloor = \frac{n-r}{m} - \frac{k-s}{m} + \left\lfloor \frac{r-s}{m} \right\rfloor = \begin{cases} \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, & \text{if } r \geq s; \\ \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor - 1, & \text{if } r < s. \end{cases}$$

□

In Chapter 4, we handle a lot of estimation involving reciprocal sums. So we recall the next lemma which plays a significant role in evaluating reciprocal sums.

Lemma 2.7. [35, Theorem 4.6] *Let a and b be integers such that $a < b$ and f a monotone function on $[a, b]$. Then*

$$\min\{f(a), f(b)\} \leq \sum_{n=a}^b f(n) - \int_a^b f(t)dt \leq \max\{f(a), f(b)\}.$$



Chapter 3

Explicit Formulas for the p -adic Valuations of Fibonomial Coefficients

We organize this chapter as follows. In Section 1, we give exact formulas for the p -adic valuations of certain integers. In Section 2, we apply the results obtained in Section 1 to Fibonomial coefficient and then use it to characterize the integers n such that $\binom{p^a n}{n}_F$ is divisible by p where p is any prime which is congruent to $\pm 2 \pmod{5}$. Finally, in Section 3, we show some examples to illustrate applications of Theorem 3.7.

3.1 The p -adic valuation of integers in special forms

In this section, we calculate the p -adic valuations of $\left\lfloor \frac{\ell p^a}{m} \right\rfloor!$ and other integers in similar forms.

Theorem 3.1. [30, Theorem 7] *Let p be a prime and let $a \geq 0$, $\ell \geq 0$, and $m \geq 1$ be integers. Assume that $p \equiv \pm 1 \pmod{m}$ and let $\delta = [\ell \not\equiv 0 \pmod{m}]$. Then*

$$\nu_p \left(\left\lfloor \frac{\ell p^a}{m} \right\rfloor! \right) = \begin{cases} \frac{\ell(p^a-1)}{m(p-1)} - a \left\{ \frac{\ell}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv 1 \pmod{m}; \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a}{2}\delta + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is even}; \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a-1}{2}\delta - \left\{ \frac{\ell}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is odd.} \end{cases}$$

We remark that if $m = 1$ or 2 , then the expressions in each case of this theorem are all equal.

We can combine every case in Theorem 3.1 into a single form as given in the next corollary.

Corollary 3.2. [30, Corollary 8] *Assume that p , a , ℓ , m , and δ satisfy the same assumptions as in Theorem 3.1. Then the p -adic valuation of $\left\lfloor \frac{\ell p^a}{m} \right\rfloor!$ is*

$$\begin{aligned} & \frac{\ell(p^a - 1)}{m(p - 1)} - \left\lfloor \frac{a}{2} \right\rfloor \delta - \left\{ \frac{\ell}{m} \right\} [a \equiv 1 \pmod{2}] \\ & + \delta \left\lfloor \frac{a}{2} \right\rfloor \left(1 - 2 \left\{ \frac{\ell}{m} \right\} \right) [p \equiv 1 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right). \end{aligned}$$

Next we deal with the p -adic valuation of an integer of the form $\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor!$ where a , b , ℓ_1 , ℓ_2 , and m are positive integers. It is natural to assume $\ell_1 p^b - \ell_2 p^a > 0$. In addition, if $a = b$, then the above expression is reduced to $\left\lfloor \frac{(\ell_1 - \ell_2)p^b}{m} \right\rfloor!$, which can be evaluated by using Theorem 3.1. We consider the case $b \geq a$ in Theorem 3.3 and the other case in Theorem 3.4.

Theorem 3.3. [30, Theorem 9] *Let p be a prime, let a be a nonnegative integer, and let b , m , ℓ_1 , ℓ_2 be positive integers satisfying $b \geq a$ and $\ell_1 p^b - \ell_2 p^a > 0$. Assume that $p \equiv \pm 1 \pmod{m}$. Then the following statements hold.*

(i) *If $p \equiv 1 \pmod{m}$, then*

$$\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) = \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p - 1)} - a \left\{ \frac{\ell_1 - \ell_2}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor! \right).$$

(ii) *If $p \equiv -1 \pmod{m}$ and $a \equiv b \pmod{2}$, then*

$$\begin{aligned} \nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p - 1)} - \left\{ \frac{\ell_1 - \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] \\ &\quad - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor! \right). \end{aligned}$$

(iii) *If $p \equiv -1 \pmod{m}$ and $a \not\equiv b \pmod{2}$, then*

$$\begin{aligned} \nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p - 1)} - \left\{ -\frac{\ell_1 + \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] \\ &\quad - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor! \right). \end{aligned}$$

We remark that if $m = 1$, the expressions in each case of this theorem are equal.

Next we replace the assumption $b \geq a$ in Theorem 3.3 by $b < a$. The calculation follows from the same idea. Although we do not use it in this research, it may be useful for future reference. So we record it in the next theorem.

Theorem 3.4. [30, Theorem 10] *Let p be a prime, let b be a nonnegative integer, and let a, m, ℓ_1, ℓ_2 be positive integers satisfying $b < a$ and $\ell_1 p^b - \ell_2 p^a > 0$. Assume that $p \equiv \pm 1 \pmod{m}$. Then the following statements hold.*

(i) *If $p \equiv 1 \pmod{m}$, then*

$$\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) = \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - b \left\{ \frac{\ell_1 - \ell_2}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell_1 - \ell_2 p^{a-b}}{m} \right\rfloor! \right).$$

(ii) *If $p \equiv -1 \pmod{m}$ and $a \equiv b \pmod{2}$, then*

$$\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) = \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - \left\{ \frac{\ell_1 - \ell_2}{m} \right\} [b \equiv 1 \pmod{2}] - \left\lfloor \frac{b}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 - \ell_2 p^{a-b}}{m} \right\rfloor! \right).$$

(iii) *If $p \equiv -1 \pmod{m}$ and $a \not\equiv b \pmod{2}$, then*

$$\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) = \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - \left\{ \frac{\ell_1 + \ell_2}{m} \right\} [b \equiv 1 \pmod{2}] - \left\lfloor \frac{b}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 - \ell_2 p^{a-b}}{m} \right\rfloor! \right).$$

3.2 The p -adic valuations of Fibonomial coefficients and some related divisibility

Recall that the *binomial coefficients* $\binom{m}{k}$ is defined by

$$\binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!}, & \text{if } 0 \leq k \leq m; \\ 0, & \text{if } k < 0 \text{ or } k > m. \end{cases}$$

A classical result of Kummer states that for $0 \leq k \leq m$, $\nu_p \left(\binom{m}{k} \right)$ is equal to the number of carries when we add k and $m - k$ in base p . From this, it is not difficult to show that for all primes p and positive integers k, b, a with $b \geq a$, we have

$$\nu_p \left(\binom{p^b}{p^a} \right) = b - a, \quad \text{or more generally, } \nu_p \left(\binom{p^a}{k} \right) = a - \nu_p(k).$$

Knuth and Wilf [19] also obtain the result analogous to that of Kummer for a C-nomial coefficient. However, our purpose is to obtain $\nu_p \left(\binom{m}{k}_F \right)$ in an explicit form. So we first express $\nu_p \left(\binom{m}{k}_F \right)$ in terms of the p -adic valuation of some binomial coefficients in Theorem 3.5. Then we write it in a form which is easy to use in Corollary 3.6. Then we apply it to obtain the p -adic valuation of Fibonomial coefficients of the form $\binom{\ell_1 p^b}{\ell_2 p^a}_F$.

Theorem 3.5. [30, Theorem 11] *Let $0 \leq k \leq m$ be integers. Then the following statements hold.*

- (i) *Let $m' = \lfloor \frac{m}{6} \rfloor$, $k' = \lfloor \frac{k}{6} \rfloor$, and let $r = m \bmod 6$ and $s = k \bmod 6$ be the least nonnegative residues of m and k modulo 6, respectively. Then*

$$\begin{aligned} \nu_2 \left(\binom{m}{k}_F \right) &= \nu_2 \left(\binom{m'}{k'} \right) + \left\lfloor \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{s+3}{6} \right\rfloor - 3 \left\lfloor \frac{r-s}{6} \right\rfloor \\ &\quad + [r < s] \nu_2 \left(\left\lfloor \frac{m-k+6}{6} \right\rfloor \right). \end{aligned}$$

- (ii) $\nu_5 \left(\binom{m}{k}_F \right) = \nu_5 \left(\binom{m}{k} \right)$.

- (iii) *Suppose that p is a prime, $p \neq 2$, and $p \neq 5$. Let $m' = \lfloor \frac{m}{z(p)} \rfloor$, $k' = \lfloor \frac{k}{z(p)} \rfloor$, and let $r = m \bmod z(p)$, and $s = k \bmod z(p)$ be the least nonnegative residues of m and k modulo $z(p)$, respectively. Then*

$$\nu_p \left(\binom{m}{k}_F \right) = \nu_p \left(\binom{m'}{k'} \right) + [r < s] \left(\nu_p \left(\left\lfloor \frac{m-k+z(p)}{z(p)} \right\rfloor \right) + \nu_p(F_{z(p)}) \right).$$

By Theorem 3.5(ii), we see that the 5-adic valuations of Fibonomial and binomial coefficients are the same. So we focus our investigation only on the p -adic valuations of Fibonomial coefficients when $p \neq 5$. Calculating r and s in Theorem 3.5(i) in every case and writing Theorem 3.5(iii) in another form, we obtain the following corollary.

Corollary 3.6. [30, Corollary 12] *Let m , k , r , and s be as in Theorem 3.5. Let*

$$A_2 = \nu_2 \left(\left\lfloor \frac{m}{6} \right\rfloor! \right) - \nu_2 \left(\left\lfloor \frac{k}{6} \right\rfloor! \right) - \nu_2 \left(\left\lfloor \frac{m-k}{6} \right\rfloor! \right),$$

and for each prime $p \neq 2, 5$, let $A_p = \nu_p \left(\left\lfloor \frac{m}{z(p)} \right\rfloor! \right) - \nu_p \left(\left\lfloor \frac{k}{z(p)} \right\rfloor! \right) - \nu_p \left(\left\lfloor \frac{m-k}{z(p)} \right\rfloor! \right)$. Then the following statements hold.

$$(i) \nu_2 \left(\binom{m}{k}_F \right) = \begin{cases} A_2, & \text{if } r \geq s \text{ and } (r, s) \neq (3, 1), (3, 2), (4, 2); \\ A_2 + 1, & \text{if } (r, s) = (3, 1), (3, 2), (4, 2); \\ A_2 + 3, & \text{if } r < s \text{ and } (r, s) \neq (0, 3), (1, 3), (2, 3), \\ & (1, 4), (2, 4), (2, 5); \\ A_2 + 2, & \text{if } (r, s) = (0, 3), (1, 3), (2, 3), (1, 4), (2, 4), \\ & (2, 5). \end{cases}$$

(ii) For $p \neq 2, 5$, we have

$$\nu_p \left(\binom{m}{k}_F \right) = \begin{cases} A_p, & \text{if } r \geq s; \\ A_p + \nu_p(F_{z(p)}), & \text{if } r < s. \end{cases}$$

In a series of papers (see [27] and references therein), Marques and Trojovský obtain a formula for $\nu_p \left(\binom{p^b}{p^a}_F \right)$ only when $b = a + 1$. Then Ballot [6] extends it to any case $b > a$. Corollary 3.6 enables us to compute $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a}_F \right)$. We illustrate this in the next theorem.

Theorem 3.7. [30, Theorem 13] *Let a, b, ℓ_1 , and ℓ_2 be positive integers and $b \geq a$. Let $p \neq 5$ be a prime. Assume that $\ell_1 p^b > \ell_2 p^a$ and let $m_p = \left\lfloor \frac{\ell_1 p^{b-a}}{z(p)} \right\rfloor$ and $k_p = \left\lfloor \frac{\ell_2}{z(p)} \right\rfloor$. Then the following statements hold.*

(i) *If $a \equiv b \pmod{2}$, then $\nu_2 \left(\binom{\ell_1 2^b}{\ell_2 2^a}_F \right)$ is equal to*

$$\begin{cases} \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv \ell_2 \pmod{3} \text{ or } \ell_2 \equiv 0 \pmod{3}; \\ a + 2 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 0 \pmod{3} \text{ and } \ell_2 \not\equiv 0 \pmod{3}; \\ \left\lceil \frac{a}{2} \right\rceil + 1 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 1 \pmod{3} \text{ and } \ell_2 \equiv 2 \pmod{3}; \\ \left\lceil \frac{a+1}{2} \right\rceil + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 2 \pmod{3} \text{ and } \ell_2 \equiv 1 \pmod{3}, \end{cases}$$

and if $a \not\equiv b \pmod{2}$, then $\nu_2 \left(\binom{\ell_1 2^b}{\ell_2 2^a} \right)_F$ is equal to

$$\begin{cases} \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv -\ell_2 \pmod{3} \text{ or } \ell_2 \equiv 0 \pmod{3}; \\ a + 2 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 0 \pmod{3} \text{ and } \ell_2 \not\equiv 0 \pmod{3}; \\ \left\lceil \frac{a+1}{2} \right\rceil + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 1 \pmod{3} \text{ and } \ell_2 \equiv 1 \pmod{3}; \\ \left\lceil \frac{a}{2} \right\rceil + 1 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 2 \pmod{3} \text{ and } \ell_2 \equiv 2 \pmod{3}. \end{cases}$$

(ii) Let $p \neq 5$ be an odd prime and let $r = \ell_1 p^b \pmod{z(p)}$ and $s = \ell_2 p^a \pmod{z(p)}$. If $p \equiv \pm 1 \pmod{5}$, then

$$\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a} \right)_F = [r < s] (a + \nu_p(m_p - k_p) + \nu_p(F_{z(p)})) + \nu_p \left(\binom{m_p}{k_p} \right),$$

and if $p \equiv \pm 2 \pmod{5}$, then $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a} \right)_F$ is equal to

$$\begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r = s \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and} \\ & \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even;} \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even;} \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd;} \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd.} \end{cases}$$

Remark 3.8. In the proof of this theorem, we also show that the condition $r = s$ in Theorem 3.7(ii) is equivalent to $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod{2}] \pmod{z(p)}$. It seems more natural to write $r = s$ in the statement of the theorem, but it is more convenient in the proof to use the condition $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod{2}] \pmod{z(p)}$.

Next, we calculate the 2-adic valuation of $\binom{2^a n}{n}_F$ and then use it to determine the integers n such that $\binom{2n}{n}_F, \binom{4n}{n}_F, \binom{8n}{n}_F$ are even. Then we determine the p -adic valuation of $\binom{p^a n}{n}_F$ for all odd primes p . For binomial coefficients, we know that $\nu_2\left(\binom{2n}{n}\right) = s_2(n)$. For Fibonomial coefficients, we have the following result.

Theorem 3.9. [32, Theorem 7] *Let a and n be positive integers, $\varepsilon = [n \not\equiv 0 \pmod{3}]$, and $A = \left\lfloor \frac{(2^a - 1)n}{3 \cdot 2^{\nu_2(n)}} \right\rfloor$. Then the following statements hold.*

(i) *If a is even, then*

$$\nu_2\left(\binom{\binom{2^a n}{n}_F}{n}\right) = \delta + A - \frac{a}{2}\varepsilon - \nu_2(A!) = \delta + s_2(A) - \frac{a}{2}\varepsilon,$$

where $\delta = [n \pmod{6} = 3, 5]$. In other words, $\delta = 1$ if $n \equiv 3, 5 \pmod{6}$ and $\delta = 0$ otherwise.

(ii) *If a is odd, then*

$$\nu_2\left(\binom{\binom{2^a n}{n}_F}{n}\right) = \delta + A - \frac{a-1}{2}\varepsilon - \nu_2(A!) = \delta + s_2(A) - \frac{a-1}{2}\varepsilon,$$

where $\delta = \frac{(n \pmod{6} - 1)}{2} [2 \nmid n] + \left\lceil \frac{\nu_2(n) + 3 - n \pmod{3}}{2} \right\rceil [n \pmod{6} = 2, 4]$. In other words, $\delta = \frac{(n \pmod{6} - 1)}{2}$ if n is odd, $\delta = 0$ if $n \equiv 0 \pmod{6}$, $\delta = \left\lceil \frac{\nu_2(n)}{2} \right\rceil + 1$ if $n \equiv 4 \pmod{6}$, and $\delta = \left\lceil \frac{\nu_2(n) + 1}{2} \right\rceil$ if $n \equiv 2 \pmod{6}$.

We can obtain the main result of Maques and Trojovský [25] as a corollary.

Corollary 3.10. [32, Corollary 8] $\binom{2n}{n}_F$ is even for all $n \geq 2$.

Corollary 3.11. [32, Corollary 9] *Let $n \geq 2$. Then $\binom{4n}{n}_F$ is even if and only if n is not a power of 2. In other words, for each $n \in \mathbb{N}$, $\binom{4n}{n}_F$ is odd if and only if $n = 2^k$ for some $k \geq 0$.*

Observe that $2, 2^2, 2^3$ are congruent to $2, 4, 1 \pmod{7}$, respectively. This implies that if $k \geq 1$ and $k \equiv 1 \pmod{3}$, then $(1 + 3 \cdot 2^k)/7$ is an integer. We can determine the integers n such that $\binom{8n}{n}_F$ is odd as follows.

Corollary 3.12. [32, Corollary 10] $\binom{8n}{n}_F$ is odd if and only if $n = \frac{1+3 \cdot 2^k}{7}$ for some $k \equiv 1 \pmod{3}$.

Theorem 3.13. [32, Theorem 11] For each $a, n \in \mathbb{N}$, $\nu_5 \left(\binom{5^a n}{n}_F \right) = \nu_5 \left(\binom{5^a n}{n} \right) = \frac{s_5((5^a-1)n)}{4}$. In particular, $\binom{5^a n}{n}_F$ is divisible by 5 for every $a, n \in \mathbb{N}$.

Theorem 3.14. [32, Theorem 12] Let $p \neq 2, 5$, $a, n \in \mathbb{N}$, $r = p^a n \bmod z(p)$, $s = n \bmod z(p)$, and $A = \left\lfloor \frac{n(p^a-1)}{p^{\nu_p(n)} z(p)} \right\rfloor$. Then the following statements hold.

(i) If $p \equiv \pm 1 \pmod{5}$, then $\nu_p \left(\binom{p^a n}{n}_F \right)$ is equal to

$$\frac{A}{p-1} - a \left\{ \frac{n}{p^{\nu_p(n)} z(p)} \right\} - \nu_p(A!) = \frac{s_p(A)}{p-1} - a \left\{ \frac{n}{p^{\nu_p(n)} z(p)} \right\}.$$

(ii) If $p \equiv \pm 2 \pmod{5}$ and a is even, then $\nu_p \left(\binom{p^a n}{n}_F \right)$ is equal to

$$\frac{A}{p-1} - \frac{a}{2} [s \neq 0] - \nu_p(A!) = \frac{s_p(A)}{p-1} - \frac{a}{2} [s \neq 0].$$

(iii) If $p \equiv \pm 2 \pmod{5}$ and a is odd, then $\nu_p \left(\binom{p^a n}{n}_F \right)$ is equal to

$$\left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - \nu_p(A!) + \delta,$$

where $\delta = \left(\left\lfloor \frac{\nu_p(n)}{2} \right\rfloor + [2 \nmid \nu_p(n)] [r > s] + [r < s] \nu_p(F_{z(p)}) \right) [r \neq s]$, or equivalently, $\delta = 0$ if $r = s$, $\delta = \left\lfloor \frac{\nu_p(n)}{2} \right\rfloor + \nu_p(F_{z(p)})$ if $r < s$, and $\delta = \left\lfloor \frac{\nu_p(n)}{2} \right\rfloor$ if $r > s$.

In the next two corollaries, we give some characterizations of the integers n such that $\binom{p^a n}{n}_F$ is divisible by p .

Corollary 3.15. [32, Corollary 13] Let p be a prime and let a and n be positive integers. If $n \equiv 0 \pmod{z(p)}$, then $p \mid \binom{p^a n}{n}_F$.

Corollary 3.16. [32, Corollary 14] Let $p \neq 2, 5$ be a prime and let a, n, r, s , and A be as in Theorem 3.14. Assume that $p \equiv \pm 2 \pmod{5}$ and $n \not\equiv 0 \pmod{z(p)}$. Then the following statements hold.

(i) Assume that a is even. Then $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) > \frac{a}{2}(p-1)$.

(ii) Assume that a is odd and $p \nmid n$. If $r < s$, then $p \mid \binom{p^a n}{n}_F$. If $r \geq s$, then $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) \geq \frac{a+1}{2}(p-1)$.

(iii) Assume that a is odd and $p \mid n$. If $r \neq s$, then $p \mid \binom{p^{an}}{n}_F$. If $r = s$, then $p \mid \binom{p^{an}}{n}_F$ if and only if $s_p(A) \geq \frac{a+1}{2}(p-1)$.

We also obtain some characterization of the integers n such that $\binom{p^{an}}{n}_F$ is divisible by p where $a = 1$ and $p \equiv \pm 1 \pmod{5}$ as follows.

Corollary 3.17. [32, Corollary 15] Let $p \neq 2, 5$ be a prime and let $A = \frac{n(p-1)}{p^{\nu_p(n)}z(p)}$. Assume that $p \equiv \pm 1 \pmod{5}$. Then $p \mid \binom{pn}{n}_F$ if and only if $s_p(A) \geq p-1$.

3.3 Examples

In this last section, we give several examples to show applications of our main results. We also recall from Remark 3.8 that the condition $r = s$ in Theorem 3.7(ii) can be replaced by $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod{2}] \pmod{z(p)}$. In the calculation given in this section, we will use this observation without further reference.

Example 3.18. Let a, b , and ℓ be positive integers and $b \geq a$. We assert that for $\ell \not\equiv 0 \pmod{3}$, we have

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = \left\lfloor \frac{a+1}{2} \right\rfloor (\varepsilon_1 \varepsilon_2 + \varepsilon'_1 \varepsilon'_2), \quad (3.1)$$

where $\varepsilon_1 = [\ell \equiv 2 \pmod{3}]$, $\varepsilon_2 = [a \equiv b \pmod{2}]$, $\varepsilon'_1 = [\ell \equiv 1 \pmod{3}]$, and $\varepsilon'_2 = [a \not\equiv b \pmod{2}]$. In addition, if $\ell \equiv 0 \pmod{3}$, then

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = b + 2 + \nu_2(\ell). \quad (3.2)$$

Proof. We apply Theorem 3.7 to verify our assertion. Here $m_2 = \left\lfloor \frac{\ell \cdot 2^{b-a}}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{1}{3} \right\rfloor = 0$. So we immediately obtain the following: if $a \equiv b \pmod{2}$, then

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod{3}; \\ a + 2 + \nu_2(m_2), & \text{if } \ell \equiv 0 \pmod{3}; \\ \left\lfloor \frac{a+1}{2} \right\rfloor, & \text{if } \ell \equiv 2 \pmod{3}, \end{cases}$$

and if $a \not\equiv b \pmod{2}$, then

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod{3}; \\ a + 2 + \nu_2(m_2), & \text{if } \ell \equiv 0 \pmod{3}; \\ \lceil \frac{a+1}{2} \rceil, & \text{if } \ell \equiv 1 \pmod{3}. \end{cases}$$

This proves (3.1). If $\ell \equiv 0 \pmod{3}$, then $m_2 = \frac{\ell}{3} \cdot 2^{b-a}$ and $\nu_2(m_2)$ is equal to

$$\nu_2(m_2) = \nu_2(\ell) + \nu_2(2^{b-a}) - \nu_2(3) = b - a + \nu_2(\ell),$$

which implies (3.2). □

Example 3.19. Substituting $\ell = 1$ in Example 3.18, we see that

$$\nu_2 \left(\binom{2^b}{2^a}_F \right) = \begin{cases} \lceil \frac{a+1}{2} \rceil & [a \not\equiv b \pmod{2}] \\ \begin{cases} 0, & \text{if } a \equiv b \pmod{2}; \\ \lceil \frac{a+1}{2} \rceil, & \text{if } a \not\equiv b \pmod{2}. \end{cases} \end{cases} \quad (3.3)$$

Our example also implies that (3.3) still holds for the 2-adic valuations of $\binom{2^{b+2c}}{2^a}_F$, $\binom{7 \cdot 2^b}{2^a}_F$, $\binom{5 \cdot 2^{b+1}}{2^a}_F$, $\binom{13 \cdot 2^b}{2^a}_F$, etc.

Example 3.20. Let a , b , and ℓ be positive integers, $b \geq a$, and p a prime distinct from 2 and 5. If $p \equiv \pm 1 \pmod{5}$, then

$$\nu_p \left(\binom{\ell p^b}{p^a}_F \right) = (b + \nu_p(F_{z(p)}) + \nu_p(\ell)) [\ell \equiv 0 \pmod{z(p)}],$$

and if $p \equiv \pm 2 \pmod{5}$, then

$$\nu_p \left(\binom{\ell p^b}{p^a}_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 - 2\varepsilon \pmod{z(p)}; \\ b + \nu_p(F_{z(p)}) + \nu_p(\ell), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is odd,} \end{cases}$$

where $\varepsilon = [a \not\equiv b \pmod{2}]$.

Proof. Similar to Example 3.18, we verify this by applying Theorem 3.7. Here $m_p = \left\lfloor \frac{\ell p^{b-a}}{z(p)} \right\rfloor$, $k_p = \left\lfloor \frac{1}{z(p)} \right\rfloor = 0$, $r = \ell p^b \pmod{z(p)}$, and $s = p^a \pmod{z(p)}$. We first assume that $p \equiv \pm 1 \pmod{5}$. Then by Lemma 2.1, we have $p \equiv 1 \pmod{z(p)}$. Therefore $s = 1$, $r \equiv \ell \pmod{z(p)}$, and

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = (a + \nu_p(m_p) + \nu_p(F_{z(p)})) [\ell \equiv 0 \pmod{z(p)}].$$

Similarly, if $p \equiv \pm 2 \pmod{5}$ and $a \equiv b \pmod{2}$, then we obtain by Lemma 2.1 and Theorem 3.7 that

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod{z(p)}; \\ a + \nu_p(m_p) + \nu_p(F_{z(p)}), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv \pmod{0, 1z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is odd.} \end{cases}$$

In addition, if $p \equiv \pm 2 \pmod{5}$ and $a \not\equiv b \pmod{2}$, then

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = \begin{cases} 0, & \text{if } \ell \equiv -1 \pmod{z(p)}; \\ a + \nu_p(m_p) + \nu_p(F_{z(p)}), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is odd.} \end{cases}$$

It remains to calculate $\nu_p(m_p)$ when $\ell \equiv 0 \pmod{z(p)}$. In this case, we have

$$\nu_p(m_p) = \nu_p \left(\frac{\ell p^{b-a}}{z(p)} \right) = \nu_p(\ell) + \nu_p(p^{b-a}) - \nu_p(z(p)) = b - a + \nu_p(\ell).$$

This implies the desired result. \square

Example 3.21. Substituting $\ell = 1$ in Example 3.20, we see that for $p \neq 2, 5$, we have

$$\nu_p \left(\left(\frac{p^b}{p^a} \right)_F \right) = \begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod{5} \text{ or } a \equiv b \pmod{2}; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod{5}, a \not\equiv b \pmod{2}, \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod{5}, a \not\equiv b \pmod{2}, \text{ and } a \text{ is odd.} \end{cases} \quad (3.4)$$

Our example also implies that (3.4) still holds for the p -adic valuations of $\binom{p^{b+2c}}{p^a}_F$ and $\binom{(z(p)+1)p^b}{p^a}_F$. Similarly, for $p \neq 2, 5$, we have

$$\nu_p \left(\binom{2p^b}{p^a}_F \right) = \begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod{5}; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod{5} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod{5} \text{ and } a \text{ is odd.} \end{cases} \quad (3.5)$$

In addition, (3.5) also holds when $\binom{2p^b}{p^a}_F$ is replaced by $\binom{\ell p^b}{p^a}_F$ for $\ell \not\equiv 0, \pm 1 \pmod{z(p)}$ and $p \neq 2, 5$. Furthermore, replacing $\binom{2p^b}{p^a}_F$ by $\binom{(z(p)-1)p^b}{p^a}_F$, the formula becomes

$$\begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod{5} \text{ or } a \not\equiv b \pmod{2}; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod{5}, a \equiv b \pmod{2}, \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod{5}, a \equiv b \pmod{2}, \text{ and } a \text{ is odd.} \end{cases}$$

Example 3.22. We know that the 5-adic valuations of Fibonomial coefficients are the same as those of binomial coefficients. For example, by Theorem 3.5(ii) and Kummer's theorem, we obtain

$$\nu_5 \left(\binom{\ell \cdot 5^b}{5^a}_F \right) = \nu_5 \left(\binom{\ell \cdot 5^b}{5^a} \right) = b - a + \nu_5(\ell),$$

for every $a, b, \ell \in \mathbb{N}$ with $b \geq a$. Similarly, $\nu_5 \left(\binom{5^b}{\ell \cdot 5^a}_F \right) = b - a - \nu_5(\ell)$ for every $a, b, \ell \in \mathbb{N}$ such that $5^b > \ell \cdot 5^a$.

Example 3.23. Let a, b , and ℓ be positive integers and $2^b > \ell \cdot 2^a$. Let $m_2 = \left\lfloor \frac{2^{b-a}}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{\ell}{3} \right\rfloor$. Then

$$\nu_2 \left(\binom{2^b}{\ell \cdot 2^a}_F \right) = \nu_2 \left(\binom{m_2}{k_2} \right) + \left(\left\lfloor \frac{a+2}{2} \right\rfloor + \nu_2(m_2 - k_2) \right) \varepsilon_1 \varepsilon_2 + \left\lfloor \frac{a+1}{2} \right\rfloor \varepsilon_3 \varepsilon_4, \quad (3.6)$$

where $\varepsilon_1 = [a \equiv b \pmod{2}]$, $\varepsilon_2 = [\ell \equiv 2 \pmod{3}]$, $\varepsilon_3 = [a \not\equiv b \pmod{2}]$, and $\varepsilon_4 = [\ell \equiv 1 \pmod{3}]$.

Proof. Similar to Example 3.18, this follows from the application of Theorem 3.7. So we leave the details to the reader. □

Example 3.24. Let $k \geq 2$. We observe that

$$\left\lfloor \frac{2^k}{3} \right\rfloor = \begin{cases} \frac{2^k-1}{3}, & \text{if } k \text{ is even;} \\ \frac{2(2^{k-1}-1)}{3}, & \text{if } k \text{ is odd,} \end{cases}$$

which implies,

$$\nu_2 \left(\left\lfloor \frac{2^k}{3} \right\rfloor \right) = [k \equiv 1 \pmod{2}]. \quad (3.7)$$

By a similar reason, we also see that for $k \geq 3$,

$$\nu_2 \left(\left\lfloor \frac{2^k}{3} \right\rfloor - 1 \right) = 2[k \equiv 0 \pmod{2}]. \quad (3.8)$$

From (3.6), (3.7), and (3.8), we obtain the following results:

(i) if $b - a \geq 2$, then $\nu_2 \left(\binom{2^b}{3 \cdot 2^a}_F \right) = [a \not\equiv b \pmod{2}]$,

(ii) if $b - a \geq 3$, then $\nu_2 \left(\binom{2^b}{5 \cdot 2^a}_F \right)$ is equal to

$$\begin{aligned} & [a \not\equiv b \pmod{2}] + \left(\left\lfloor \frac{a+2}{2} \right\rfloor + 2[a \equiv b \pmod{2}] \right) [a \equiv b \pmod{2}] \\ & = 1 + \left\lfloor \frac{a+4}{2} \right\rfloor [a \equiv b \pmod{2}], \end{aligned}$$

(iii) if $b - a \geq 3$, then $\nu_2 \left(\binom{2^b}{6 \cdot 2^a}_F \right) = [a \equiv b \pmod{2}]$,

(iv) if $b - a \geq 4$, then $\nu_2 \left(\binom{2^b}{7 \cdot 2^a}_F \right) = [a \equiv b \pmod{2}] + \left\lfloor \frac{a+1}{2} \right\rfloor [a \not\equiv b \pmod{2}]$.

Example 3.25. Let $p \neq 5$ be an odd prime and let a, b , and ℓ be positive integers, $p^b > \ell p^a$, $m_p = \left\lfloor \frac{p^b - a}{z(p)} \right\rfloor$, and $k_p = \left\lfloor \frac{\ell}{z(p)} \right\rfloor$. Then the following statements hold.

(i) If $p \equiv \pm 1 \pmod{5}$, then

$$\nu_p \left(\binom{p^b}{\ell p^a}_F \right) = (a + \nu_p(m_p - k_p) + \nu_p(F_{z(p)})) [\ell \not\equiv 0, 1 \pmod{z(p)}] + \nu_p \left(\binom{m_p}{k_p} \right),$$

(ii) If $p \equiv \pm 2 \pmod{5}$, then $\nu_p \left(\binom{p^b}{\ell p^a}_F \right)$ is equal to

$$\begin{aligned} & \nu_p \left(\binom{m_p}{k_p} \right) + \varepsilon_1 \varepsilon_2 \varepsilon_5 \left(\left\lfloor \frac{a}{2} \right\rfloor + \nu_p(m_p - k_p) + \varepsilon_3 \nu_p(F_{z(p)}) \right) \\ & + \varepsilon_1 \varepsilon_4 (1 - \varepsilon_5) \left(\left\lfloor \frac{a}{2} \right\rfloor + \varepsilon_3 \nu_p(F_{z(p)}) \right) \end{aligned} \quad (3.9)$$

where $\varepsilon_1 = [\ell \not\equiv 0 \pmod{z(p)}]$, $\varepsilon_2 = [\ell \not\equiv 1 \pmod{z(p)}]$, $\varepsilon_3 = [b \equiv 0 \pmod{2}]$, $\varepsilon_4 = [\ell \not\equiv -1 \pmod{z(p)}]$, and $\varepsilon_5 = [a \equiv b \pmod{2}]$.

Proof. Similar to Example 3.20, this follows from the application of Lemma 2.1 and Theorem 3.7. Since (i) is easily verified, we only give the proof of (ii). The calculation is done in two cases. If $p \equiv \pm 2 \pmod{5}$ and $a \equiv b \pmod{2}$, then $\nu_p \left(\binom{p^b}{\ell p^a}_F \right)$ is equal to

$$\begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \equiv 0, 1 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is even;} \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is odd,} \end{cases}$$

$$= \nu_p \left(\binom{m_p}{k_p} \right) + \varepsilon_1 \varepsilon_2 \left(\left\lfloor \frac{a}{2} \right\rfloor + \nu_p(m_p - k_p) + \varepsilon_3 \nu_p(F_{z(p)}) \right),$$

where $\varepsilon_1 = [\ell \not\equiv 0 \pmod{z(p)}]$, $\varepsilon_2 = [\ell \not\equiv 1 \pmod{z(p)}]$, and $\varepsilon_3 = [b \equiv 0 \pmod{2}]$.

If $p \equiv \pm 2 \pmod{5}$ and $a \not\equiv b \pmod{2}$, then

$$\nu_p \left(\binom{p^b}{\ell p^a}_F \right) = \begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \equiv 0, -1 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is even;} \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is odd,} \end{cases}$$

$$= \nu_p \left(\binom{m_p}{k_p} \right) + \varepsilon_1 \varepsilon_4 \left(\left\lfloor \frac{a}{2} \right\rfloor + \varepsilon_3 \nu_p(F_{z(p)}) \right),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are as above and $\varepsilon_4 = [\ell \not\equiv -1 \pmod{z(p)}]$. Let $\varepsilon_5 = [a \equiv b \pmod{2}]$. Then both cases can be combined to obtain (ii). \square

Example 3.26. Let $k \geq 2$. We observe that $z(7) = 8$ and

$$\left\lfloor \frac{7^k}{8} \right\rfloor = \begin{cases} \frac{7^k - 1}{8}, & \text{if } k \text{ is even;} \\ \frac{7(7^{k-1} - 1)}{8}, & \text{if } k \text{ is odd.} \end{cases}$$

Therefore

$$\nu_7 \left(\left\lfloor \frac{7^k}{8} \right\rfloor \right) = [k \equiv 1 \pmod{2}] \quad \text{and} \quad \nu_7 \left(\left\lfloor \frac{7^k}{8} \right\rfloor - 1 \right) = 0. \quad (3.10)$$

From (3.9) and (3.10), we obtain the following results:

- (i) if $b - a \geq 2$, then $\nu_7 \left(\binom{7^b}{8 \cdot 7^a}_F \right) = [a \not\equiv b \pmod{2}]$,
- (ii) if $b - a \geq 2$, then $\nu_7 \left(\binom{7^b}{9 \cdot 7^a}_F \right) = \left(\left\lfloor \frac{a+2}{2} \right\rfloor + [b \equiv 0 \pmod{2}] \right) [a \not\equiv b \pmod{2}]$,

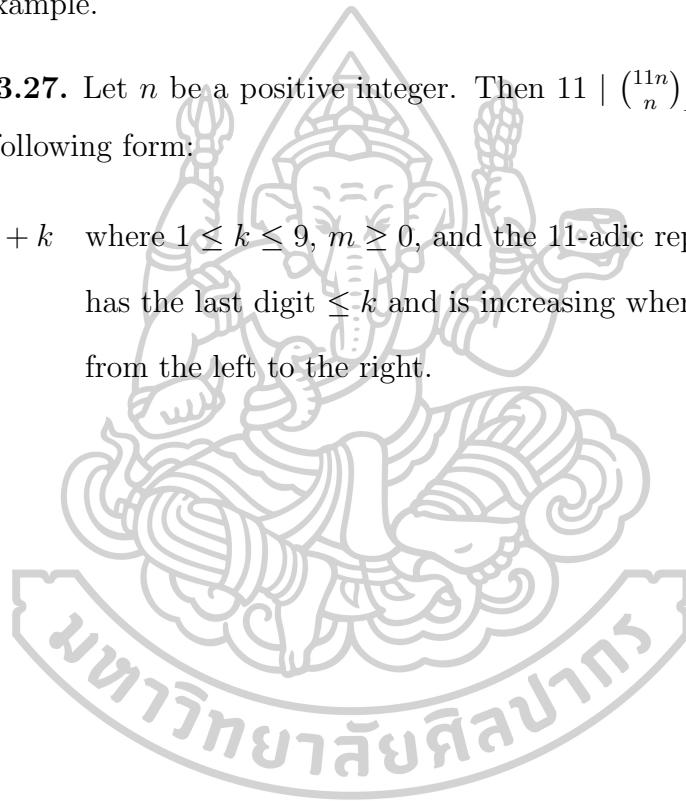
(iii) if $b - a \geq 2$, then $\nu_7 \left(\binom{7^b}{15 \cdot 7^a}_F \right)$ is equal to

$$[a \not\equiv b \pmod{2}] + \left(\left\lceil \frac{a}{2} \right\rceil + [b \equiv 0 \pmod{2}] \right) [a \equiv b \pmod{2}].$$

In Corollaries 3.16 and 3.17, we have the characterization of the integers n such that $\binom{p^n}{n}_F$ is divisible by p for $p \neq 5$. However, these do not give the such characterization in the digital representation. We will show our solution of this problem in the near future but, for now, we give our result when $p = 11$ in the following example.

Example 3.27. Let n be a positive integer. Then $11 \mid \binom{11^n}{n}_F$ if and only if n is not of the following form:

$n = 10m + k$ where $1 \leq k \leq 9$, $m \geq 0$, and the 11-adic representation of m has the last digit $\leq k$ and is increasing when we read it from the left to the right.



Chapter 4

Reciprocal Sum of Palindromes

Throughout this section, x, y, z are positive real numbers, a, b, m, n, k, ℓ are positive integers, $b \geq 2$,

$$x_b = \sum_{m=1}^{b-1} \frac{1}{m}, \quad y_b = \sum_{m=b}^{b^2-1} \frac{1}{m}, \quad \text{and} \quad z_b = \sum_{m=b+1}^{b^2} \frac{1}{m}.$$

Note that $x_b = s_{b,1}$, $x_b/(b+1) = s_{b,2}$, and that

$$z_b - y_b = \frac{1}{b^2} - \frac{1}{b} = \frac{1-b}{b^2}.$$

Theorem 4.1. [31, Theorem 1] *We have*

$$\frac{y_b}{b} - \frac{x_b}{b^3} \leq s_{b,3} \leq \frac{y_b}{b} \quad \text{and} \quad \frac{z_b}{b^{\lfloor \frac{k}{2} \rfloor}} \leq s_{b,k} \leq \frac{y_b}{b^{\lfloor \frac{k}{2} \rfloor}} \quad \text{for every } k \geq 4.$$

Theorem 4.2. [31, Theorem 2] *For every $b, \ell \geq 2$, we have*

$$\left(\frac{b+2}{b+1}\right) x_b + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2z_b}{(b-1)b^{\ell-1}} \leq s_b \leq \left(\frac{b+2}{b+1}\right) x_b + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2y_b}{(b-1)b^{\ell-1}}. \quad (4.1)$$

In particular,

$$\left(\frac{b+2}{b+1}\right) x_b + \frac{y_b}{b} - \frac{x_b}{b^3} + \frac{2z_b}{b(b-1)} \leq s_b \leq \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b(b-1)}\right) y_b.$$

Let $b \geq 2$ and let $U_b = U_b(\ell)$ and $L_b = L_b(\ell)$ be the upper and lower bounds of s_b given in (4.1). We see that $y_b - z_b = \frac{b-1}{b^2}$ and

$$0 \leq U_b - s_b \leq U_b - L_b = \frac{2(y_b - z_b)}{(b-1)b^{\ell-1}} = \frac{2}{b^{\ell+1}},$$

which converges to 0. We apply (4.1) and run the computation in a computer to approximate s_b by U_b and L_b with errors less than 10^{-8} as the following table below. In fact, if ℓ is large enough, we can find the first n digits of s_b where n is arbitrary.

Table 4.1: Upper and lower bounds for s_b

b	Lower bound for s_b	Upper bound for s_b
2	2.378795704268652	2.378795711719233
3	2.616761112331746	2.616761117494096
4	2.785771526362811	2.785771533813391
5	2.920048244568022	2.920048252760022
6	3.033033181096604	3.033033186609329
7	3.131376859365746	3.131376866446013
8	3.218887858788806	3.218887860651451
9	3.297976950072639	3.297976955234989
10	3.370283258515688	3.370283260515688
11	3.436981687017363	3.436981696347511
12	3.498948958553883	3.498948963205244
13	3.556860134337803	3.556860136789592
14	3.611248238723658	3.611248240078865
15	3.662542857892273	3.662542858672642
16	3.711096160282906	3.711096167733487
17	3.757201045822589	3.757201050696611
18	3.801104117340674	3.801104120607473
19	3.843015238782920	3.843015241020377
20	3.883114678846523	3.883114680409023

According to the table, we see that the upper bound for s_b is less than the lower bound for s_{b+1} , that is, $U_b < L_{b+1}$ for $b = 2, 3, \dots, 19$ and large enough ℓ . So we have $s_2 < s_3 < s_4 < \dots < s_{20}$. This observation leads us to obtain the next theorem.

Theorem 4.3. [31, Theorem 3] *The sequence $(s_b)_{b \geq 2}$ is strictly increasing.*

Recall that if we write $f(b) = g(b) + O^*(h(b))$, then it means that $f(b) = g(b) + O(h(b))$ and the implied constant can be taken to be 1. In addition, $f(b) =$

$g(b) + \Omega_+(h(b))$ means $\limsup_{b \rightarrow \infty} \frac{f(b)-g(b)}{h(b)} > 0$.

Theorem 4.4. [31, Theorem 4] *Uniformly for $b \geq 2$,*

$$s_b = \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b^2}\right) y_b + O^* \left(\frac{5 \log b}{b^3}\right). \quad (4.2)$$

This estimate is sharp in the sense that $O^ \left(\frac{5 \log b}{b^3}\right)$ can be replaced by $\Omega_+ \left(\frac{\log b}{b^3}\right)$.*

Let $A_b = \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b^2}\right) y_b$ be the main term given in (4.2). We can use Theorem 4.4 to evaluate s_b with errors not exceeding $\frac{\log b}{b^3}$ as the following table below compared with the upper and lower bounds for s_b .

Table 4.2: Estimate for s_b

b	Lower bounds for s_b	Upper bounds for s_b	Approximations A_b for s_b
2	2.378795704268652	2.378795711719233	2.166666666666667
3	2.616761112331746	2.616761117494096	2.551587301587301
4	2.785771526362811	2.785771533813391	2.756835872460873
5	2.920048244568022	2.920048252760022	2.904490511993204
6	3.033033181096604	3.033033186609329	3.023623384120149
7	3.131376859365746	3.131376866446013	3.125212746440348
8	3.218887858788806	3.218887860651451	3.214609999834978
9	3.297976950072639	3.297976955234989	3.294875492907945
10	3.370283258515688	3.370283260515688	3.367956297787750
11	3.436981687017363	3.436981696347511	3.435186961914700
12	3.498948958553883	3.498948963205244	3.497532930652704
13	3.556860134337803	3.556860136789592	3.555721442141313
14	3.611248238723658	3.611248240078865	3.610317641886406
15	3.662542857892273	3.662542858672642	3.661771673430161
16	3.711096160282906	3.711096167733487	3.710449304915721
17	3.757201045822589	3.757201050696611	3.756652678610918
18	3.801104117340674	3.801104120607473	3.800634853047960
19	3.843015238782920	3.843015241020377	3.842610288601568
20	3.883114678846523	3.883114680409023	3.882762592688996

Recall that by applying Euler–Maclaurin summation formula, we get

$$\sum_{m \leq n} \frac{1}{m} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\theta_n}{60n^4}, \quad (4.3)$$

where γ is Euler’s constant and $\theta_n \in [0, 1]$. The calculation of (4.3) can be found in Tenenbaum [44, p. 6]. From this, we obtain another form of Theorem 4.4 as follows.

Theorem 4.5. [31, Theorem 5] *Uniformly for $b \geq 2$,*

$$s_b = \log b + \gamma + \left(\frac{1}{b} + \frac{1}{b+1} \right) \log b + \frac{\gamma}{b+1} - \frac{1}{2b} + \frac{2 \log b}{b^2} - \frac{1}{12b(b+1)} + O^* \left(\frac{6 \log b}{b^3} \right). \quad (4.4)$$

This estimate is sharp in the sense that $O^ \left(\frac{6 \log b}{b^3} \right)$ is also $\Omega_+ \left(\frac{\log b}{b^3} \right)$.*

Corollary 4.6. [31, Corollary 6] *The sequence $(s_b)_{b \geq 2}$ diverges to $+\infty$ and the sequence $(s_b - s_{b-1})_{b \geq 3}$ converges to zero as $b \rightarrow \infty$.*

Recall that a sequence $(a_n)_{n \geq 0}$ is said to be log–concave if $a_n^2 - a_{n-1}a_{n+1} > 0$ for every $n \geq 1$ and is said to be log–convex if $a_n^2 - a_{n-1}a_{n+1} < 0$ for every $n \geq 1$. For a survey article concerning the log–concavity and log–convexity of sequences, we refer the reader to Stanley [43]. See also Pongsriiam [34] for some combinatorial sequences which are log–concave or log–convex, and some open problems concerning the log–properties of a certain sequence.

Theorem 4.7. [31, Theorem 7] *The sequence $(s_b)_{b \geq 2}$ is log–concave.*

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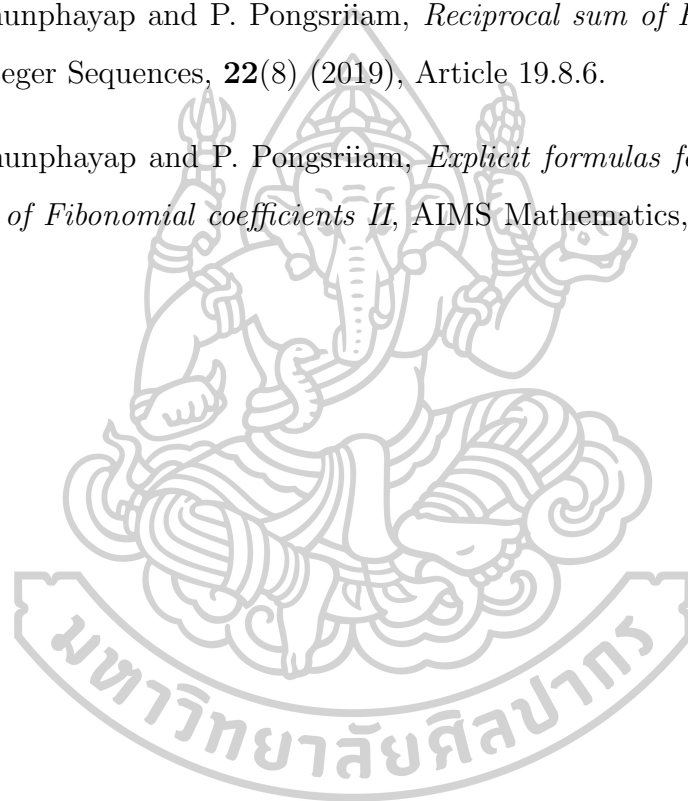
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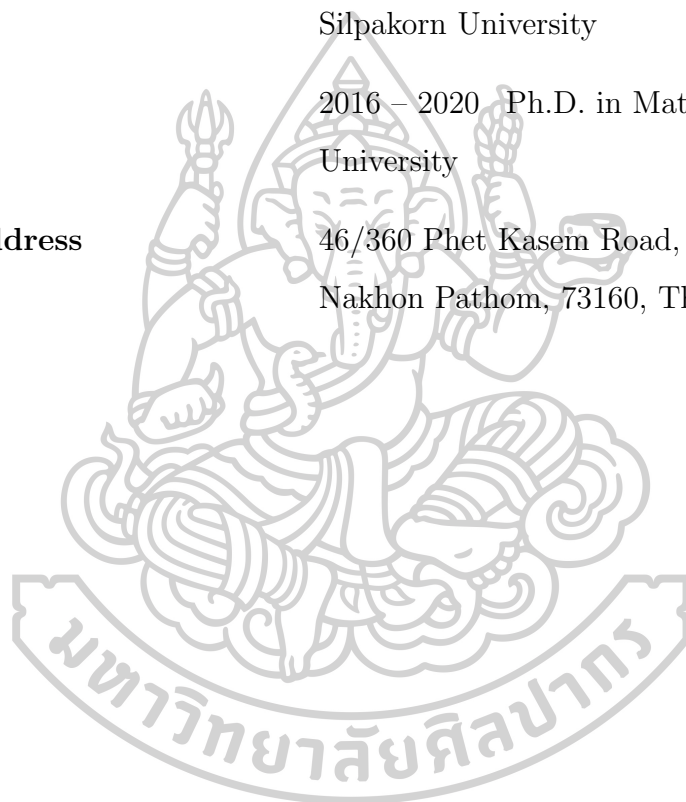
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1. P. Phunphayap and P. Pongsriiam, *Explicit formulas for the p -adic valuations of Fibonomial coefficients*, Journal of Integer Sequences, **21**(3) (2018), Article 18.3.1.
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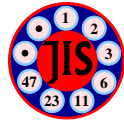


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Explicit Formulas for the p -adic Valuations of Fibonomial Coefficients

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Abstract

We obtain explicit formulas for the p -adic valuations of Fibonomial coefficients which extend some results in the literature.

1 Introduction

The *Fibonacci sequence* $(F_n)_{n \geq 1}$ is given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with the initial values $F_1 = F_2 = 1$. For each $m \geq 1$ and $1 \leq k \leq m$, the *Fibonomial coefficients* $\binom{m}{k}_F$ are defined by

$$\binom{m}{k}_F = \frac{F_1 F_2 F_3 \cdots F_m}{(F_1 F_2 F_3 \cdots F_k)(F_1 F_2 F_3 \cdots F_{m-k})} = \frac{F_{m-k+1} F_{m-k+2} \cdots F_m}{F_1 F_2 F_3 \cdots F_k},$$

where F_n is the n th Fibonacci number. If $k = 0$, we define $\binom{m}{k}_F = 1$ and if $k > m$, we define $\binom{m}{k}_F = 0$. It is well known that $\binom{m}{k}_F$ is an integer for all positive integers m and k . So it

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is natural to consider the divisibility properties and the p -adic valuation of $\binom{m}{k}_F$. As usual, p always denotes a prime and the p -adic valuation (or p -adic order) of a positive integer n , denoted by $\nu_p(n)$, is the exponent of p in the prime factorization of n . In addition, the order (or the rank) of appearance of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer k such that $n \mid F_k$. The Fibonacci sequence and the triangle of Fibonomial coefficients are, respectively, [A000045](#) and [A010048](#) in OEIS [25]. Also see [A055870](#) and [A003267](#) for signed Fibonomial triangle and central Fibonomial coefficients, respectively.

In 1989, Knuth and Wilf [8] gave a short description of the p -adic valuation of $\binom{m}{k}_C$ where C is a regularly divisible sequence. However, this does not give explicit formulas for $\binom{m}{k}_F$. Then recently, there has been some interest in explicitly evaluating the p -adic valuation of Fibonomial coefficients of the form $\binom{p^b}{p^a}_F$. For example, Marques and Trojovský [10, 11] and Marques, Sellers, and Trojovský [12] deal with the case $b = a + 1$, $a \geq 1$. Ballot [2, Theorem 4.2] extends the Kummer-like theorem of Knuth and Wilf [8, Theorem 2], which gives the p -adic valuation of Fibonomials, to all Lucasnomials, and, in particular, uses it to determine explicitly the p -adic valuation of Lucasnomials of the form $\binom{p^b}{p^a}_U$, for all nondegenerate fundamental Lucas sequences U and all integers $b > a \geq 0$, [2, Theorem 7.1].

Note that in the formula given by Marques and Trojovský [11, Theorem 1] for $U = F$ and $b = a + 1$, only the case of a even is actually explicitly computed. It appears, using the theorem of Ballot [1, Theorem 7.1], that their stated result for a odd is correct only for primes p for which p^2 does not divide $F_{z(p)}$, where $z(p)$ is the rank of appearance of p in the Fibonacci sequence. Also see Examples 16 and 18 in this article.

Our purpose is to extend Ballot's theorem, Theorem 7.1, in the case $U = F$ and $b \geq a > 0$ and obtain explicit formulas for $\binom{\ell_1 p^b}{\ell_2 p^a}_F$, where ℓ_1 and ℓ_2 are arbitrary positive integers such that $\ell_1 p^b > \ell_2 p^a$. This leads us to study the p -adic valuations of integers of the forms

$$\left\lfloor \frac{\ell p^a}{m} \right\rfloor! \quad \text{or} \quad \left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor!, \quad (1)$$

where $p \equiv \pm 1 \pmod{m}$. For instance, we obtain in Example 17 the following result: for positive integers a, b, ℓ with $b \geq a$, and a prime p distinct from 2 and 5, if $p \equiv \pm 1 \pmod{5}$, then

$$\nu_p \left(\binom{\ell p^b}{p^a}_F \right) = \begin{cases} b + \nu_p(F_{z(p)}) + \nu_p(\ell), & \text{if } z(p) \mid \ell; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $p \equiv \pm 2 \pmod{5}$, then

$$\nu_p \left(\binom{\ell p^b}{p^a}_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 - 2\varepsilon \pmod{z(p)}; \\ b + \nu_p(F_{z(p)}) + \nu_p(\ell), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is odd,} \end{cases}$$

where $\varepsilon = 1$ if a and b have different parity and $\varepsilon = 0$ otherwise. We also obtain the corresponding results for $p \in \{2, 5\}$ in Examples 15 and 19. These extend all the main results in [10, 11, 12] and Ballot's theorem, Theorem 7.1, in the case $U = F$.

Recall that for each $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x , $\{x\}$ is the *fractional part* of x given by $\{x\} = x - \lfloor x \rfloor$, and $\lceil x \rceil$ is the smallest integer larger than or equal to x . In addition, we write $a \bmod m$ to denote the least nonnegative residue of a modulo m . We also use the *Iverson notation*: if P is a mathematical statement, then

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

For example, $[5 \equiv -1 \pmod{4}] = 0$ and $[3 \equiv -1 \pmod{4}] = 1$.

We organize this article as follows. In Section 2, we give some preliminaries and useful results which are needed in the proof of the main theorems. In Section 3, we give exact formulas for the p -adic valuations of integers (1). In Section 4, we apply the results obtained in Section 3 to Fibonomial coefficients. Our most general theorem is Theorem 13. Finally, in Section 5, we give the p -adic valuations of some specific sub-families of Fibonomial coefficients of type (1), since generally, the more specific the family, the shortest the formula becomes.

For more information related to Fibonacci numbers, we invite the readers to visit the second author's Researchgate account [23] which contains some freely downloadable versions of his publications [5, 6, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

2 Preliminaries and lemmas

Recall that for each odd prime p and $a \in \mathbb{Z}$, the *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \mid a; \\ 1, & \text{if } a \text{ is a quadratic residue of } p; \\ -1, & \text{if } a \text{ is a quadratic nonresidue of } p. \end{cases}$$

Then we have the following result.

Lemma 1. *Let $p \neq 5$ be a prime and let m and n be positive integers. Then the following statements hold.*

- (i) *If $p > 2$, then $F_{p-\left(\frac{5}{p}\right)} \equiv 0 \pmod{p}$.*
- (ii) *$n \mid F_m$ if and only if $z(n) \mid m$.*
- (iii) *$z(p) \mid p + 1$ if and only if $p \equiv 2$ or $-2 \pmod{5}$, and $z(p) \mid p - 1$ otherwise.*
- (iv) *$\gcd(z(p), p) = 1$.*

Proof. These are well known results. For example, (i) and (ii) can be found in [4, p. 410] and [26], respectively. Then (iii) follows from (i) and (ii). By (iii), $z(p) \mid p \pm 1$. Since $\gcd(p, p \pm 1) = 1$, we obtain $\gcd(z(p), p) = 1$. This proves (iv). \square

Lengyel's result and Legendre's formula given in the following lemmas are important tools in evaluating the p -adic valuation of Fibonomial coefficients. We also refer the reader to [10, 11, 12, 15] for other similar applications of Lengyel's result.

Lemma 2. (Lengyel [9]) *For $n \geq 1$, we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{6}, \end{cases}$$

$\nu_5(F_n) = \nu_5(n)$, and if p is a prime distinct from 2 and 5, then

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}, \end{cases}$$

Lemma 3. (Legendre's formula) *Let n be a positive integer and let p be a prime. Then*

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

In the proof of the main results, we will deal with a lot of calculation involving the floor function. So it is useful to recall the following results.

Lemma 4. *For $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, the following holds*

- (i) $\lfloor n + x \rfloor = n + \lfloor x \rfloor$,
- (ii) $\{n + x\} = \{x\}$,
- (iii) $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$
- (iv) $\{-x\} = \begin{cases} 1 - \{x\}, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$
- (v) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\} + \{y\} \geq 1, \end{cases}$
- (vi) $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$ for $n \geq 1$.

Proof. These are well-known results and can be proved easily. For more details, see in [1, Exercise 13, p. 72] or in [3, Chapter 3]. We also refer the reader to [14] for a nice application of (v). \square

The next lemma is used often in counting the number of positive integers $n \leq x$ lying in a residue class $a \pmod q$, see for instance in [24, Proof of Lemma 2.6].

Lemma 5. For $x \in [1, \infty)$, $a, q \in \mathbb{Z}$ and $q \geq 1$, we have

$$\sum_{\substack{1 \leq n \leq x \\ n \equiv a \pmod q}} 1 = \left\lfloor \frac{x-a}{q} \right\rfloor - \left\lfloor -\frac{a}{q} \right\rfloor. \quad (2)$$

Proof. Replacing a by $a+q$ and applying Lemma 4, we see that the value on the right-hand side of (2) is not changed. Obviously, the left-hand side is also invariant when we replace a by $a+q$. So it is enough to consider only the case $1 \leq a \leq q$. Since $n \equiv a \pmod q$, we write $n = a + kq$ where $k \geq 0$ and $a + kq \leq x$. So $k \leq \frac{x-a}{q}$. Therefore

$$\sum_{\substack{1 \leq n \leq x \\ n \equiv a \pmod q}} 1 = \sum_{0 \leq k \leq \frac{x-a}{q}} 1 = \left\lfloor \frac{x-a}{q} \right\rfloor + 1 = \left\lfloor \frac{x-a}{q} \right\rfloor - \left\lfloor -\frac{a}{q} \right\rfloor.$$

\square

It is convenient to use the Iverson notation and to denote the least nonnegative residue of a modulo m by $a \pmod m$. Therefore we will do so from this point on.

Lemma 6. Let n and k be integers, m a positive integer, $r = n \pmod m$, and $s = k \pmod m$. Then

$$\left\lfloor \frac{n-k}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor - [r < s].$$

Proof. By Lemma 4(i) and the fact that $0 \leq r < m$, we obtain

$$\left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{n-r}{m} + \frac{r}{m} \right\rfloor = \frac{n-r}{m} + \left\lfloor \frac{r}{m} \right\rfloor = \frac{n-r}{m}.$$

Similarly, $\left\lfloor \frac{k}{m} \right\rfloor = \frac{k-s}{m}$. Therefore $\left\lfloor \frac{n-k}{m} \right\rfloor$ is equal to

$$\left\lfloor \frac{n-r}{m} - \frac{k-s}{m} + \frac{r-s}{m} \right\rfloor = \frac{n-r}{m} - \frac{k-s}{m} + \left\lfloor \frac{r-s}{m} \right\rfloor = \begin{cases} \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, & \text{if } r \geq s; \\ \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor - 1, & \text{if } r < s. \end{cases}$$

\square

3 The p -adic valuation of integers in special forms

In this section, we calculate the p -adic valuation of $\lfloor \frac{\ell p^a}{m} \rfloor!$ and other integers in similar forms.

Theorem 7. *Let p be a prime and let $a \geq 0$, $\ell \geq 0$, and $m \geq 1$ be integers. Assume that $p \equiv \pm 1 \pmod{m}$ and let $\delta = [\ell \not\equiv 0 \pmod{m}]$. Then*

$$\nu_p \left(\left\lfloor \frac{\ell p^a}{m} \right\rfloor! \right) = \begin{cases} \frac{\ell(p^a-1)}{m(p-1)} - a \left\{ \frac{\ell}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv 1 \pmod{m}; \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a}{2}\delta + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is even}; \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a-1}{2}\delta - \left\{ \frac{\ell}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is odd}. \end{cases}$$

We remark that if $m = 1$ or 2 , then the expressions in each case of this theorem are all equal.

Proof. The result is easily verified when $a = 0$ or $\ell = 0$. So we assume throughout that $a \geq 1$ and $\ell \geq 1$. We also use Lemmas 4(i), 4(vi), and 5 repeatedly without reference. By Legendre's formula, we obtain

$$\nu_p \left(\left\lfloor \frac{\ell p^a}{m} \right\rfloor! \right) = \sum_{j=1}^{\infty} \left\lfloor \frac{\ell p^a}{m p^j} \right\rfloor = \sum_{j=1}^a \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor + \sum_{j=a+1}^{\infty} \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor = \sum_{j=1}^a \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right). \quad (3)$$

From (3), it is immediate that for $m = 1$, we obtain

$$\nu_p((\ell p^a)!) = \frac{\ell(p^a-1)}{p-1} + \nu_p(\ell!).$$

So we assume throughout that $m \geq 2$.

Case 1. $p \equiv 1 \pmod{m}$. Then, for every $k \geq 0$, $p^k \equiv 1 \pmod{m}$ and

$$\left\lfloor \frac{\ell p^k}{m} \right\rfloor = \left\lfloor \frac{\ell(p^k-1)}{m} + \frac{\ell}{m} \right\rfloor = \frac{\ell(p^k-1)}{m} + \left\lfloor \frac{\ell}{m} \right\rfloor.$$

Therefore the sum $\sum_{j=1}^a \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor$ appearing in (3) is equal to

$$\sum_{j=1}^a \left(\frac{\ell(p^{a-j}-1)}{m} + \left\lfloor \frac{\ell}{m} \right\rfloor \right) = \left(\frac{\ell}{m} \sum_{j=1}^a p^{a-j} \right) - a \left(\frac{\ell}{m} - \left\lfloor \frac{\ell}{m} \right\rfloor \right) = \frac{\ell(p^a-1)}{m(p-1)} - a \left\{ \frac{\ell}{m} \right\}.$$

Case 2. $p \equiv -1 \pmod{m}$. Then for $k \geq 0$, we have $p^k \equiv 1 \pmod{m}$ if k is even, and $p^k \equiv -1 \pmod{m}$ if k is odd. Therefore

$$\begin{aligned} \left\lfloor \frac{\ell p^k}{m} \right\rfloor &= \left\lfloor \frac{\ell(p^k-1)}{m} + \frac{\ell}{m} \right\rfloor = \frac{\ell(p^k-1)}{m} + \left\lfloor \frac{\ell}{m} \right\rfloor && \text{if } k \geq 0 \text{ and } k \text{ is even,} \\ \left\lfloor \frac{\ell p^k}{m} \right\rfloor &= \left\lfloor \frac{\ell(p^k+1)}{m} - \frac{\ell}{m} \right\rfloor = \frac{\ell(p^k+1)}{m} + \left\lfloor -\frac{\ell}{m} \right\rfloor && \text{if } k \geq 0 \text{ and } k \text{ is odd.} \end{aligned}$$

Therefore the sum $\sum_{j=1}^a \left\lfloor \frac{\ell p^{a-j}}{m} \right\rfloor$ appearing in (3) is equal to

$$\begin{aligned}
& \sum_{\substack{1 \leq j \leq a \\ a-j \equiv 0 \pmod{2}}} \left(\frac{\ell(p^{a-j} - 1)}{m} + \left\lfloor \frac{\ell}{m} \right\rfloor \right) + \sum_{\substack{1 \leq j \leq a \\ a-j \equiv 1 \pmod{2}}} \left(\frac{\ell(p^{a-j} + 1)}{m} + \left\lfloor -\frac{\ell}{m} \right\rfloor \right) \\
&= \frac{\ell}{m} \sum_{1 \leq j \leq a} p^{a-j} - \sum_{\substack{1 \leq j \leq a \\ j \equiv a \pmod{2}}} \left(\frac{\ell}{m} - \left\lfloor \frac{\ell}{m} \right\rfloor \right) + \sum_{\substack{1 \leq j \leq a \\ j \equiv a-1 \pmod{2}}} \left(\frac{\ell}{m} + \left\lfloor -\frac{\ell}{m} \right\rfloor \right) \\
&= \frac{\ell(p^a - 1)}{m(p-1)} + \left\lfloor -\frac{a}{2} \right\rfloor \left(\frac{\ell}{m} - \left\lfloor \frac{\ell}{m} \right\rfloor \right) - \left\lfloor -\frac{a-1}{2} \right\rfloor \left(\frac{\ell}{m} + \left\lfloor -\frac{\ell}{m} \right\rfloor \right). \tag{4}
\end{aligned}$$

By Lemma 4(iii), we see that

$$\left\lfloor \frac{\ell}{m} \right\rfloor + \left\lfloor -\frac{\ell}{m} \right\rfloor = -[\ell \not\equiv 0 \pmod{m}] = -\delta.$$

Therefore if a is even, then (4) is equal to

$$\begin{aligned}
& \frac{\ell(p^a - 1)}{m(p-1)} - \frac{a}{2} \left(\frac{\ell}{m} - \left\lfloor \frac{\ell}{m} \right\rfloor \right) + \frac{a}{2} \left(\frac{\ell}{m} + \left\lfloor -\frac{\ell}{m} \right\rfloor \right) \\
&= \frac{\ell(p^a - 1)}{m(p-1)} + \frac{a}{2} \left(\left\lfloor \frac{\ell}{m} \right\rfloor + \left\lfloor -\frac{\ell}{m} \right\rfloor \right) = \frac{\ell(p^a - 1)}{m(p-1)} - \frac{a}{2} \delta,
\end{aligned}$$

and if a is odd, then (4) is equal to

$$\begin{aligned}
& \frac{\ell(p^a - 1)}{m(p-1)} - \frac{a+1}{2} \left(\frac{\ell}{m} - \left\lfloor \frac{\ell}{m} \right\rfloor \right) + \frac{a-1}{2} \left(\frac{\ell}{m} + \left\lfloor -\frac{\ell}{m} \right\rfloor \right) \\
&= \frac{\ell(p^a - 1)}{m(p-1)} + \frac{a-1}{2} \left(\left\lfloor \frac{\ell}{m} \right\rfloor + \left\lfloor -\frac{\ell}{m} \right\rfloor \right) + \left\lfloor \frac{\ell}{m} \right\rfloor - \frac{\ell}{m} \\
&= \frac{\ell(p^a - 1)}{m(p-1)} - \left(\frac{a-1}{2} \right) \delta - \left\{ \frac{\ell}{m} \right\}.
\end{aligned}$$

This completes the proof. \square

We can combine every case in Theorem 7 into a single form as given in the next corollary.

Corollary 8. *Assume that p , a , ℓ , m , and δ satisfy the same assumptions as in Theorem 7. Then the p -adic valuation of $\left\lfloor \frac{\ell p^a}{m} \right\rfloor!$ is*

$$\begin{aligned}
& \frac{\ell(p^a - 1)}{m(p-1)} - \left\lfloor \frac{a}{2} \right\rfloor \delta - \left\{ \frac{\ell}{m} \right\} [a \equiv 1 \pmod{2}] \\
& + \delta \left\lfloor \frac{a}{2} \right\rfloor \left(1 - 2 \left\{ \frac{\ell}{m} \right\} \right) [p \equiv 1 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right). \tag{5}
\end{aligned}$$

Proof. This is merely a combination of each case from Theorem 7. For example, when $p \equiv -1 \pmod{m}$, the right-hand side of (5) reduces to

$$\begin{aligned} & \frac{\ell(p^a - 1)}{m(p-1)} - \left\lfloor \frac{a}{2} \right\rfloor \delta - \left\{ \frac{\ell}{m} \right\} [a \equiv 1 \pmod{2}] + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right) \\ &= \begin{cases} \frac{\ell(p^a - 1)}{m(p-1)} - \frac{a}{2} \delta + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } a \text{ is even;} \\ \frac{\ell(p^a - 1)}{m(p-1)} - \left(\frac{a-1}{2} \right) \delta - \left\{ \frac{\ell}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } a \text{ is odd,} \end{cases} \end{aligned}$$

which is the same as Theorem 7. The other cases are similar. We leave the details to the reader. \square

Next we deal with the p -adic valuation of an integer of the form $\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor!$ where a, b, ℓ_1, ℓ_2 , and m are positive integers. It is natural to assume $\ell_1 p^b - \ell_2 p^a > 0$. In addition, if $a = b$, then the above expression is reduced to $\left\lfloor \frac{(\ell_1 - \ell_2) p^b}{m} \right\rfloor!$, which can be evaluated by using Theorem 7. We consider the case $b \geq a$ in Theorem 9 and the other case in Theorem 10.

Theorem 9. *Let p be a prime, let a be a nonnegative integer, and let b, m, ℓ_1, ℓ_2 be positive integers satisfying $b \geq a$ and $\ell_1 p^b - \ell_2 p^a > 0$. Assume that $p \equiv \pm 1 \pmod{m}$. Then the following statements hold.*

(i) *If $p \equiv 1 \pmod{m}$, then*

$$\nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) = \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - a \left\{ \frac{\ell_1 - \ell_2}{m} \right\} + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor! \right).$$

(ii) *If $p \equiv -1 \pmod{m}$ and $a \equiv b \pmod{2}$, then*

$$\begin{aligned} \nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{ \frac{\ell_1 - \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] \\ &\quad - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor! \right). \end{aligned}$$

(iii) *If $p \equiv -1 \pmod{m}$ and $a \not\equiv b \pmod{2}$, then*

$$\begin{aligned} \nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{m} \right\rfloor! \right) &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{ -\frac{\ell_1 + \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] \\ &\quad - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{m}] + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{m} \right\rfloor! \right). \end{aligned}$$

We remark that if $m = 1$, the expressions in each case of this theorem are equal.

Proof. The result is easily checked when $a = 0$, and as discussed above, if $b = a$, then the result can be verified using Theorem 7. So we assume throughout that $a \geq 1$ and $b > a$. Similar to the proof of Theorem 7, we use Lemmas 4(i), 4(vi), and 5 repeatedly without reference. Then, as for (3), we obtain

$$\begin{aligned} \nu_p \left(\left[\frac{\ell_1 p^b - \ell_2 p^a}{m} \right]! \right) &= \sum_{j=1}^a \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right] + \sum_{j=a+1}^{\infty} \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right] \\ &= \sum_{j=1}^a \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right] + \nu_p \left(\left[\frac{\ell_1 p^{b-a} - \ell_2}{m} \right]! \right). \end{aligned} \quad (6)$$

We see that when $m = 1$, (6) becomes

$$\nu_p \left((\ell_1 p^b - \ell_2 p^a)! \right) = \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{p-1} + \nu_p \left((\ell_1 p^{b-a} - \ell_2)! \right).$$

So assume throughout that $m \geq 2$. We begin with the proof of (i). Suppose that $p \equiv 1 \pmod{m}$. For each $1 \leq j \leq a$, we have

$$\begin{aligned} \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right] &= \left[\frac{\ell_1 p^{b-j} - \ell_1}{m} - \frac{\ell_2 p^{a-j} - \ell_2}{m} + \frac{\ell_1 - \ell_2}{m} \right] \\ &= \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{\ell_1 - \ell_2}{m} + \left[\frac{\ell_1 - \ell_2}{m} \right]. \end{aligned}$$

Then the sum $\sum_{j=1}^a \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right]$ appearing in (6) is equal to

$$\begin{aligned} &\frac{\ell_1}{m} \sum_{1 \leq j \leq a} p^{b-j} - \frac{\ell_2}{m} \sum_{1 \leq j \leq a} p^{a-j} - a \left(\frac{\ell_1 - \ell_2}{m} - \left[\frac{\ell_1 - \ell_2}{m} \right] \right) \\ &= \frac{\ell_1}{m} \left(\frac{p^{b-a}(p^a - 1)}{p-1} \right) - \frac{\ell_2}{m} \left(\frac{p^a - 1}{p-1} \right) - a \left\{ \frac{\ell_1 - \ell_2}{m} \right\} \\ &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - a \left\{ \frac{\ell_1 - \ell_2}{m} \right\}. \end{aligned}$$

This proves (i). So from this point on, we assume that $p \equiv -1 \pmod{m}$. For each $1 \leq j \leq a$, we have

$$\begin{aligned} \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right] &= \left[\frac{\ell_1 p^{b-j} - (-1)^{b-j} \ell_1}{m} - \frac{\ell_2 p^{a-j} - (-1)^{a-j} \ell_2}{m} + \frac{(-1)^{b-j} \ell_1 - (-1)^{a-j} \ell_2}{m} \right] \\ &= \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{(-1)^{b-j} \ell_1 - (-1)^{a-j} \ell_2}{m} + \left[\frac{(-1)^{b-j} \ell_1 - (-1)^{a-j} \ell_2}{m} \right] \\ &= \begin{cases} \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{(-1)^{b-j}(\ell_1 - \ell_2)}{m} + \left[\frac{(-1)^{b-j}(\ell_1 - \ell_2)}{m} \right], & \text{if } a \equiv b \pmod{2}; \\ \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} - \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} + \left[\frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right], & \text{if } a \not\equiv b \pmod{2}. \end{cases} \end{aligned}$$

Case 1. $a \equiv b \pmod{2}$. Then the sum $\sum_{j=1}^a \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor$ appearing in (6) is equal to

$$\begin{aligned} & \frac{\ell_1}{m} \sum_{1 \leq j \leq a} p^{b-j} - \frac{\ell_2}{m} \sum_{1 \leq j \leq a} p^{a-j} - \left(\frac{\ell_1 - \ell_2}{m} \right) \sum_{1 \leq j \leq a} (-1)^{b-j} + \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j} (\ell_1 - \ell_2)}{m} \right\rfloor \\ &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left(\frac{\ell_1 - \ell_2}{m} \right) \sum_{1 \leq j \leq a} (-1)^{b-j} + \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j} (\ell_1 - \ell_2)}{m} \right\rfloor. \end{aligned} \quad (7)$$

Observe that

$$\sum_{1 \leq j \leq a} (-1)^{b-j} = \begin{cases} 0, & \text{if } a \text{ is even;} \\ 1, & \text{if } a \text{ is odd.} \end{cases}$$

So we have

$$\left(\frac{\ell_1 - \ell_2}{m} \right) \sum_{1 \leq j \leq a} (-1)^{b-j} = \left(\frac{\ell_1 - \ell_2}{m} \right) [a \equiv 1 \pmod{2}].$$

It remains to calculate the last term in (7). If $\ell_1 \equiv \ell_2 \pmod{m}$, then we obtain by Lemma 4(iii) that

$$\begin{aligned} \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j} (\ell_1 - \ell_2)}{m} \right\rfloor &= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor, & \text{if } a \text{ is odd;} \end{cases} \\ &= \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor [a \equiv 1 \pmod{2}]. \end{aligned}$$

Similarly, if $\ell_1 \not\equiv \ell_2 \pmod{m}$, then we obtain by Lemma 4(iii) that

$$\begin{aligned} \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j} (\ell_1 - \ell_2)}{m} \right\rfloor &= \begin{cases} -\frac{a}{2}, & \text{if } a \text{ is even;} \\ \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor - \frac{a-1}{2}, & \text{if } a \text{ is odd;} \end{cases} \\ &= \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor [a \equiv 1 \pmod{2}] - \left\lfloor \frac{a}{2} \right\rfloor. \end{aligned}$$

In any case,

$$\sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j} (\ell_1 - \ell_2)}{m} \right\rfloor = \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor [a \equiv 1 \pmod{2}] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}].$$

Therefore (7) is equal to

$$\begin{aligned} & \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left(\frac{\ell_1 - \ell_2}{m} \right) [a \equiv 1 \pmod{2}] + \left\lfloor \frac{\ell_1 - \ell_2}{m} \right\rfloor [a \equiv 1 \pmod{2}] \\ & - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}] \\ &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{ \frac{\ell_1 - \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv \ell_2 \pmod{m}]. \end{aligned}$$

This proves (ii). Next we prove (iii).

Case 2. $a \not\equiv b \pmod{2}$. Similar to Case 1, the sum $\sum_{j=1}^a \left\lfloor \frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right\rfloor$ appearing in (6) is equal to

$$\begin{aligned} & \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left(\frac{\ell_1 + \ell_2}{m} \right) \sum_{1 \leq j \leq a} (-1)^{b-j} + \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor \\ &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} + \left(\frac{\ell_1 + \ell_2}{m} \right) [a \equiv 1 \pmod{2}] + \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor. \end{aligned} \quad (8)$$

If $\ell_1 \equiv -\ell_2 \pmod{m}$, then we obtain by Lemma 4(iii) that

$$\begin{aligned} \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor &= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor, & \text{if } a \text{ is odd;} \end{cases} \\ &= \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor [a \equiv 1 \pmod{2}]. \end{aligned}$$

Similarly, if $\ell_1 \not\equiv -\ell_2 \pmod{m}$, then we obtain by Lemma 4(iii) that

$$\begin{aligned} \sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor &= \begin{cases} \left\lfloor \frac{a}{2} \right\rfloor, & \text{if } a \text{ is even;} \\ \left\lfloor \frac{a-1}{2} \right\rfloor, & \text{if } a \text{ is odd;} \end{cases} \\ &= \left\lfloor \frac{a}{2} \right\rfloor [a \equiv 1 \pmod{2}] - \left\lfloor \frac{a}{2} \right\rfloor. \end{aligned}$$

In any case, $\sum_{1 \leq j \leq a} \left\lfloor \frac{(-1)^{b-j}(\ell_1 + \ell_2)}{m} \right\rfloor = \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor [a \equiv 1 \pmod{2}] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{m}]$. Therefore (8) is equal to

$$\begin{aligned} & \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} + \left(\frac{\ell_1 + \ell_2}{m} \right) [a \equiv 1 \pmod{2}] + \left\lfloor -\frac{\ell_1 + \ell_2}{m} \right\rfloor [a \equiv 1 \pmod{2}] \\ & - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{m}] \\ &= \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{m(p-1)} - \left\{ \frac{\ell_1 + \ell_2}{m} \right\} [a \equiv 1 \pmod{2}] - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{m}]. \end{aligned}$$

This completes the proof. \square

Next we replace the assumption $b \geq a$ in Theorem 9 by $b < a$. The calculation follows from the same idea so we skip the details of the proof. Although we do not use it in this article, it may be useful for future reference. So we record it in the next theorem.

Theorem 10. *Let p be a prime, let b be a nonnegative integer, and let a, m, ℓ_1, ℓ_2 be positive integers satisfying $b < a$ and $\ell_1 p^b - \ell_2 p^a > 0$. Assume that $p \equiv \pm 1 \pmod{m}$. Then the following statements hold.*

(i) If $p \equiv 1 \pmod{m}$, then

$$\nu_p \left(\left[\frac{\ell_1 p^b - \ell_2 p^a}{m} \right]! \right) = \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - b \left\{ \frac{\ell_1 - \ell_2}{m} \right\} + \nu_p \left(\left[\frac{\ell_1 - \ell_2 p^{a-b}}{m} \right]! \right).$$

(ii) If $p \equiv -1 \pmod{m}$ and $a \equiv b \pmod{2}$, then

$$\begin{aligned} \nu_p \left(\left[\frac{\ell_1 p^b - \ell_2 p^a}{m} \right]! \right) &= \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - \left\{ \frac{\ell_1 - \ell_2}{m} \right\} [b \equiv 1 \pmod{2}] \\ &\quad - \left[\frac{b}{2} \right] [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p \left(\left[\frac{\ell_1 - \ell_2 p^{a-b}}{m} \right]! \right). \end{aligned}$$

(iii) If $p \equiv -1 \pmod{m}$ and $a \not\equiv b \pmod{2}$, then

$$\begin{aligned} \nu_p \left(\left[\frac{\ell_1 p^b - \ell_2 p^a}{m} \right]! \right) &= \frac{(\ell_1 - \ell_2 p^{a-b})(p^b - 1)}{m(p-1)} - \left\{ \frac{\ell_1 + \ell_2}{m} \right\} [b \equiv 1 \pmod{2}] \\ &\quad - \left[\frac{b}{2} \right] [\ell_1 \not\equiv \ell_2 \pmod{m}] + \nu_p \left(\left[\frac{\ell_1 - \ell_2 p^{a-b}}{m} \right]! \right). \end{aligned}$$

Proof. We begin by writing $\nu_p \left(\left[\frac{\ell_1 p^b - \ell_2 p^a}{m} \right]! \right)$ as

$$\sum_{j=1}^b \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right] + \sum_{j=b+1}^{\infty} \left[\frac{\ell_1 p^{b-j} - \ell_2 p^{a-j}}{m} \right].$$

The second sum above is $\nu_p \left(\left[\frac{\ell_1 - \ell_2 p^{a-b}}{m} \right]! \right)$. The first sum can be evaluated in the same way as in Theorem 9. We leave the details to the reader. \square

When we put more restrictions on the range of ℓ_1 and ℓ_2 , the expression $\nu_p \left(\left[\frac{\ell_1 p^{b-a} - \ell_2}{m} \right]! \right)$ appearing in Theorems 9 and 10 can be evaluated further. Nevertheless, since we do not need it in our application, we do not give them here. In the future, we plan to put it in the second author's Researchgate account. So the interested reader can find it there.

4 The p -adic valuations of Fibonomial coefficients

Recall that the *binomial coefficients* $\binom{m}{k}$ is defined by

$$\binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!}, & \text{if } 0 \leq k \leq m; \\ 0, & \text{if } k < 0 \text{ or } k > m. \end{cases}$$

A classical result of Kummer states that for $0 \leq k \leq m$, $\nu_p \left(\binom{m}{k} \right)$ is equal to the number of carries when we add k and $m - k$ in base p . From this, it is not difficult to show that for all primes p and positive integers k, b, a with $b \geq a$, we have

$$\nu_p \left(\binom{p^b}{p^a} \right) = b - a, \quad \text{or more generally, } \nu_p \left(\binom{p^a}{k} \right) = a - \nu_p(k).$$

Knuth and Wilf [8] also obtain the result analogous to that of Kummer for a C-nomial coefficient. However, our purpose is to obtain $\nu_p \left(\binom{m}{k}_F \right)$ in an explicit form. So we first express $\nu_p \left(\binom{m}{k}_F \right)$ in terms of the p -adic valuation of some binomial coefficients in Theorem 11. Then we write it in a form which is easy to use in Corollary 12. Then we apply it to obtain the p -adic valuation of Fibonomial coefficients of the form $\binom{\ell_1 p^b}{\ell_2 p^a}_F$.

Theorem 11. *Let $0 \leq k \leq m$ be integers. Then the following statements hold.*

- (i) *Let $m' = \lfloor \frac{m}{6} \rfloor$, $k' = \lfloor \frac{k}{6} \rfloor$, and let $r = m \bmod 6$ and $s = k \bmod 6$ be the least nonnegative residues of m and k modulo 6, respectively. Then*

$$\begin{aligned} \nu_2 \left(\binom{m}{k}_F \right) &= \nu_2 \left(\binom{m'}{k'} \right) + \left\lfloor \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{s+3}{6} \right\rfloor - 3 \left\lfloor \frac{r-s}{6} \right\rfloor \\ &\quad + [r < s] \nu_2 \left(\left\lfloor \frac{m-k+6}{6} \right\rfloor \right). \end{aligned}$$

- (ii) $\nu_5 \left(\binom{m}{k}_F \right) = \nu_5 \left(\binom{m}{k} \right)$.

- (iii) *Suppose that p is a prime, $p \neq 2$, and $p \neq 5$. Let $m' = \lfloor \frac{m}{z(p)} \rfloor$, $k' = \lfloor \frac{k}{z(p)} \rfloor$, and let $r = m \bmod z(p)$, and $s = k \bmod z(p)$ be the least nonnegative residues of m and k modulo $z(p)$, respectively. Then*

$$\nu_p \left(\binom{m}{k}_F \right) = \nu_p \left(\binom{m'}{k'} \right) + [r < s] \left(\nu_p \left(\left\lfloor \frac{m-k+z(p)}{z(p)} \right\rfloor \right) + \nu_p(F_{z(p)}) \right).$$

Proof. We will use Lemmas 4(i) and 5 repeatedly without reference. In addition, it is useful to recall that for every $a, b \in \mathbb{N}$, $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ and if $b \mid a$, then $\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b)$. Since the formulas to prove clearly hold when $k = 0$ or m , we assume $m \geq 2$ and $1 \leq k < m$.

By Lemma 2, we obtain, for every $\ell \geq 1$,

$$\begin{aligned}
\nu_2(F_1 F_2 F_3 \cdots F_\ell) &= \sum_{\substack{1 \leq n \leq \ell \\ n \equiv 3 \pmod{6}}} \nu_2(F_n) + \sum_{\substack{1 \leq n \leq \ell \\ n \equiv 0 \pmod{6}}} \nu_2(F_n) \\
&= \sum_{\substack{1 \leq n \leq \ell \\ n \equiv 3 \pmod{6}}} 1 + \sum_{\substack{1 \leq n \leq \ell \\ n \equiv 0 \pmod{6}}} (\nu_2(n) + 2) \\
&= \left\lfloor \frac{\ell+3}{6} \right\rfloor + 2 \left\lfloor \frac{\ell}{6} \right\rfloor + \sum_{1 \leq j \leq \frac{\ell}{6}} \nu_2(6j) \\
&= \left\lfloor \frac{\ell+3}{6} \right\rfloor + 3 \left\lfloor \frac{\ell}{6} \right\rfloor + \sum_{1 \leq j \leq \frac{\ell}{6}} \nu_2(j) \\
&= \left\lfloor \frac{\ell+3}{6} \right\rfloor + 3 \left\lfloor \frac{\ell}{6} \right\rfloor + \nu_2 \left(\left\lfloor \frac{\ell}{6} \right\rfloor! \right). \tag{9}
\end{aligned}$$

Then we obtain from the definition of $\binom{m}{k}_F$ and from (9) that

$$\begin{aligned}
\nu_2 \left(\binom{m}{k}_F \right) &= \nu_2(F_1 F_2 \cdots F_m) - \nu_2(F_1 F_2 \cdots F_{m-k}) - \nu_2(F_1 F_2 \cdots F_k) \\
&= \left(\left\lfloor \frac{m+3}{6} \right\rfloor - \left\lfloor \frac{m-k+3}{6} \right\rfloor - \left\lfloor \frac{k+3}{6} \right\rfloor \right) + 3 \left(\left\lfloor \frac{m}{6} \right\rfloor - \left\lfloor \frac{m-k}{6} \right\rfloor - \left\lfloor \frac{k}{6} \right\rfloor \right) \\
&\quad + \nu_2 \left(\left\lfloor \frac{m}{6} \right\rfloor! \right) - \nu_2 \left(\left\lfloor \frac{m-k}{6} \right\rfloor! \right) - \nu_2 \left(\left\lfloor \frac{k}{6} \right\rfloor! \right). \tag{10}
\end{aligned}$$

The expression in the first parenthesis in (10) is equal to

$$\begin{aligned}
&\left\lfloor \frac{m-r}{6} + \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{(m-r) - (k-s)}{6} + \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{k-s}{6} + \frac{s+3}{6} \right\rfloor \\
&= \frac{m-r}{6} + \left\lfloor \frac{r+3}{6} \right\rfloor - \frac{(m-r) - (k-s)}{6} - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \frac{k-s}{6} - \left\lfloor \frac{s+3}{6} \right\rfloor \\
&= \left\lfloor \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{s+3}{6} \right\rfloor.
\end{aligned}$$

Similarly, the expression in the second parenthesis is

$$3 \left(\left\lfloor \frac{r}{6} \right\rfloor - \left\lfloor \frac{r-s}{6} \right\rfloor - \left\lfloor \frac{s}{6} \right\rfloor \right) = -3 \left\lfloor \frac{r-s}{6} \right\rfloor.$$

Therefore (10) becomes

$$\nu_2 \left(\binom{m}{k}_F \right) = \left\lfloor \frac{r+3}{6} \right\rfloor - \left\lfloor \frac{r-s+3}{6} \right\rfloor - \left\lfloor \frac{s+3}{6} \right\rfloor - 3 \left\lfloor \frac{r-s}{6} \right\rfloor + \nu_2 \left(\frac{[x+y]!}{[x]![y]!} \right) \tag{11}$$

where $x = \frac{m-k}{6}$ and $y = \frac{k}{6}$. By Lemma 4(v), we see that

$$\begin{aligned} \frac{[x+y]!}{[x]![y]!} &= \begin{cases} \binom{[x+y]}{[y]}, & \text{if } \{x\} + \{y\} < 1; \\ \binom{[x+y]}{[y]}([x] + 1), & \text{if } \{x\} + \{y\} \geq 1; \end{cases} \\ &= \begin{cases} \binom{m'}{k'}, & \text{if } \{x\} + \{y\} < 1; \\ \binom{m'}{k'} \binom{[m-k+6]}{6}, & \text{if } \{x\} + \{y\} \geq 1. \end{cases} \end{aligned}$$

By Lemma 4(ii), we obtain

$$\{x\} = \left\{ \frac{(m-r) - (k-s)}{6} + \frac{r-s}{6} \right\} = \left\{ \frac{r-s}{6} \right\} \text{ and } \{y\} = \left\{ \frac{k-s}{6} + \frac{s}{6} \right\} = \frac{s}{6}.$$

If $r \geq s$, then $\{x\} + \{y\} = \left\{ \frac{r-s}{6} \right\} + \frac{s}{6} = \frac{r-s}{6} + \frac{s}{6} = \frac{r}{6} < 1$. If $r < s$, then we obtain by Lemma 4(iv) that $\{x\} + \{y\} = \left\{ -\frac{s-r}{6} \right\} + \frac{s}{6} = 1 - \frac{s-r}{6} + \frac{s}{6} = 1 + \frac{r}{6} \geq 1$. Therefore

$$\frac{[x+y]!}{[x]![y]!} = \begin{cases} \binom{m'}{k'}, & \text{if } r \geq s; \\ \binom{m'}{k'} \binom{[m-k+6]}{6}, & \text{if } r < s. \end{cases} \quad (12)$$

Substituting (12) in (11), we obtain part (i) of this theorem. The calculation in parts (ii) and (iii) are similar, so we give fewer details than given in part (i). By Lemma 2, for every $\ell \geq 1$, we have

$$\nu_5(F_1 F_2 \cdots F_\ell) = \sum_{1 \leq n \leq \ell} \nu_5(F_n) = \sum_{1 \leq n \leq \ell} \nu_5(n) = \nu_5(\ell!),$$

which implies

$$\nu_5 \left(\binom{m}{k}_F \right) = \nu_5(m!) - \nu_5(k!) - \nu_5((m-k)!) = \nu_5 \left(\binom{m}{k} \right).$$

For (iii), we apply Lemmas 2 and 1(iv) to obtain

$$\begin{aligned} \nu_p(F_1 F_2 \cdots F_\ell) &= \sum_{\substack{1 \leq n \leq \ell \\ n \equiv 0 \pmod{z(p)}}} \nu_p(F_n) = \sum_{\substack{1 \leq n \leq \ell \\ n \equiv 0 \pmod{z(p)}}} (\nu_p(n) + \nu_p(F_{z(p)})) \\ &= \sum_{1 \leq k \leq \frac{\ell}{z(p)}} \nu_p(kz(p)) + \left\lfloor \frac{\ell}{z(p)} \right\rfloor \nu_p(F_{z(p)}) \\ &= \nu_p \left(\left\lfloor \frac{\ell}{z(p)} \right\rfloor! \right) + \left\lfloor \frac{\ell}{z(p)} \right\rfloor \nu_p(F_{z(p)}). \end{aligned}$$

As in part (i), the above implies that

$$\nu_p \left(\binom{m}{k}_F \right) = \nu_p \left(\frac{[x+y]!}{[x]![y]!} \right) + ([x+y] - [x] - [y]) \nu_p(F_{z(p)}), \quad (13)$$

where $x = \frac{m-k}{z(p)}$ and $y = \frac{k}{z(p)}$. In addition, if $r \geq s$, then $\{x\} + \{y\} < 1$ and if $r < s$, then $\{x\} + \{y\} \geq 1$. Therefore (13) can be simplified to the desired result. This completes the proof. \square

By Theorem 11(ii), we see that the 5-adic valuations of Fibonomial and binomial coefficients are the same. So we focus our investigation only on the p -adic valuations of Fibonomial coefficients when $p \neq 5$. Calculating r and s in Theorem 11(i) in every case and writing Theorem 11(iii) in another form, we obtain the following corollary.

Corollary 12. *Let m, k, r , and s be as in Theorem 11. Let*

$$A_2 = \nu_2 \left(\left\lfloor \frac{m}{6} \right\rfloor! \right) - \nu_2 \left(\left\lfloor \frac{k}{6} \right\rfloor! \right) - \nu_2 \left(\left\lfloor \frac{m-k}{6} \right\rfloor! \right),$$

and for each prime $p \neq 2, 5$, let $A_p = \nu_p \left(\left\lfloor \frac{m}{z(p)} \right\rfloor! \right) - \nu_p \left(\left\lfloor \frac{k}{z(p)} \right\rfloor! \right) - \nu_p \left(\left\lfloor \frac{m-k}{z(p)} \right\rfloor! \right)$. Then the following statements hold.

$$(i) \quad \nu_2 \left(\binom{m}{k}_F \right) = \begin{cases} A_2, & \text{if } r \geq s \text{ and } (r, s) \neq (3, 1), (3, 2), (4, 2); \\ A_2 + 1, & \text{if } (r, s) = (3, 1), (3, 2), (4, 2); \\ A_2 + 3, & \text{if } r < s \text{ and } (r, s) \neq (0, 3), (1, 3), (2, 3), \\ & (1, 4), (2, 4), (2, 5); \\ A_2 + 2, & \text{if } (r, s) = (0, 3), (1, 3), (2, 3), (1, 4), (2, 4), \\ & (2, 5). \end{cases}$$

(ii) For $p \neq 2, 5$, we have

$$\nu_p \left(\binom{m}{k}_F \right) = \begin{cases} A_p, & \text{if } r \geq s; \\ A_p + \nu_p(F_{z(p)}), & \text{if } r < s. \end{cases}$$

Proof. For (i), we have $0 \leq r \leq 5$ and $0 \leq s \leq 5$, so we can directly consider every case and reduce Theorem 11(i) to the result in this corollary. In addition, (ii) follows directly from (13). \square

In a series of papers (see [11] and references therein), Marques and Trojovský obtain a formula for $\nu_p \left(\binom{p^b}{p^a}_F \right)$ only when $b = a + 1$. Then Ballot [2] extends it to any case $b > a$. Corollary 12 enables us to compute $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a}_F \right)$. We illustrate this in the next theorem.

Theorem 13. *Let a, b, ℓ_1 , and ℓ_2 be positive integers and $b \geq a$. Let $p \neq 5$ be a prime. Assume that $\ell_1 p^b > \ell_2 p^a$ and let $m_p = \left\lfloor \frac{\ell_1 p^{b-a}}{z(p)} \right\rfloor$ and $k_p = \left\lfloor \frac{\ell_2}{z(p)} \right\rfloor$. Then the following statements hold.*

(i) If $a \equiv b \pmod{2}$, then $\nu_2 \left(\binom{\ell_1 2^b}{\ell_2 2^a} \right)_F$ is equal to

$$\begin{cases} \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv \ell_2 \pmod{3} \text{ or } \ell_2 \equiv 0 \pmod{3}; \\ a + 2 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 0 \pmod{3} \text{ and } \ell_2 \not\equiv 0 \pmod{3}; \\ \left[\frac{a}{2} \right] + 1 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 1 \pmod{3} \text{ and } \ell_2 \equiv 2 \pmod{3}; \\ \left[\frac{a+1}{2} \right] + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 2 \pmod{3} \text{ and } \ell_2 \equiv 1 \pmod{3}, \end{cases}$$

and if $a \not\equiv b \pmod{2}$, then $\nu_2 \left(\binom{\ell_1 2^b}{\ell_2 2^a} \right)_F$ is equal to

$$\begin{cases} \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv -\ell_2 \pmod{3} \text{ or } \ell_2 \equiv 0 \pmod{3}; \\ a + 2 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 0 \pmod{3} \text{ and } \ell_2 \not\equiv 0 \pmod{3}; \\ \left[\frac{a+1}{2} \right] + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 1 \pmod{3} \text{ and } \ell_2 \equiv 1 \pmod{3}; \\ \left[\frac{a}{2} \right] + 1 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 2 \pmod{3} \text{ and } \ell_2 \equiv 2 \pmod{3}. \end{cases}$$

(ii) Let $p \neq 5$ be an odd prime and let $r = \ell_1 p^b \pmod{z(p)}$ and $s = \ell_2 p^a \pmod{z(p)}$. If $p \equiv \pm 1 \pmod{5}$, then

$$\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a} \right)_F = [r < s] (a + \nu_p(m_p - k_p) + \nu_p(F_{z(p)})) + \nu_p \left(\binom{m_p}{k_p} \right),$$

and if $p \equiv \pm 2 \pmod{5}$, then $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a} \right)_F$ is equal to

$$\begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r = s \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and} \\ & \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even}; \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd.} \end{cases}$$

Remark 14. In the proof of this theorem, we also show that the condition $r = s$ in Theorem 13(ii) is equivalent to $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod{2}] \pmod{z(p)}$. It seems more natural to write $r = s$ in the statement of the theorem, but it is more convenient in the proof to use the condition $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod{2}] \pmod{z(p)}$.

Proof of Theorem 13. We apply Corollary 12 to calculate $\nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$ with $m = \ell_1 2^b$, $k = \ell_2 2^a$, $r = \ell_1 2^b \pmod{6}$, and $s = \ell_2 2^a \pmod{6}$. For convenience, we also let $r' = \ell_1 \pmod{3}$, and $s' = \ell_2 \pmod{3}$. Therefore A_2 given in Corollary 12 is

$$A_2 = \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-1}}{3} \right\rfloor!\right) - \nu_2\left(\left\lfloor \frac{\ell_2 2^{a-1}}{3} \right\rfloor!\right) - \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-1} - \ell_2 2^{a-1}}{3} \right\rfloor!\right). \quad (14)$$

By Corollary 8, the first term on the right-hand side of (14) is equal to

$$\begin{aligned} & \frac{\ell_1(2^{b-1} - 1)}{3} - \left\lfloor \frac{b-1}{2} \right\rfloor [\ell_1 \not\equiv 0 \pmod{3}] - \left\{ \frac{\ell_1}{3} \right\} [b \equiv 0 \pmod{2}] + \nu_2\left(\left\lfloor \frac{\ell_1}{3} \right\rfloor!\right) \\ &= \frac{\ell_1(2^{b-1} - 1)}{3} - \left\lfloor \frac{b-1}{2} \right\rfloor [r' \neq 0] - \frac{r'}{3} [b \equiv 0 \pmod{2}] + \nu_2\left(\left\lfloor \frac{\ell_1}{3} \right\rfloor!\right). \end{aligned} \quad (15)$$

Similarly, the second term is

$$\frac{\ell_2(2^{a-1} - 1)}{3} - \left\lfloor \frac{a-1}{2} \right\rfloor [s' \neq 0] - \frac{s'}{3} [a \equiv 0 \pmod{2}] + \nu_2\left(\left\lfloor \frac{\ell_2}{3} \right\rfloor!\right). \quad (16)$$

To evaluate the third term on the right-hand side of (14), we divide the proof into two cases according to the parity of a and b .

Case 1. $a \equiv b \pmod{2}$. Observe that $\ell_1 \equiv \ell_2 \pmod{3}$ if and only if $r' = s'$. In addition, $\left\{\frac{\ell_1 - \ell_2}{3}\right\} = \left\{\frac{r' - s'}{3}\right\}$ and $\left\lfloor \frac{r' - s'}{3} \right\rfloor = -[r' < s']$. Then by Theorem 9, the third term on the right-hand side of (14) is equal to

$$\begin{aligned} & \frac{(\ell_1 2^{b-a} - \ell_2)(2^{a-1} - 1)}{3} - \left\{ \frac{r' - s'}{3} \right\} [a \equiv 0 \pmod{2}] - \left\lfloor \frac{a-1}{2} \right\rfloor [r' \neq s'] + \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor!\right) \\ &= \frac{(\ell_1 2^{b-a} - \ell_2)(2^{a-1} - 1)}{3} - \left(\frac{r' - s'}{3} + [r' < s']\right) [a \equiv 0 \pmod{2}] - \left\lfloor \frac{a-1}{2} \right\rfloor [r' \neq s'] \\ &+ \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor!\right). \end{aligned} \quad (17)$$

Recall that $m_2 = \left\lfloor \frac{\ell_1 2^{b-a}}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{\ell_2}{3} \right\rfloor$. Since $b - a$ is even, $2^{b-a} \equiv 1 \pmod{3}$ and we obtain by Lemma 6 that

$$\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor = m_2 - k_2 - [r' < s'].$$

Therefore $\nu_2 \left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor! \right)$ is equal to

$$\nu_2(m_2!) - [r' < s']\nu_2(m_2 - k_2) - \nu_2 \left(\frac{m_2!}{(m_2 - k_2)!} \right).$$

By Corollary 8, $\nu_2(m_2!)$ is equal to

$$\frac{\ell_1(2^{b-a} - 1)}{3} - \frac{b-a}{2}[r' \neq 0] + \nu_2 \left(\left\lfloor \frac{\ell_1}{3} \right\rfloor! \right).$$

We substitute the value of $\nu_2 \left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor! \right)$ in (17) and then substitute (15), (16), and (17) in (14) to obtain A_2 . We see that there are some cancellations. For instance,

$$\frac{r'}{3} \left([a \equiv 0 \pmod{2}] - [b \equiv 0 \pmod{2}] \right) = 0.$$

and

$$\nu_2 \left(\frac{m_2!}{(m_2 - k_2)!} \right) - \nu_2 \left(\left\lfloor \frac{\ell_2}{3} \right\rfloor! \right) = \nu_2 \left(\binom{m_2}{k_2} \right)$$

Then we obtain

$$\begin{aligned} A_2 = & - \left\lfloor \frac{b-1}{2} \right\rfloor [r' \neq 0] + \left\lfloor \frac{a-1}{2} \right\rfloor [s' \neq 0] + [r' < s'] [a \equiv 0 \pmod{2}] \\ & + \left\lfloor \frac{a-1}{2} \right\rfloor [r' \neq s'] + \frac{b-a}{2} [r' \neq 0] + [r' < s'] \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right). \end{aligned} \quad (18)$$

Next we divide the calculation of A_2 into 4 cases:

- Case 1.1. $\ell_1 \equiv \ell_2 \pmod{3}$ or $\ell_2 \equiv 0 \pmod{3}$,
- Case 1.2. $\ell_1 \equiv 0 \pmod{3}$ and $\ell_2 \not\equiv 0 \pmod{3}$,
- Case 1.3. $\ell_1 \equiv 1 \pmod{3}$ and $\ell_2 \equiv 2 \pmod{3}$,
- Case 1.4. $\ell_1 \equiv 2 \pmod{3}$ and $\ell_2 \equiv 1 \pmod{3}$.

Since the calculation in each case is similar, we only show the details in Case 1.1 and Case 1.2. So assume that $\ell_1 \equiv \ell_2 \pmod{3}$. Then $r' = s'$ and (18) becomes

$$A_2 = - \left\lfloor \frac{b-1}{2} \right\rfloor [r' \neq 0] + \left\lfloor \frac{a-1}{2} \right\rfloor [r' \neq 0] + \frac{b-a}{2} [r' \neq 0] + \nu_2 \left(\binom{m_2}{k_2} \right)$$

Since $-\left\lfloor \frac{b-1}{2} \right\rfloor + \left\lfloor \frac{a-1}{2} \right\rfloor + \frac{b-a}{2} = 0$, we see that $A_2 = \nu_2 \left(\binom{m_2}{k_2} \right)$. Next if $\ell_2 \equiv 0 \pmod{3}$, then $s' = 0$ and the same calculation leads to $A_2 = \nu_2 \left(\binom{m_2}{k_2} \right)$. Next assume that $\ell_1 \equiv 0 \pmod{3}$ and $\ell_2 \not\equiv 0 \pmod{3}$. Then $r' = 0$, $s' \neq 0$, and (18) becomes

$$A_2 = \left\lfloor \frac{a-1}{2} \right\rfloor + [a \equiv 0 \pmod{2}] + \left\lfloor \frac{a-1}{2} \right\rfloor + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right).$$

Observing that the sum of the first three terms above is equal to $a - 1$, we obtain $A_2 = a - 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right)$. The other cases are similar. Therefore A_2 is

$$\begin{cases} \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv \ell_2 \pmod{3} \text{ or } \ell_2 \equiv 0 \pmod{3}; \\ a - 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 0 \pmod{3} \text{ and } \ell_2 \not\equiv 0 \pmod{3}; \\ \lfloor \frac{a}{2} \rfloor + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 1 \pmod{3} \text{ and } \ell_2 \equiv 2 \pmod{3}; \\ \lfloor \frac{a-1}{2} \rfloor + \nu_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 2 \pmod{3} \text{ and } \ell_2 \equiv 1 \pmod{3}. \end{cases}$$

Recall that $r = \ell_1 2^b \pmod{6}$ and $s = \ell_2 2^a \pmod{6}$. Therefore

$$r = \begin{cases} 0, & \text{if } \ell_1 \equiv 0 \pmod{3}; \\ 2, & \text{if } b \text{ is even and } \ell_1 \equiv 2 \pmod{3} \text{ or if } b \text{ is odd and } \ell_1 \equiv 1 \pmod{3}; \\ 4 & \text{if } b \text{ is even and } \ell_1 \equiv 1 \pmod{3} \text{ or if } b \text{ is odd and } \ell_1 \equiv 2 \pmod{3}, \end{cases}$$

and

$$s = \begin{cases} 0, & \text{if } \ell_2 \equiv 0 \pmod{3}; \\ 2, & \text{if } a \text{ is even and } \ell_2 \equiv 2 \pmod{3} \text{ or if } a \text{ is odd and } \ell_2 \equiv 1 \pmod{3}; \\ 4 & \text{if } a \text{ is even and } \ell_2 \equiv 1 \pmod{3} \text{ or if } a \text{ is odd and } \ell_2 \equiv 2 \pmod{3}. \end{cases}$$

To obtain the formula for $\nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$, we divide the calculation into 4 cases: Case 1.1 to Case 1.4 as before. Then we consider the values of r and s in each case, and substitute A_2 in Corollary 12. This leads to the desired result. Since the calculation in each case is similar, we only give the details in Case 1.3. In this case, $A_2 = \lfloor \frac{a}{2} \rfloor + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right)$, $(r, s) = (2, 4)$ if a and b are odd, and $(r, s) = (4, 2)$ if a and b are even. By Corollary 12, we obtain

$$\begin{aligned} \nu_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right) &= \begin{cases} A_2 + 2, & \text{if } a \text{ and } b \text{ are odd;} \\ A_2 + 1, & \text{if } a \text{ and } b \text{ are even,} \end{cases} \\ &= \lfloor \frac{a}{2} \rfloor + 1 + \nu_2(m_2 - k_2) + \nu_2\left(\binom{m_2}{k_2}\right), \end{aligned}$$

as required. The other cases are similar.

Case 2. $a \not\equiv b \pmod{2}$. The calculation in this case is similar to Case 1, so we omit some details. By Theorem 9, the third term on the right-hand side of (14) is equal to

$$\begin{aligned} &\frac{(\ell_1 2^{b-a} - \ell_2)(2^{a-1} - 1)}{3} - \left\{ -\frac{r' + s'}{3} \right\} [a \equiv 0 \pmod{2}] - \left\lfloor \frac{a-1}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{3}] \\ &+ \nu_2\left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor!\right). \end{aligned} \tag{19}$$

Since $b - a$ is odd, $\ell_1 2^{b-a} \equiv -r' \pmod{3}$ and we obtain by Lemma 6 that

$$\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor = m_2 - k_2 - B$$

where $B = [(r', s') \in \{(0, 1), (0, 2), (2, 2)\}]$. Similar to Case 1, $\nu_2 \left(\left\lfloor \frac{\ell_1 2^{b-a} - \ell_2}{3} \right\rfloor! \right)$ is

$$\nu_2(m_2!) - B\nu_2(m_2 - k_2) - \nu_2 \left(\frac{m_2!}{(m_2 - k_2)!} \right).$$

Then we evaluate $\nu_2(m_2!)$ by Corollary 8, and substitute all of these in (14) to obtain that A_2 is equal to

$$\begin{aligned} & \left(\frac{b-a-1}{2} - \left\lfloor \frac{b-1}{2} \right\rfloor \right) [r' \neq 0] - \frac{r'}{3} [b \equiv 0 \pmod{2}] + \left\lfloor \frac{a-1}{2} \right\rfloor [s' \neq 0] \\ & + \left(\frac{s'}{3} + \left\{ -\frac{r'+s'}{3} \right\} \right) [a \equiv 0 \pmod{2}] + \left\lfloor \frac{a-1}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{3}] + \frac{r'}{3} \\ & + B\nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right). \end{aligned}$$

Then we divide the calculation into 4 cases and obtain that A_2 is

$$\begin{cases} \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv -\ell_2 \pmod{3} \text{ or } \ell_2 \equiv 0 \pmod{3}; \\ a - 1 + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 0 \pmod{3} \text{ and } \ell_2 \not\equiv 0 \pmod{3}; \\ \left\lfloor \frac{a-1}{2} \right\rfloor + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 1 \pmod{3} \text{ and } \ell_2 \equiv 1 \pmod{3}; \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_2(m_2 - k_2) + \nu_2 \left(\binom{m_2}{k_2} \right), & \text{if } \ell_1 \equiv 2 \pmod{3} \text{ and } \ell_2 \equiv 2 \pmod{3}. \end{cases}$$

We illustrate the calculation of A_2 above only for the case $\ell_2 \equiv 0 \pmod{3}$ since the other cases are similar. So suppose $\ell_2 \equiv 0 \pmod{3}$. So $s' = 0$. If $r' = 0$, then it is easy to see that A_2 is equal to $\nu_2 \left(\binom{m_2}{k_2} \right)$. So assume that $r' \neq 0$. Then A_2 is equal to $x + y + \nu_2 \left(\binom{m_2}{k_2} \right)$, where

$$x = \frac{b-a-1}{2} - \left\lfloor \frac{b-1}{2} \right\rfloor + \left\lfloor \frac{a-1}{2} \right\rfloor = \begin{cases} 0, & \text{if } a \text{ is odd;} \\ -1, & \text{if } a \text{ is even,} \end{cases}$$

$$y = \frac{-r'}{3} [b \equiv 0 \pmod{2}] + \left\{ -\frac{r'}{3} \right\} [a \equiv 0 \pmod{2}] + \frac{r'}{3} = \begin{cases} 0, & \text{if } a \text{ is odd;} \\ 1, & \text{if } a \text{ is even.} \end{cases}$$

Therefore $A_2 = \nu_2 \left(\binom{m_2}{k_2} \right)$, as required. As in Case 1, we divide the calculation of $\nu_2 \left(\binom{\ell_1 2^b}{\ell_2 2^a} \right)_F$ into 4 cases according to the value of A_2 , which leads to the desired result. This proves (i).

For (ii), we apply Corollary 12 with $m = \ell_1 p^b$ and $k = \ell_2 p^a$. For convenience, we let $r' = \ell_1 \pmod{z(p)}$ and $s' = \ell_2 \pmod{z(p)}$. The calculation of this part is similar to that of part (i), so we omit some details. We have

$$A_p = \nu_p \left(\left\lfloor \frac{\ell_1 p^b}{z(p)} \right\rfloor! \right) - \nu_p \left(\left\lfloor \frac{\ell_2 p^a}{z(p)} \right\rfloor! \right) - \nu_p \left(\left\lfloor \frac{\ell_1 p^b - \ell_2 p^a}{z(p)} \right\rfloor! \right). \quad (20)$$

Case 1. $p \equiv \pm 1 \pmod{5}$. Then by Lemma 1(iii), $p \equiv 1 \pmod{z(p)}$. By Corollary 8, the first term on the right-hand side of (20) is equal to

$$\begin{aligned} & \frac{\ell_1(p^b - 1)}{z(p)(p - 1)} + \nu_p \left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor! \right) - \left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0] - \frac{r'}{z(p)} [b \equiv 1 \pmod{2}] + \left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0] \left(1 - \frac{2r'}{z(p)} \right) \\ &= \frac{\ell_1(p^b - 1)}{z(p)(p - 1)} - \frac{br'}{z(p)} + \nu_p \left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor! \right), \end{aligned}$$

and similarly, the second term is

$$\frac{\ell_2(p^a - 1)}{z(p)(p - 1)} - \frac{as'}{z(p)} + \nu_p \left(\left\lfloor \frac{\ell_2}{z(p)} \right\rfloor! \right).$$

By Theorem 9, the third term is

$$\frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{z(p)(p - 1)} - a \left(\frac{r' - s'}{z(p)} + [r' < s'] \right) + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor! \right).$$

Since $p \equiv 1 \pmod{z(p)}$, we obtain by Lemma 6 that

$$\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor = m_p - k_p - [r' < s'].$$

Therefore $\nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor! \right)$ is equal to

$$\nu_p(m_p!) - [r' < s'] \nu_p(m_p - k_p) - \nu_p \left(\frac{m_p!}{(m_p - k_p)!} \right).$$

As usual, the first term above can be evaluated by Corollary 8 and is equal to

$$\frac{\ell_1(p^{b-a} - 1)}{z(p)(p - 1)} - (b - a) \frac{r'}{z(p)} + \nu_p \left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor! \right).$$

We substitute all of these in (20) to obtain

$$A_p = [r' < s'] (a + \nu_p(m_p - k_p)) + \nu_p \left(\binom{m_p}{k_p} \right).$$

Since $p \equiv 1 \pmod{z(p)}$, $r = r'$ and $s = s'$. Substituting A_p and applying Corollary 12, we obtain the desired result.

Case 2. $p \equiv \pm 2 \pmod{5}$. Then by Lemma 1(iii), $p \equiv -1 \pmod{z(p)}$. By Corollary 8, the first term on the right-hand side of (20) is equal to

$$\frac{\ell_1(p^b - 1)}{z(p)(p - 1)} - \left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0] - \frac{r'}{z(p)} [b \equiv 1 \pmod{2}] + \nu_p \left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor! \right).$$

Similarly, the second term is

$$\frac{\ell_2(p^a - 1)}{z(p)(p - 1)} - \left\lfloor \frac{a}{2} \right\rfloor [s' \neq 0] - \frac{s'}{z(p)} [a \equiv 1 \pmod{2}] + \nu_p \left(\left\lfloor \frac{\ell_2}{z(p)} \right\rfloor! \right).$$

For the third term, we divide the proof into two cases according to the parity of a and b .

Case 2.1. $a \equiv b \pmod{2}$. Then by Theorem 9, the third term on the right-hand side of (20) is equal to

$$\begin{aligned} & \frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{z(p)(p - 1)} - \left(\frac{r' - s'}{z(p)} + [r' < s'] \right) [a \equiv 1 \pmod{2}] - \left\lfloor \frac{a}{2} \right\rfloor [r' \neq s'] \\ & + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor! \right). \end{aligned}$$

As in Case 1, we apply Lemma 6 to write

$$\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor = m_p - k_p - [r' < s'],$$

and then use Corollary 8 to show that $\nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor! \right)$ is equal to

$$\frac{\ell_1(p^{b-a} - 1)}{z(p)(p - 1)} - \frac{b - a}{2} [r' \neq 0] + \nu_p \left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor! \right) - [r' < s'] \nu_p(m_p - k_p) - \nu_p \left(\frac{m_p!}{(m_p - k_p)!} \right).$$

Substituting all of these in (20), we see that A_p is equal to

$$\begin{aligned} & - \left\lfloor \frac{b}{2} \right\rfloor [r' \neq 0] + \left\lfloor \frac{a}{2} \right\rfloor [s' \neq 0] + [r' < s'] [b \equiv 1 \pmod{2}] + \left\lfloor \frac{a}{2} \right\rfloor [r' \neq s'] + \frac{b - a}{2} [r' \neq 0] \\ & + [r' < s'] \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right) \\ & = \begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv \ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' > s'; \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' < s'. \end{cases} \end{aligned}$$

Recall that $r = \ell_1 p^b \pmod{z(p)}$ and $s = \ell_2 p^a \pmod{z(p)}$. If a and b are even, then $p^b \equiv p^a \equiv 1 \pmod{z(p)}$, $r = r'$, and $s = s'$, and we can obtain $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a}_F \right)$ by substituting A_p in Corollary 12. Suppose a and b are odd. Then $r \equiv -r' \pmod{z(p)}$ and $s \equiv -s' \pmod{z(p)}$ and thus when r and s are both nonzero or are both zero, we have

$$r \geq s \quad \text{if and only if} \quad r' \leq s'.$$

Similar to the above, we can obtain $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a}_F \right)$ by the substitution of A_p in Corollary 12.

We see that $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a} \right)$ is equal to

$$\begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv \ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}; \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}. \end{cases}$$

Since $a \equiv b \pmod{2}$, we see that $p^a \equiv p^b \pmod{z(p)}$ and therefore

$$\ell_1 \equiv \ell_2 \pmod{z(p)} \Leftrightarrow r = s \tag{21}$$

So the condition $\ell_1 \equiv \ell_2 \pmod{z(p)}$ can be replaced by $r = s$.

Case 2.2. $a \not\equiv b \pmod{2}$. The calculation in this case is similar to that given before. So we skip some details. By Theorem 9, the third term on the right-hand side of (20) is equal to

$$\frac{(\ell_1 p^{b-a} - \ell_2)(p^a - 1)}{z(p)(p-1)} - [a \equiv 1 \pmod{2}] B_1 - \left\lfloor \frac{a}{2} \right\rfloor [\ell_1 \not\equiv -\ell_2 \pmod{z(p)}] + \nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor ! \right),$$

where $B_1 = \left\{ -\frac{r'+s'}{z(p)} \right\} = -\frac{r'+s'}{z(p)} + [r'+s' > 0] + [r'+s' > z(p)]$. Since $p \equiv -1 \pmod{z(p)}$, we obtain by Lemma 6 and a straightforward verification that

$$\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor = m_p - k_p - \varepsilon,$$

where $\varepsilon = [-r' \pmod{z(p)} < s'] = [r' = 0 \text{ and } s' \neq 0] + [r' + s' > z(p)]$. Then by Corollary 8, $\nu_p \left(\left\lfloor \frac{\ell_1 p^{b-a} - \ell_2}{z(p)} \right\rfloor ! \right)$ is equal to

$$\frac{\ell_1(p^{b-a} - 1)}{z(p)(p-1)} - \left\lfloor \frac{b-a}{2} \right\rfloor [r' \neq 0] - \frac{r'}{z(p)} + \nu_p \left(\left\lfloor \frac{\ell_1}{z(p)} \right\rfloor ! \right) - B_2 - \nu_p \left(\frac{m_p!}{(m_p - k_p)!} \right),$$

where $B_2 = \varepsilon \nu_p(m_p - k_p)$. Since $a \not\equiv b \pmod{2}$, $[b \equiv 1 \pmod{2}] = 1 - [a \equiv 1 \pmod{2}]$ and $\lfloor \frac{b-a}{2} \rfloor + \lfloor \frac{a}{2} \rfloor - \lfloor \frac{b}{2} \rfloor + [a \equiv 1 \pmod{2}]$. We substitute all of these in (20) to obtain that A_p is equal to

$$\begin{aligned} & \left(\left\lfloor \frac{b-a}{2} \right\rfloor - \left\lfloor \frac{b}{2} \right\rfloor \right) [r' \neq 0] + \left\lfloor \frac{a}{2} \right\rfloor \left([s' \neq 0] + [\ell_1 \not\equiv -\ell_2 \pmod{z(p)}] \right) \\ & + ([r' + s' > 0] + [r' + s' > z(p)]) [a \equiv 1 \pmod{2}] + B_2 + \nu_p \left(\binom{m_p}{k_p} \right) \\ & = \begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv -\ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' + s' < z(p); \\ \left\lfloor \frac{a}{2} \right\rfloor + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)} \text{ and } r' + s' > z(p). \end{cases} \end{aligned}$$

Recall that $r = \ell_1 p^b \pmod{z(p)}$ and $s = \ell_2 p^a \pmod{z(p)}$. Suppose that a is odd and b is even. Then $r = r'$ and $s \equiv -s' \pmod{z(p)}$. Moreover, if $s' \neq 0$, then $s = z(p) - s'$ and thus

$$r < s \Leftrightarrow r' + s' < z(p) \quad \text{and} \quad r > s \Leftrightarrow r' + s' > z(p).$$

Similarly, if a is even and b is odd, then $r \equiv -r' \pmod{z(p)}$ and $s = s'$, and for $r' \neq 0$, we have

$$r < s \Leftrightarrow r' + s' > z(p) \quad \text{and} \quad r > s \Leftrightarrow r' + s' < z(p).$$

From the above observation and the substitution of A_p in Corollary 12, we see that $\nu_p \left(\binom{\ell_1 p^b}{\ell_2 p^a} \right)$ is equal to

$$\begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv -\ell_2 \pmod{z(p)} \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and } \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is odd}; \\ \frac{a}{2} + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \text{ and } a \text{ is even}. \end{cases}$$

Since $a \not\equiv b \pmod{2}$, we see that $p^a \equiv -p^b \pmod{z(p)}$ and therefore

$$\ell_1 \equiv -\ell_2 \pmod{z(p)} \Leftrightarrow r = s.$$

Combining this with (21), we conclude that

$$\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod{2}] \pmod{z(p)} \Leftrightarrow r = s.$$

This completes the proof. \square

5 Examples

In this last section, we give several examples to show applications of our main results. We also recall from Remark 14 that the condition $r = s$ in Theorem 13(ii) can be replaced by $\ell_1 \equiv \ell_2 - 2\ell_2[a \not\equiv b \pmod{2}] \pmod{z(p)}$. In the calculation given in this section, we will use this observation without further reference.

Example 15. Let a, b , and ℓ be positive integers and $b \geq a$. We assert that for $\ell \not\equiv 0 \pmod{3}$, we have

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = \left\lceil \frac{a+1}{2} \right\rceil (\varepsilon_1 \varepsilon_2 + \varepsilon'_1 \varepsilon'_2), \quad (22)$$

where $\varepsilon_1 = [\ell \equiv 2 \pmod{3}]$, $\varepsilon_2 = [a \equiv b \pmod{2}]$, $\varepsilon'_1 = [\ell \equiv 1 \pmod{3}]$, and $\varepsilon'_2 = [a \not\equiv b \pmod{2}]$. In addition, if $\ell \equiv 0 \pmod{3}$, then

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = b + 2 + \nu_2(\ell). \quad (23)$$

Proof. We apply Theorem 13 to verify our assertion. Here $m_2 = \left\lfloor \frac{\ell \cdot 2^{b-a}}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{1}{3} \right\rfloor = 0$. So we immediately obtain the following: if $a \equiv b \pmod{2}$, then

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod{3}; \\ a + 2 + \nu_2(m_2), & \text{if } \ell \equiv 0 \pmod{3}; \\ \left\lceil \frac{a+1}{2} \right\rceil, & \text{if } \ell \equiv 2 \pmod{3}, \end{cases}$$

and if $a \not\equiv b \pmod{2}$, then

$$\nu_2 \left(\binom{\ell \cdot 2^b}{2^a}_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod{3}; \\ a + 2 + \nu_2(m_2), & \text{if } \ell \equiv 0 \pmod{3}; \\ \left\lceil \frac{a+1}{2} \right\rceil, & \text{if } \ell \equiv 2 \pmod{3}. \end{cases}$$

This proves (22). If $\ell \equiv 0 \pmod{3}$, then $m_2 = \frac{\ell}{3} \cdot 2^{b-a}$ and $\nu_2(m_2)$ is equal to

$$\nu_2(m_2) = \nu_2(\ell) + \nu_2(2^{b-a}) - \nu_2(3) = b - a + \nu_2(\ell),$$

which implies (23). □

Example 16. Substituting $\ell = 1$ in Example 15, we see that

$$\begin{aligned} \nu_2 \left(\binom{2^b}{2^a}_F \right) &= \left\lceil \frac{a+1}{2} \right\rceil [a \not\equiv b \pmod{2}] \\ &= \begin{cases} 0, & \text{if } a \equiv b \pmod{2}; \\ \left\lceil \frac{a+1}{2} \right\rceil, & \text{if } a \not\equiv b \pmod{2}. \end{cases} \end{aligned} \quad (24)$$

Our example also implies that (24) still holds for the 2-adic valuations of $\binom{2^{b+2c}}{2^a}_F$, $\binom{7 \cdot 2^b}{2^a}_F$, $\binom{5 \cdot 2^{b+1}}{2^a}_F$, $\binom{13 \cdot 2^b}{2^a}_F$, etc.

Example 17. Let a, b , and ℓ be positive integers, $b \geq a$, and p a prime distinct from 2 and 5. If $p \equiv \pm 1 \pmod{5}$, then

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = (b + \nu_p(F_{z(p)}) + \nu_p(\ell)) [\ell \equiv 0 \pmod{z(p)}],$$

and if $p \equiv \pm 2 \pmod{5}$, then

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 - 2\varepsilon \pmod{z(p)}; \\ b + \nu_p(F_{z(p)}) + \nu_p(\ell), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1 - 2\varepsilon \pmod{z(p)} \text{ and } a \text{ is odd,} \end{cases}$$

where $\varepsilon = [a \not\equiv b \pmod{2}]$.

Proof. Similar to Example 15, we verify this by applying Theorem 13. Here $m_p = \left\lfloor \frac{\ell p^{b-a}}{z(p)} \right\rfloor$, $k_p = \left\lfloor \frac{1}{z(p)} \right\rfloor = 0$, $r = \ell p^b \pmod{z(p)}$, and $s = p^a \pmod{z(p)}$. We first assume that $p \equiv \pm 1 \pmod{5}$. Then by Lemma 1, we have $p \equiv 1 \pmod{z(p)}$. Therefore $s = 1$, $r \equiv \ell \pmod{z(p)}$, and

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = (a + \nu_p(m_p) + \nu_p(F_{z(p)})) [\ell \equiv 0 \pmod{z(p)}].$$

Similarly, if $p \equiv \pm 2 \pmod{5}$ and $a \equiv b \pmod{2}$, then we obtain by Lemma 1 and Theorem 13 that

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = \begin{cases} 0, & \text{if } \ell \equiv 1 \pmod{z(p)}; \\ a + \nu_p(m_p) + \nu_p(F_{z(p)}), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv \pmod{0, 1z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is odd.} \end{cases}$$

In addition, if $p \equiv \pm 2 \pmod{5}$ and $a \not\equiv b \pmod{2}$, then

$$\nu_p \left(\left(\frac{\ell p^b}{p^a} \right)_F \right) = \begin{cases} 0, & \text{if } \ell \equiv -1 \pmod{z(p)}; \\ a + \nu_p(m_p) + \nu_p(F_{z(p)}), & \text{if } \ell \equiv 0 \pmod{z(p)}; \\ \frac{a}{2}, & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is odd.} \end{cases}$$

It remains to calculate $\nu_p(m_p)$ when $\ell \equiv 0 \pmod{z(p)}$. In this case, we have

$$\nu_p(m_p) = \nu_p \left(\frac{\ell p^{b-a}}{z(p)} \right) = \nu_p(\ell) + \nu_p(p^{b-a}) - \nu_p(z(p)) = b - a + \nu_p(\ell).$$

This implies the desired result. □

Example 18. Substituting $\ell = 1$ in Example 17, we see that for $p \neq 2, 5$, we have

$$\nu_p \left(\binom{p^b}{p^a}_F \right) = \begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod{5} \text{ or } a \equiv b \pmod{2}; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod{5}, a \not\equiv b \pmod{2}, \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod{5}, a \not\equiv b \pmod{2}, \text{ and } a \text{ is odd.} \end{cases} \quad (25)$$

Our example also implies that (25) still holds for the p -adic valuations of $\binom{p^{b+2c}}{p^a}_F$ and $\binom{(z(p)+1) \cdot p^b}{p^a}_F$. Similarly, for $p \neq 2, 5$, we have

$$\nu_p \left(\binom{2p^b}{p^a}_F \right) = \begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod{5}; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod{5} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod{5} \text{ and } a \text{ is odd.} \end{cases} \quad (26)$$

In addition, (26) also holds when $\binom{2p^b}{p^a}_F$ is replaced by $\binom{\ell p^b}{p^a}_F$ for $\ell \not\equiv 0, \pm 1 \pmod{z(p)}$ and $p \neq 2, 5$. Furthermore, replacing $\binom{2p^b}{p^a}_F$ by $\binom{(z(p)-1)p^b}{p^a}_F$, the formula becomes

$$\begin{cases} 0, & \text{if } p \equiv \pm 1 \pmod{5} \text{ or } a \not\equiv b \pmod{2}; \\ \frac{a}{2}, & \text{if } p \equiv \pm 2 \pmod{5}, a \equiv b \pmod{2}, \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}), & \text{if } p \equiv \pm 2 \pmod{5}, a \equiv b \pmod{2}, \text{ and } a \text{ is odd.} \end{cases}$$

Example 19. We know that the 5-adic valuations of Fibonomial coefficients are the same as those of binomial coefficients. For example, by Theorem 11(ii) and Kummer's theorem, we obtain

$$\nu_5 \left(\binom{\ell \cdot 5^b}{5^a}_F \right) = \nu_5 \left(\binom{\ell \cdot 5^b}{5^a} \right) = b - a + \nu_5(\ell),$$

for every $a, b, \ell \in \mathbb{N}$ with $b \geq a$. Similarly, $\nu_5 \left(\binom{5^b}{\ell \cdot 5^a}_F \right) = b - a - \nu_5(\ell)$ for every $a, b, \ell \in \mathbb{N}$ such that $5^b > \ell \cdot 5^a$.

Example 20. Let a, b , and ℓ be positive integers and $2^b > \ell \cdot 2^a$. Let $m_2 = \left\lfloor \frac{2^b - a}{3} \right\rfloor$ and $k_2 = \left\lfloor \frac{\ell}{3} \right\rfloor$. Then

$$\nu_2 \left(\binom{2^b}{\ell \cdot 2^a}_F \right) = \nu_2 \left(\binom{m_2}{k_2} \right) + \left(\left\lfloor \frac{a+2}{2} \right\rfloor + \nu_2(m_2 - k_2) \right) \varepsilon_1 \varepsilon_2 + \left\lfloor \frac{a+1}{2} \right\rfloor \varepsilon_3 \varepsilon_4, \quad (27)$$

where $\varepsilon_1 = [a \equiv b \pmod{2}]$, $\varepsilon_2 = [\ell \equiv 2 \pmod{3}]$, $\varepsilon_3 = [a \not\equiv b \pmod{2}]$, and $\varepsilon_4 = [\ell \equiv 1 \pmod{3}]$.

Proof. Similar to Example 15, this follows from the application of Theorem 13. So we leave the details to the reader. \square

Example 21. Let $k \geq 2$. We observe that

$$\left\lfloor \frac{2^k}{3} \right\rfloor = \begin{cases} \frac{2^k-1}{3}, & \text{if } k \text{ is even;} \\ \frac{2(2^{k-1}-1)}{3}, & \text{if } k \text{ is odd,} \end{cases}$$

which implies,

$$\nu_2 \left(\left\lfloor \frac{2^k}{3} \right\rfloor \right) = [k \equiv 1 \pmod{2}]. \quad (28)$$

By a similar reason, we also see that for $k \geq 3$,

$$\nu_2 \left(\left\lfloor \frac{2^k}{3} \right\rfloor - 1 \right) = 2[k \equiv 0 \pmod{2}]. \quad (29)$$

From (27), (28), and (29), we obtain the following results:

(i) if $b - a \geq 2$, then $\nu_2 \left(\binom{2^b}{3 \cdot 2^a}_F \right) = [a \not\equiv b \pmod{2}]$,

(ii) if $b - a \geq 3$, then $\nu_2 \left(\binom{2^b}{5 \cdot 2^a}_F \right)$ is equal to

$$\begin{aligned} & [a \not\equiv b \pmod{2}] + \left(\left\lfloor \frac{a+2}{2} \right\rfloor + 2[a \equiv b \pmod{2}] \right) [a \equiv b \pmod{2}] \\ & = 1 + \left\lfloor \frac{a+4}{2} \right\rfloor [a \equiv b \pmod{2}], \end{aligned}$$

(iii) if $b - a \geq 3$, then $\nu_2 \left(\binom{2^b}{6 \cdot 2^a}_F \right) = [a \equiv b \pmod{2}]$,

(iv) if $b - a \geq 4$, then $\nu_2 \left(\binom{2^b}{7 \cdot 2^a}_F \right) = [a \equiv b \pmod{2}] + \left\lfloor \frac{a+1}{2} \right\rfloor [a \not\equiv b \pmod{2}]$.

Example 22. Let $p \neq 5$ be an odd prime and let a, b , and ℓ be positive integers, $p^b > \ell p^a$, $m_p = \left\lfloor \frac{p^{b-a}}{z(p)} \right\rfloor$, and $k_p = \left\lfloor \frac{\ell}{z(p)} \right\rfloor$. Then the following statements hold.

(i) If $p \equiv \pm 1 \pmod{5}$, then

$$\nu_p \left(\binom{p^b}{\ell p^a}_F \right) = (a + \nu_p(m_p - k_p) + \nu_p(F_{z(p)})) [\ell \not\equiv 0, 1 \pmod{z(p)}] + \nu_p \left(\binom{m_p}{k_p} \right),$$

(ii) If $p \equiv \pm 2 \pmod{5}$, then $\nu_p \left(\binom{p^b}{\ell p^a}_F \right)$ is equal to

$$\nu_p \left(\binom{m_p}{k_p} \right) + \varepsilon_1 \varepsilon_2 \varepsilon_3 \left(\left\lfloor \frac{a}{2} \right\rfloor + \nu_p(m_p - k_p) + \varepsilon_3 \nu_p(F_{z(p)}) \right) + \varepsilon_1 \varepsilon_4 (1 - \varepsilon_5) \left(\left\lfloor \frac{a}{2} \right\rfloor + \varepsilon_3 \nu_p(F_{z(p)}) \right) \quad (30)$$

where $\varepsilon_1 = [\ell \not\equiv 0 \pmod{z(p)}]$, $\varepsilon_2 = [\ell \not\equiv 1 \pmod{z(p)}]$, $\varepsilon_3 = [b \equiv 0 \pmod{2}]$, $\varepsilon_4 = [\ell \not\equiv -1 \pmod{z(p)}]$, and $\varepsilon_5 = [a \equiv b \pmod{2}]$.

Proof. Similar to Example 17, this follows from the application of Lemma 1 and Theorem 13. Since (i) is easily verified, we only give the proof of (ii). The calculation is done in two cases. If $p \equiv \pm 2 \pmod{5}$ and $a \equiv b \pmod{2}$, then $\nu_p \left(\binom{p^b}{\ell p^a}_F \right)$ is equal to

$$\begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \equiv 0, 1 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p(F_{z(p)}) + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a+1}{2} + \nu_p(m_p - k_p) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, 1 \pmod{z(p)} \text{ and } a \text{ is odd}, \end{cases}$$

$$= \nu_p \left(\binom{m_p}{k_p} \right) + \varepsilon_1 \varepsilon_2 \left(\left\lfloor \frac{a}{2} \right\rfloor + \nu_p(m_p - k_p) + \varepsilon_3 \nu_p(F_{z(p)}) \right),$$

where $\varepsilon_1 = [\ell \not\equiv 0 \pmod{z(p)}]$, $\varepsilon_2 = [\ell \not\equiv 1 \pmod{z(p)}]$, and $\varepsilon_3 = [b \equiv 0 \pmod{2}]$. If $p \equiv \pm 2 \pmod{5}$ and $a \not\equiv b \pmod{2}$, then

$$\nu_p \left(\binom{p^b}{\ell p^a}_F \right) = \begin{cases} \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \equiv 0, -1 \pmod{z(p)}; \\ \frac{a}{2} + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is even}; \\ \frac{a-1}{2} + \nu_p(F_{z(p)}) + \nu_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell \not\equiv 0, -1 \pmod{z(p)} \text{ and } a \text{ is odd}, \end{cases}$$

$$= \nu_p \left(\binom{m_p}{k_p} \right) + \varepsilon_1 \varepsilon_4 \left(\left\lfloor \frac{a}{2} \right\rfloor + \varepsilon_3 \nu_p(F_{z(p)}) \right),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are as above and $\varepsilon_4 = [\ell \not\equiv -1 \pmod{z(p)}]$. Let $\varepsilon_5 = [a \equiv b \pmod{2}]$. Then both cases can be combined to obtain (ii). \square

Example 23. Let $k \geq 2$. We observe that $z(7) = 8$ and

$$\left\lfloor \frac{7^k}{8} \right\rfloor = \begin{cases} \frac{7^k - 1}{8}, & \text{if } k \text{ is even}; \\ \frac{7(7^{k-1} - 1)}{8}, & \text{if } k \text{ is odd}. \end{cases}$$

Therefore

$$\nu_7 \left(\left\lfloor \frac{7^k}{8} \right\rfloor \right) = [k \equiv 1 \pmod{2}] \quad \text{and} \quad \nu_7 \left(\left\lfloor \frac{7^k}{8} \right\rfloor - 1 \right) = 0. \quad (31)$$

From (30) and (31), we obtain the following results:

- (i) if $b - a \geq 2$, then $\nu_7 \left(\binom{7^b}{(8 \cdot 7^a)}_F \right) = [a \not\equiv b \pmod{2}]$,
- (ii) if $b - a \geq 2$, then $\nu_7 \left(\binom{7^b}{(9 \cdot 7^a)}_F \right) = \left(\left\lfloor \frac{a+2}{2} \right\rfloor + [b \equiv 0 \pmod{2}] \right) [a \not\equiv b \pmod{2}]$,
- (iii) if $b - a \geq 2$, then $\nu_7 \left(\binom{7^b}{(15 \cdot 7^a)}_F \right)$ is equal to

$$[a \not\equiv b \pmod{2}] + \left(\left\lfloor \frac{a}{2} \right\rfloor + [b \equiv 0 \pmod{2}] \right) [a \equiv b \pmod{2}].$$

To keep this article not too lengthy, we plan to give more applications of our main results in the next article.

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Reciprocal Sum of Palindromes

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Abstract

A positive integer n is a b -adic palindrome if the representation of n in base b reads the same backward as forward. Let s_b be the reciprocal sum of all b -adic palindromes. In this article, we obtain upper and lower bounds, and an asymptotic formula for s_b . We also show that the sequence $(s_b)_{b \geq 2}$ is strictly increasing and log-concave.

1 Introduction

Let $n \geq 1$ and $b \geq 2$ be integers. We call n a *palindrome in base b* (or *b -adic palindrome*) if the b -adic expansion of $n = (a_k a_{k-1} \cdots a_0)_b$ with $a_k \neq 0$ has the symmetric property $a_{k-i} = a_i$ for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$. As usual, if we write a number without specifying the base, then it is always in base 10. So, for example, $9 = (1001)_2 = (100)_3$ is a palindrome in bases 2 and 10 but not in base 3.

In recent years, there has been an increasing interest in the importance of palindromes in mathematics [1, 2, 3, 13, 17, 25], theoretical computer science [4, 9, 12], and theoretical physics [11, 14]. There are also some discussions on the reciprocal sum of palindromes on the internet but as far as we are aware, our observation has not appeared in the literature. Throughout this article, we let $b \geq 2$, s_b the reciprocal sum of all b -adic palindromes, and $s_{b,k}$ the reciprocal sum of all b -adic palindromes which have k digits in their b -adic expansions.

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The set of all b -adic palindromes is infinite but quite sparse, so it is not difficult to see that s_b converges. In fact, Shallit proposed the convergence of s_b as a problem in the *Fibonacci Quarterly* in 1980 [26, 27].

In this article, we obtain upper and lower bounds for s_b , which enable us to show that $s_{b+1} > s_b$ for all $b \geq 2$ and $s_b^2 - s_{b-1}s_{b+1} > 0$ for all $b \geq 3$. That is, the sequence $(s_b)_{b \geq 2}$ is strictly increasing and log-concave. Furthermore, we give an asymptotic formula for s_b of the form $s_b = g(b) + O(h(b))$ where the implied constant can be taken to be 1 and the order of magnitude of $h(b)$ is $\frac{\log b}{b^3}$ as $b \rightarrow \infty$. Our result $s_{b+1} > s_b$ for all $b \geq 2$ also implies that if $b_1 > b_2 \geq 2$ and if we use the logarithmic measure, then we can say that the palindromes in base b_1 occur more often than those in base b_2 . On the other hand, if we use the usual counting measure, then we obtain from Pongsriiam and Subwattanachai's exact formula [22] that the number of palindromes in different bases which are less than or equal to N are not generally comparable. It seems that there are races between palindromes in different bases which may be similar to races between primes in different residue classes. We will get back to this problem in the near future.

The reciprocal sum of an integer sequence is also of general interest in mathematics and theoretical physics as proposed by Bayless and Klyve [8], and by Roggero, Nardelli, and Di Noto [24]. See also the work of Nguyen and Pomerance [19] on the reciprocal sum of the amicable numbers, the preprint of Kinlaw, Kobayashi, and Pomerance [15] on the reciprocal sum of the positive integers n satisfying $\varphi(n) = \varphi(n+1)$, and the article by Lichtman [16] on the reciprocal sum of primitive nondeficient numbers. In addition, Banks [5], Cilleruelo, Luca, and Baxter [10], and Rajasekaran, Shallit, and Smith [23] have recently investigated some additive properties of palindromes while Banks, Hart, and Sakata [6] and Banks and Shparlinski [7] show some of their multiplicative properties. For more information concerning palindromes, we refer the reader to the entry [A002113](#) in the On-Line Encyclopedia of Integer Sequences (OEIS) [28].

2 Results

Throughout this section, a, c, m, n, k, ℓ denote positive integers, and x, y, z denote positive real numbers. Furthermore,

- $\lfloor x \rfloor$ is the greatest integer less than or equal to x ;
- $\lceil x \rceil$ is the least integer greater than or equal to x ;
- $\log x$ is the natural logarithm of x ;
- $x_b = \sum_{m=1}^{b-1} \frac{1}{m}$;
- $y_b = \sum_{m=b}^{b^2-1} \frac{1}{m}$; and
- $z_b = \sum_{m=b+1}^{b^2} \frac{1}{m}$.

Note that $x_b = s_{b,1}$ and $x_b/(b+1) = s_{b,2}$ and that

$$z_b - y_b = \frac{1}{b^2} - \frac{1}{b} = \frac{1-b}{b^2}.$$

Theorem 1. *We have*

$$\frac{y_b}{b} - \frac{x_b}{b^3} \leq s_{b,3} \leq \frac{y_b}{b} \quad \text{and} \quad \frac{z_b}{b^{\lfloor \frac{k}{2} \rfloor}} \leq s_{b,k} \leq \frac{y_b}{b^{\lfloor \frac{k}{2} \rfloor}} \quad \text{for every } k \geq 4.$$

Proof. We first consider the case $k = 3$. The b -adic palindromes which have 3 digits are of the form $(aca)_b$ where $1 \leq a \leq b-1$ and $0 \leq c \leq b-1$. Since

$$\left| \frac{1}{(aca)_b} - \frac{1}{(ac0)_b} \right| = \left| \frac{-a}{(ab^2 + cb + a)(ab^2 + cb)} \right| \leq \frac{a}{(ab^2)^2} = \frac{1}{ab^4},$$

we obtain

$$\frac{1}{(ac0)_b} - \frac{1}{ab^4} \leq \frac{1}{(aca)_b} \leq \frac{1}{(ac0)_b}. \quad (1)$$

Observe that

$$\sum_{\substack{1 \leq a \leq b-1 \\ 0 \leq c \leq b-1}} \frac{1}{(ac0)_b} = \frac{1}{b} \sum_{\substack{1 \leq a \leq b-1 \\ 0 \leq c \leq b-1}} \frac{1}{ab+c} = \frac{y_b}{b} \quad \text{and} \quad \sum_{\substack{1 \leq a \leq b-1 \\ 0 \leq c \leq b-1}} \frac{1}{ab^4} = \frac{x_b}{b^3}.$$

So by summing (1) over all $a = 1, 2, \dots, b-1$ and $c = 0, 1, \dots, b-1$, we obtain the inequality $\frac{y_b}{b} - \frac{x_b}{b^3} \leq s_{b,3} \leq \frac{y_b}{b}$. For $k = 4$, $1 \leq a \leq b-1$, and $0 \leq c \leq b-1$, we have

$$\frac{1}{b^2(ab+c+1)} = \frac{1}{(ac00)_b + b^2} \leq \frac{1}{(acca)_b} \leq \frac{1}{(ac00)_b} = \frac{1}{b^2(ab+c)}.$$

Summing over all $a = 1, 2, \dots, b-1$ and $c = 0, 1, \dots, b-1$ leads to

$$\frac{z_b}{b^2} = \frac{1}{b^2} \sum_{\substack{1 \leq a \leq b-1 \\ 0 \leq c \leq b-1}} \frac{1}{ab+c+1} \leq s_{b,4} \leq \frac{1}{b^2} \sum_{\substack{1 \leq a \leq b-1 \\ 0 \leq c \leq b-1}} \frac{1}{ab+c} = \frac{y_b}{b^2}.$$

Let $k \geq 5$. The b -adic palindromes which have k digits are of the form $(aa_1a_2 \cdots a_{k-2}a)_b$ where $1 \leq a \leq b-1$, $0 \leq a_i \leq b-1$ for all $i \in \{1, 2, \dots, k-2\}$ with the usual symmetric property on a_i . We fix a and a_1 and count the number of palindromes in this form. There are b choices for $a_2 \in \{0, 1, 2, \dots, b-1\}$ and so there is only 1 choice for $a_{k-3} = a_2$. Similarly, there are b choices for a_3 and 1 choice for a_{k-4} . By continuing this counting, we see that the number of palindromes in this form (when a and a_1 are already chosen) is equal to $b^{\lfloor \frac{k-4}{2} \rfloor}$. Therefore the reciprocal sum of such palindromes satisfies

$$\sum_{a_2, \dots, a_{k-3}} \frac{1}{(aa_1a_2 \cdots a_{k-2}a)_b} \leq \frac{b^{\lfloor \frac{k-4}{2} \rfloor}}{(aa_10 \cdots 0a_1a)_b} \leq \frac{b^{\lfloor \frac{k-4}{2} \rfloor}}{b^{k-2}(ab+a_1)} = \frac{1}{b^{\lfloor \frac{k}{2} \rfloor}(ab+a_1)},$$

where a_2, \dots, a_{k-3} run over all integers $0, 1, 2, \dots, b-1$ with the symmetric condition of palindromes. Hence

$$s_{b,k} = \sum_{\substack{1 \leq a \leq b-1 \\ 0 \leq a_1 \leq b-1}} \sum_{a_2, \dots, a_{k-3}} \frac{1}{(aa_1a_2 \cdots a_{k-2}a)_b} \leq \sum_{\substack{1 \leq a \leq b-1 \\ 0 \leq a_1 \leq b-1}} \frac{1}{b^{\lfloor \frac{k}{2} \rfloor} (ab + a_1)} = \frac{y_b}{b^{\lfloor \frac{k}{2} \rfloor}}.$$

Similarly, if a and a_1 are fixed, then

$$\sum_{a_2, \dots, a_{k-3}} \frac{1}{(aa_1a_2 \cdots a_{k-2}a)_b} \geq \frac{b^{\lceil \frac{k-4}{2} \rceil}}{ab^{k-1} + a_1b^{k-2} + b^{k-2}} = \frac{1}{b^{\lfloor \frac{k}{2} \rfloor} (ab + a_1 + 1)}.$$

Summing the above over all $a = 1, 2, \dots, b-1$ and $a_1 = 0, 1, \dots, b-1$, we obtain the desired lower bound for $s_{b,k}$. This completes the proof. \square

Theorem 2. For every $b, \ell \geq 2$, we have

$$\left(\frac{b+2}{b+1}\right) x_b + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2z_b}{(b-1)b^{\ell-1}} \leq s_b \leq \left(\frac{b+2}{b+1}\right) x_b + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2y_b}{(b-1)b^{\ell-1}}.$$

In particular,

$$\left(\frac{b+2}{b+1}\right) x_b + \frac{y_b}{b} - \frac{x_b}{b^3} + \frac{2z_b}{b(b-1)} \leq s_b \leq \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b(b-1)}\right) y_b. \quad (2)$$

Proof. For simplicity, we write x, y, z instead of x_b, y_b, z_b , respectively. We consider $s_{b,k}$ for each k as follows. Obviously $s_{b,1} = 1 + \frac{1}{2} + \cdots + \frac{1}{b-1} = x$. For $k = 2$, $s_{b,k}$ is

$$\sum_{a=1}^{b-1} \frac{1}{(aa)_b} = \sum_{a=1}^{b-1} \frac{1}{a(b+1)} = \frac{x}{b+1}.$$

By writing $s_b = x + \frac{x}{b+1} + \sum_{k=3}^{2\ell-1} s_{b,k} + \sum_{k=2\ell}^{\infty} s_{b,k}$ and applying Theorem 1, we obtain

$$s_b \leq \frac{b+2}{b+1} x + \sum_{k=3}^{2\ell-1} s_{b,k} + \sum_{k=2\ell}^{\infty} \frac{y}{b^{\lfloor \frac{k}{2} \rfloor}} = \frac{b+2}{b+1} x + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2y}{(b-1)b^{\ell-1}}.$$

Similarly,

$$s_b \geq \frac{b+2}{b+1} x + \sum_{k=3}^{2\ell-1} s_{b,k} + \sum_{k=2\ell}^{\infty} \frac{z}{b^{\lfloor \frac{k}{2} \rfloor}} = \left(\frac{b+2}{b+1}\right) x + \sum_{k=3}^{2\ell-1} s_{b,k} + \frac{2z}{(b-1)b^{\ell-1}}.$$

This proves the first part of this theorem. The second part follows from Theorem 1 and the substitution $\ell = 2$ in the first part. \square

Theorem 3. *The sequence $(s_b)_{b \geq 2}$ is strictly increasing.*

Proof. We first verify that $s_{b+1} > s_b$ for $2 \leq b \leq 16$. Myers gives the decimal expansion of s_2 in the entry [A244162](#) in the OEIS [28]. Myers also describe the algorithm in his calculation, which can be found in the web page [18]. So we know that $s_2 < 2.3787957$. Alternatively, substituting $\ell = 3$ and $b = 2, 3$ in Theorem 2 and running the computation in a computer, we obtain $2.32137259 \leq s_2 \leq 2.44637260$ and $2.60503980 \leq s_3 \leq 2.62973117$, which implies that $s_2 < s_3$. Similarly, we apply (2) to obtain upper and lower bounds for s_b and we see that $s_b < s_{b+1}$ for $3 \leq b \leq 16$. So we assume throughout that $b \geq 16$. We first observe that

$$y_b = z_b + \frac{1}{b} - \frac{1}{b^2} \quad \text{and} \quad \frac{y_b}{b} - \frac{z_b}{b} = \frac{b-1}{b^3} \geq \frac{x_b}{b^3}.$$

Therefore $\frac{y_b}{b} - \frac{x_b}{b^3} \geq \frac{z_b}{b}$ for all $b \geq 2$. So the term $\frac{y_b}{b} - \frac{x_b}{b^3}$ in (2) can be replaced by $\frac{z_b}{b}$. Therefore we obtain by (2) that $s_{b+1} - s_b$ is larger than

$$\left(\frac{b+3}{b+2}\right)x_{b+1} - \left(\frac{b+2}{b+1}\right)x_b + \left(\frac{1}{b+1} + \frac{2}{(b+1)b}\right)z_{b+1} - \left(\frac{1}{b} + \frac{2}{b(b-1)}\right)y_b. \quad (3)$$

In addition, $z_{b+1} - z_b$ is equal to

$$-\frac{1}{b+1} + \sum_{m=b^2+1}^{b^2+2b+1} \frac{1}{m} \geq -\frac{1}{b+1} + \frac{2b+1}{b^2+2b+1} = \frac{b}{(b+1)^2} > 0.$$

Since $x_{b+1} = x_b + \frac{1}{b}$ and $z_{b+1} > z_b = y_b - \frac{1}{b} + \frac{1}{b^2}$, we obtain from (3) that $s_{b+1} - s_b$ is larger than or equal to

$$\begin{aligned} & \left(\frac{b+3}{b+2}\right)\left(\frac{1}{b}\right) + x_b \left(\frac{b+3}{b+2} - \frac{b+2}{b+1}\right) + y_b \left(\frac{1}{b+1} + \frac{2}{b(b+1)} - \frac{1}{b} - \frac{2}{b(b-1)}\right) \\ & + \left(\frac{1}{b^2} - \frac{1}{b}\right)\left(\frac{1}{b+1} + \frac{2}{b(b+1)}\right) \\ & = \frac{1}{b} + \frac{1}{b(b+2)} - \frac{x_b}{(b+1)(b+2)} - \frac{(b+3)y_b}{b(b-1)(b+1)} - \frac{(b-1)(b+2)}{b^3(b+1)}. \end{aligned}$$

Recall that if a and b are integers, $a < b$, and f is monotone on $[a, b]$, then

$$\min\{f(a), f(b)\} \leq \sum_{n=a}^b f(n) - \int_a^b f(t)dt \leq \max\{f(a), f(b)\}. \quad (4)$$

From (4), we obtain

$$\begin{aligned} x_b &= \sum_{m=1}^{b-1} \frac{1}{m} \leq 1 + \log(b-1) \leq \frac{3}{2} \log b, \\ y_b &= -\frac{1}{b-1} + \sum_{m=b-1}^{b^2-1} \frac{1}{m} \leq \log(b+1) \leq \frac{5}{4} \log b. \end{aligned}$$

In addition, it is straightforward to verify that

$$\frac{1}{b(b+2)} - \frac{(b-1)(b+2)}{b^3(b+1)} > -\frac{1}{b^2}.$$

Therefore $s_{b+1} - s_b$ is larger than

$$\frac{1}{b} - \frac{3 \log b}{2(b+1)(b+2)} - \frac{5(b+3) \log b}{4b(b-1)(b+1)} - \frac{1}{b^2} > \frac{1}{b} - \frac{3 \log b}{2b^2} - \frac{3 \log b}{2b^2} - \frac{1}{b^2} = \frac{1}{b} - \frac{1}{b^2} - \frac{3 \log b}{b^2}. \quad (5)$$

Observe that the function $x \mapsto \frac{\log x}{x}$ is decreasing on $[3, \infty)$. Since $b \geq 16$, we obtain

$$\frac{3 \log b}{b} \leq \frac{3 \log 16}{16} < \frac{7}{10} \quad \text{and} \quad \frac{1}{b} < \frac{1}{10}.$$

Hence we obtain from (5) that

$$s_{b+1} - s_b > \frac{1}{b} - \frac{1}{b^2} - \frac{3 \log b}{b^2} > \frac{1}{b} - \frac{1}{10b} - \frac{7}{10b} = \frac{1}{5b} > 0.$$

This completes the proof. \square

Recall that if we write $f(b) = g(b) + O^*(h(b))$, then it means that $f(b) = g(b) + O(h(b))$ and the implied constant can be taken to be 1. In addition, $f(b) = g(b) + \Omega_+(h(b))$ means $\limsup_{b \rightarrow \infty} \frac{f(b) - g(b)}{h(b)} > 0$. From this point on, we use (4) without further reference.

Theorem 4. *Uniformly for $b \geq 2$,*

$$s_b = \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b^2}\right) y_b + O^*\left(\frac{5 \log b}{b^3}\right).$$

This estimate is sharp in the sense that $O^\left(\frac{5 \log b}{b^3}\right)$ can be replaced by $\Omega_+\left(\frac{\log b}{b^3}\right)$.*

Proof. Let $g(b) = \left(\frac{b+2}{b+1}\right) x_b + \left(\frac{1}{b} + \frac{2}{b^2}\right) y_b$ be the main term above. Since $y_b \leq \log(b+1)$, we obtain by (2) that

$$s_b - g(b) \leq \frac{2y_b}{b^2(b-1)} \leq \frac{2 \log(b+1)}{b^2(b-1)}.$$

If $b = 2$, then it is easy to check that $\frac{2y_b}{b^2(b-1)} = \frac{5}{12} < \frac{5 \log b}{b^3}$. If $b \geq 3$, then we assert that $\frac{2 \log(b+1)}{b^2(b-1)} \leq \frac{5 \log b}{b^3}$. To verify this assertion, we observe that it is equivalent to $\left(\frac{5}{2} \log b\right) \left(\frac{b-1}{b}\right) \geq \log(b+1)$. Since $b \geq 3$, we obtain

$$\left(\frac{5}{2} \log b\right) \left(\frac{b-1}{b}\right) \geq \frac{5}{3} \log b \geq \log 2 + \log b = \log(2b) \geq \log(b+1), \text{ as desired.}$$

So in any case,

$$s_b - g(b) \leq \frac{5 \log b}{b^3}. \quad (6)$$

We also obtain by (2) that $s_b - g(b)$ is larger than or equal to

$$\frac{2z_b}{b(b-1)} - \frac{2y_b}{b^2} - \frac{x_b}{b^3} = \frac{2z_b}{b(b-1)} - \frac{2(z_b + \frac{1}{b} - \frac{1}{b^2})}{b^2} - \frac{x_b}{b^3} = \frac{2z_b}{b^2(b-1)} - \frac{x_b}{b^3} - \frac{2}{b^3} + \frac{2}{b^4}. \quad (7)$$

We have

$$x_b = \sum_{m=1}^{b-1} \frac{1}{m} \leq 1 + \log(b-1) \leq 2 \log b, \quad (8)$$

$$z_b = \sum_{m=b}^{b^2} \frac{1}{m} - \frac{1}{b} \geq \int_b^{b^2} \frac{1}{t} dt + \frac{1}{b^2} - \frac{1}{b} \geq \log b - \frac{1}{b} \geq \frac{\log b}{4}. \quad (9)$$

Therefore (7) implies that

$$s_b - g(b) \geq \frac{\log b}{2b^2(b-1)} - \frac{2 \log b}{b^3} - \frac{2}{b^3} \geq \frac{\log b}{2b^3} - \frac{2 \log b}{b^3} - \frac{3 \log b}{b^3} > -\frac{5 \log b}{b^3}. \quad (10)$$

By (6) and (10), we obtain $|s_b - g(b)| \leq \frac{5 \log b}{b^3}$. This proves the first part of this theorem. For the Ω_+ result, we only need to observe that as $b \rightarrow \infty$, (7) and the inequalities $x_b \leq 1 + \log(b-1)$ and $z_b \geq \log b - \frac{1}{b}$ given in (8) and (9) imply that

$$s_b - g(b) \geq \frac{2(\log b - \frac{1}{b})}{b^2(b-1)} - \frac{1 + \log(b-1)}{b^3} - \frac{2}{b^3} + \frac{2}{b^4} > \frac{3 \log b}{2b^2(b-1)} - \frac{11 \log b}{10b^3} - \frac{2}{b^3} + \frac{2}{b^4},$$

so

$$\limsup_{b \rightarrow \infty} \frac{s_b - g(b)}{\left(\frac{\log b}{b^3}\right)} \geq \frac{3}{2} - \frac{11}{10} > 0.$$

This completes the proof. \square

Recall that by applying Euler-Maclaurin summation formula, we get

$$\sum_{m \leq n} \frac{1}{m} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\theta_n}{60n^4}, \quad (11)$$

where γ is Euler's constant and $\theta_n \in [0, 1]$. The calculation of (11) can be found in Tenenbaum [30, p. 6]. From this, we obtain another form of Theorem 4 as follows.

Theorem 5. *Uniformly for $b \geq 2$,*

$$s_b = \log b + \gamma + \left(\frac{1}{b} + \frac{1}{b+1}\right) \log b + \frac{\gamma}{b+1} - \frac{1}{2b} + \frac{2 \log b}{b^2} - \frac{1}{12b(b+1)} + O^*\left(\frac{6 \log b}{b^3}\right). \quad (12)$$

This estimate is sharp in the sense that $O^\left(\frac{6 \log b}{b^3}\right)$ is also $\Omega_+\left(\frac{\log b}{b^3}\right)$.*

Proof. By (11), we have

$$\begin{aligned}
x_b &= \sum_{m \leq b} \frac{1}{m} - \frac{1}{b} = \log b + \gamma - \frac{1}{2b} - \frac{1}{12b^2} + \frac{\theta_b}{60b^4}, \\
z_b &= \sum_{m \leq b^2} \frac{1}{m} - \sum_{m \leq b} \frac{1}{m} = \left(\log b^2 + \gamma + \frac{1}{2b^2} - \frac{1}{12b^4} + \frac{\theta_{b^2}}{60b^8} \right) - \left(\log b + \gamma + \frac{1}{2b} - \frac{1}{12b^2} + \frac{\theta_b}{60b^4} \right) \\
&= \log b - \frac{1}{2b} + \frac{7}{12b^2} - \frac{5 + \theta_b}{60b^4} + \frac{\theta_{b^2}}{60b^8}, \\
y_b &= z_b + \frac{1}{b} - \frac{1}{b^2} = \log b + \frac{1}{2b} - \frac{5}{12b^2} - \frac{5 + \theta_b}{60b^4} + \frac{\theta_{b^2}}{60b^8}.
\end{aligned}$$

Writing $\frac{b+2}{b+1} = 1 + \frac{1}{b+1}$ and substituting x_b and y_b in Theorem 4, we obtain

$$s_b = h(b) + h_1(b) + O^* \left(\frac{5 \log b}{b^3} \right) \quad (13)$$

where $h(b)$ is the main term given in (12) and

$$h_1(b) = \frac{11b^2 - 3b - 10}{12b^4(b+1)} + \frac{\theta_b}{60b^4} + \frac{\theta_b}{60b^4(b+1)} + \frac{\theta_{b^2}}{60b^8} + \frac{\theta_{b^2}}{30b^{10}} - \frac{5 + \theta_b}{60b^5} - \frac{5 + \theta_b}{30b^6}.$$

It is not difficult to see that $h_1(b) \geq 0$ and

$$h_1(b) \leq \frac{11b^2}{12b^4(b+1)} + \frac{1}{60b^4} + \frac{1}{60b^4(b+1)} \leq \frac{11 \log b}{12b^3} + \frac{\log b}{60b^3} + \frac{\log b}{60b^3} \leq \frac{\log b}{b^3}. \quad (14)$$

Therefore (13) implies that $s_b = h(b) + O^* \left(\frac{6 \log b}{b^3} \right)$, which is the same as (12). In addition, by the first inequality given in (14), we see that $h_1(b) \ll \frac{1}{b^3}$. Since $O^* \left(\frac{5 \log b}{b^3} \right)$ in (13) is $\Omega_+ \left(\frac{\log b}{b^3} \right)$ and $h_1(b) \ll \frac{1}{b^3}$, $h_1(b) + O^* \left(\frac{5 \log b}{b^3} \right)$ in (13) is $\Omega_+ \left(\frac{\log b}{b^3} \right)$. This completes the proof. \square

Corollary 6. *The sequence $(s_b)_{b \geq 2}$ diverges to $+\infty$ and the sequence $(s_b - s_{b-1})_{b \geq 3}$ converges to zero as $b \rightarrow \infty$.*

Proof. The first assertion follows immediately from Theorem 5. Recall that $\log(b-1) = \log b + O \left(\frac{1}{b} \right)$. So we obtain by Theorem 5 that as $b \rightarrow \infty$,

$$0 < s_b - s_{b-1} = \log b - \log(b-1) + O \left(\frac{\log b}{b} \right) \ll \frac{\log b}{b},$$

which implies our assertion. \square

Recall that a sequence $(a_n)_{n \geq 0}$ is said to be log-concave if $a_n^2 - a_{n-1}a_{n+1} > 0$ for every $n \geq 1$ and is said to be log-convex if $a_n^2 - a_{n-1}a_{n+1} < 0$ for every $n \geq 1$. For a survey article concerning the log-concavity and log-convexity of sequences, we refer the reader to Stanley [29]. See also Pongsriam [21] for some combinatorial sequences which are log-concave or log-convex, and some open problems concerning the log-properties of a certain sequence.

Theorem 7. *The sequence $(s_b)_{b \geq 2}$ is log-concave.*

Proof. For each $b = 2, 3, \dots, 15$, we use Theorem 2 with $\ell = 4$ to get an upper bound C_b and a lower bound D_b for s_b . In addition, for each $b \geq 13$, let U_b and L_b be the upper and lower bounds of s_b given in (2), respectively. Then

$s_b^2 - s_{b-1}s_{b+1} > D_b^2 - C_{b-1}C_{b+1}$ for $3 \leq b \leq 14$ and $s_b^2 - s_{b-1}s_{b+1} > L_b^2 - U_{b-1}U_{b+1}$ for $b \geq 14$.

We use MATLAB to check that $D_b^2 - C_{b-1}C_{b+1} > 0$ for $3 \leq b \leq 14$ and $L_b^2 - U_{b-1}U_{b+1} > 0$ for $14 \leq b \leq 1500$. So $s_b^2 - s_{b-1}s_{b+1} > 0$ for $3 \leq b \leq 1500$. So we assume throughout that $b > 1500$. Then

$$\begin{aligned} U_{b-1}U_{b+1} &= \frac{(b+1)(b+3)}{b(b+2)}x_{b-1}x_{b+1} + \frac{b+2}{(b-2)(b-1)(b+1)}y_{b-1}y_{b+1} \\ &\quad + \frac{b(b+3)}{(b-2)(b-1)(b+2)}x_{b+1}y_{b-1} + \frac{b+2}{b^2}x_{b-1}y_{b+1} \\ &= A_1 + A_2 + A_3 + A_4, \text{ say.} \end{aligned} \tag{15}$$

Since

$$\begin{aligned} z_b &= y_b + \frac{1}{b^2} - \frac{1}{b}, \quad b^4 + 2b^3 - b - 1 \geq b^4 + 2b^3 - 2b - 1 = (b-1)(b+1)^3, \text{ and} \\ L_b &= \frac{b^4 + 2b^3 - b - 1}{b^3(b+1)}x_b + \frac{y_b}{b} + \frac{2z_b}{b(b-1)}, \end{aligned}$$

we obtain

$$L_b = \frac{b^4 + 2b^3 - b - 1}{b^3(b+1)}x_b + \frac{b+1}{b(b-1)}y_b - \frac{2}{b^3} \geq \frac{(b-1)(b+1)^2}{b^3}x_b + \frac{b+1}{b(b-1)}y_b - \frac{2}{b^3}.$$

Therefore L_b^2 is larger than or equal to

$$\begin{aligned} &\frac{(b-1)^2(b+1)^4}{b^6}x_b^2 + \frac{(b+1)^2}{b^2(b-1)^2}y_b^2 + \frac{4}{b^6} + \frac{2(b+1)^3}{b^4}x_b y_b - \frac{4(b-1)(b+1)^2}{b^6}x_b - \frac{4(b+1)}{b^4(b-1)}y_b \\ &\geq \frac{(b-1)^2(b+1)^4}{b^6}x_b^2 + \frac{(b+1)^2}{b^2(b-1)^2}y_b^2 + \frac{2(b+1)^3}{b^4}x_b y_b - \frac{4(b-1)(b+1)^2}{b^6}x_b - \frac{4(b+1)}{b^4(b-1)}y_b \\ &= B_1 + B_2 + B_3 - B_4 - B_5, \text{ say.} \end{aligned} \tag{16}$$

In addition, we see that

$$\begin{aligned} y_{b-1} &= y_b + \frac{1}{b-1} - \sum_{m=(b-1)^2}^{b^2-1} \frac{1}{m} \leq y_b + \frac{1}{b-1} - \frac{2b-1}{b^2-1} \leq y_b - \frac{b-2}{b^2}, \\ y_{b+1} &= y_b - \frac{1}{b} + \sum_{m=b^2}^{b^2+2b} \frac{1}{m} \leq y_b - \frac{1}{b} + \frac{2b+1}{b^2} = y_b + \frac{b+1}{b^2}, \\ x_{b-1} &= x_b - \frac{1}{b-1}, \quad \text{and} \quad x_{b+1} = x_b + \frac{1}{b}. \end{aligned}$$

From these, we obtain the following inequalities:

$$\begin{aligned}
A_1 &= \frac{(b+1)(b+3)}{b(b+2)}x_b^2 - \frac{(b+1)(b+3)}{b^2(b-1)(b+2)}x_b - \frac{(b+1)(b+3)}{b^2(b-1)(b+2)} \\
&\leq \frac{(b+1)(b+3)}{b(b+2)}x_b^2 - \frac{(b+1)(b+3)}{b^2(b-1)(b+2)}x_b - \frac{1}{b^2}, \\
A_2 &\leq \frac{b+2}{(b-2)(b-1)(b+1)}y_b^2 + \frac{3(b+2)}{b^2(b-2)(b-1)(b+1)}y_b - \frac{b+2}{b^4(b-1)}, \\
A_3 &\leq \frac{b(b+3)}{(b-2)(b-1)(b+2)}x_b y_b + \frac{b+3}{(b-2)(b-1)(b+2)}y_b - \frac{b+3}{b(b-1)(b+2)}x_b - \frac{b+3}{b^2(b-1)(b+2)} \\
&\leq \frac{b(b+3)}{(b-2)(b-1)(b+2)}x_b y_b + \frac{b+3}{(b-2)(b-1)(b+2)}y_b - \frac{b+3}{b(b-1)(b+2)}x_b, \\
A_4 &\leq \frac{b+2}{b^2}x_b y_b - \frac{b+2}{b^2(b-1)}y_b + \frac{(b+1)(b+2)}{b^4}x_b - \frac{(b+1)(b+2)}{b^4(b-1)} \\
&\leq \frac{b+2}{b^2}x_b y_b - \frac{b+2}{b^2(b-1)}y_b + \frac{(b+1)(b+2)}{b^4}x_b.
\end{aligned}$$

Since $b > 1500$, it is not difficult to verify that

$$\begin{aligned}
B_1 - B_4 - A_1 &\geq -\frac{(b+1)(6b^3 + 3b^2 - 3b - 2)}{b^6(b+2)}x_b^2 + \frac{b^6 - 5b^4 + 8b^3 + 16b^2 - 4b - 8}{b^6(b-1)(b+2)}x_b + \frac{1}{b^2} \\
&\geq -\frac{7}{b^3}x_b^2 + \frac{1}{b^2}, \\
B_2 - B_5 - A_2 &\geq -\frac{b^2 + 5b + 2}{b^2(b-2)(b-1)^2(b+1)}y_b^2 - \frac{7b^3 + 6b^2 - 12b - 8}{b^4(b-2)(b-1)(b+1)}y_b + \frac{b+2}{b^4(b-1)} \\
&\geq -\frac{1}{b^3}y_b^2 - \frac{1}{b^3}y_b, \\
B_3 - A_3 - A_4 &\geq -\frac{2(b^4 + 8b^3 + 5b^2 - 8b - 4)}{b^4(b-2)(b-1)(b+2)}x_b y_b - \frac{b^2 + 4b + 8}{b^2(b-2)(b-1)(b+2)}y_b \\
&\quad - \frac{b^3 + 3b^2 - 4b - 4}{b^4(b-1)(b+2)}x_b \\
&\geq -\frac{4}{b^3}x_b y_b - \frac{3}{b^3}y_b - \frac{3}{b^3}x_b.
\end{aligned}$$

From (15), (16), and the above inequalities, we obtain

$$L_b^2 - U_{b-1}U_{b+1} \geq \frac{1}{b^2} - \frac{7}{b^3}x_b^2 - \frac{1}{b^3}y_b^2 - \frac{4}{b^3}x_b y_b - \frac{4}{b^3}y_b - \frac{3}{b^3}x_b.$$

Since $x_b \leq \frac{4}{3} \log b$ and $y_b \leq \frac{5}{4} \log b$,

$$L_b^2 - U_{b-1}U_{b+1} \geq \frac{1}{b^2} - \frac{2977}{144} \frac{(\log b)^2}{b^3} - \frac{9 \log b}{b^3} \geq \frac{b - 23(\log b)^2}{b^3}.$$

Observe that the function $x \mapsto x - 23(\log x)^2$ is strictly increasing on $[300, \infty)$. Since $b \geq 1500$,

$$b - 23(\log b)^2 \geq 1500 - 23(\log 1500)^2 > 0.$$

Therefore $s_b^2 - s_{b-1}s_{b+1} \geq L_b^2 - U_{b-1}U_{b+1} > 0$. Hence $(s_b)_{b \geq 2}$ is log-concave, as desired. \square

Remark 8. We have uploaded the numerical data on the computation of s_b in the second author's ResearchGate account [20] which are freely downloadable by everyone.

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Research article

Explicit formulas for the p -adic valuations of Fibonomial coefficients II

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Abstract: In this article, we give explicit formulas for the p -adic valuations of the Fibonomial coefficients $\binom{p^n}{n}_F$ for all primes p and positive integers a and n . This is a continuation from our previous article extending some results in the literature, which deal only with $p = 2, 3, 5, 7$ and $a = 1$. Then we use these formulas to characterize the positive integers n such that $\binom{pn}{n}_F$ is divisible by p , where p is any prime which is congruent to $\pm 2 \pmod{5}$.

Keywords: Fibonacci number; binomial coefficient; Fibonomial coefficient; p -adic valuation; p -adic order; divisibility

Mathematics Subject Classification: 11B39; 11B65; 11A63

1. Introduction

The *Fibonacci sequence* $(F_n)_{n \geq 1}$ is given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with the initial values $F_1 = F_2 = 1$. For each $m \geq 1$ and $1 \leq k \leq m$, the *Fibonomial coefficients* $\binom{m}{k}_F$ is defined by

$$\binom{m}{k}_F = \frac{F_1 F_2 F_3 \cdots F_m}{(F_1 F_2 F_3 \cdots F_k)(F_1 F_2 F_3 \cdots F_{m-k})} = \frac{F_{m-k+1} F_{m-k+2} \cdots F_m}{F_1 F_2 F_3 \cdots F_k}.$$

Similar to the binomial coefficients, we define $\binom{m}{k}_F = 1$ if $k = 0$ and $\binom{m}{k}_F = 0$ if $k > m$, and it is well-known that $\binom{m}{k}_F$ is always an integer for every $m \geq 1$ and $k \geq 0$.

Recently, there has been an increasing interest in the study of Fibonomial coefficients. Marques and Trojovský [25, 26] start the investigation on the divisibility of Fibonomial coefficients by determining the integers $n \geq 1$ such that $\binom{pn}{n}_F$ is divisible by p for $p = 2, 3$. Marques, Sellers, and Trojovský [24] show that p divides $\binom{p^{a+1}}{p^a}_F$ for $p \equiv \pm 2 \pmod{5}$ and $a \geq 1$. Marques and Trojovský [27] and Trojovský [42] extend their results further and obtained the p -adic valuation of $\binom{p^{a+1}}{p^a}_F$ in [42]. Then Ballot [2, Theorem 2] generalizes the Kummer-like theorem of Knuth and

Wilf [22] and uses it to give a generalization of Marques and Trojovský's results. In particular, Ballot [2, Theorems 3.6, 5.2, and 5.3] finds all integers n such that $p \mid \binom{pn}{n}_U$ for any nondegenerate fundamental Lucas sequence U and $p = 2, 3$ and for $p = 5, 7$ in the case $U = F$. Phunphayap and Pongsriiam [31] provide the most general formula for the p -adic valuation of Fibonomial coefficients in the most general form $\binom{m}{n}_F$. For other recent results on the divisibility properties of the Fibonacci numbers, the Fibonomial coefficients, and other combinatorial numbers, see for example [3–5, 11–13, 16, 17, 28, 30, 32–34, 37, 38, 41, 43]. For some identities involving Fibonomial coefficients and generalizations, we refer the reader to the work of Kilic and his coauthors [7, 8, 18–21]. For the p -adic valuations of Eulerian, Bernoulli, and Stirling numbers, see [6, 9, 14, 23, 40]. Hence the relation $p \mid \binom{p^n}{n}_F$ has been studied only in the case $p = 2, 3, 5, 7$ and $a = 1$.

In this article, we extend the investigation on $\binom{p^n}{n}_F$ to the case of any prime p and any positive integer a . Replacing n by p^a and p^a by p , this becomes Marques and Trojovský's results [27, 42]. Substituting $a = 1$, $p \in \{2, 3, 5, 7\}$, and letting n be arbitrary, this reduces to Ballot's theorems [2]. So our results are indeed an extension of those previously mentioned. To obtain such the general result for all p and a , the calculation is inevitably long but we try to make it as simple as possible. As a reward, we can easily show in Corollaries 9 and 10 that $\binom{4n}{n}_F$ is odd if and only if n is a nonnegative power of 2, and $\binom{8n}{n}_F$ is odd if and only if $n = (1 + 3 \cdot 2^k)/7$ for some $k \equiv 1 \pmod{3}$.

We organize this article as follows. In Section 2, we give some preliminaries and results which are needed in the proof of the main theorems. In Section 3, we calculate the p -adic valuation of $\binom{p^n}{n}_F$ for all a , p , and n , and use it to give a characterization of the positive integers n such that $\binom{p^n}{n}_F$ is divisible by p where p is any prime which is congruent to $\pm 2 \pmod{5}$. Remark that there also is an interesting pattern in the p -adic representation of the integers n such that $\binom{pn}{n}_F$ is divisible by p . The proof is being prepared but it is a bit too long to include in this paper. We are trying to make it simpler and shorter and will publish it in the future. For more information and some recent articles related to the Fibonacci numbers, we refer the readers to [15, 35, 36, 39] and references therein.

2. Preliminaries and lemmas

Throughout this article, unless stated otherwise, x is a real number, p is a prime, a, b, k, m, n, q are integers, $m, n \geq 1$, and $q \geq 2$. The p -adic valuation (or p -adic order) of n , denoted by $v_p(n)$, is the exponent of p in the prime factorization of n . In addition, the order (or the rank) of appearance of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer m such that $n \mid F_m$, $\lfloor x \rfloor$ is the largest integer less than or equal to x , $\{x\}$ is the fractional part of x given by $\{x\} = x - \lfloor x \rfloor$, $\lceil x \rceil$ is the smallest integer larger than or equal to x , and $a \bmod m$ is the least nonnegative residue of a modulo m . Furthermore, for a mathematical statement P , the Iverson notation $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

We define $s_q(n)$ to be the sum of digits of n when n is written in base q , that is, if $n = (a_k a_{k-1} \dots a_0)_q = a_k q^k + a_{k-1} q^{k-1} + \dots + a_0$ where $0 \leq a_i < q$ for every i , then $s_q(n) = a_k + a_{k-1} + \dots + a_0$. Next, we recall some well-known and useful results for the reader's convenience.

Lemma 1. Let $p \neq 5$ be a prime. Then the following statements hold.

- (i) $n \mid F_m$ if and only if $z(n) \mid m$
- (ii) $z(p) \mid p + 1$ if and only if $p \equiv \pm 2 \pmod{5}$ and $z(p) \mid p - 1$, otherwise.
- (iii) $\gcd(z(p), p) = 1$.

Proof. These are well-known. See, for example, in [31, Lemma 1] for more details. \square

Lemma 2. (Legendre's formula) Let n be a positive integer and let p be a prime. Then

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n - s_p(n)}{p - 1}.$$

We will deal with a lot of calculations involving the floor function. So we recall the following results, which will be used throughout this article, sometimes without reference.

Lemma 3. For $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, the following holds

- (i) $\lfloor k + x \rfloor = k + \lfloor x \rfloor$,
- (ii) $\{k + x\} = \{x\}$,
- (iii) $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$
- (iv) $0 \leq \{x\} < 1$ and $\{x\} = 0$ if and only if $x \in \mathbb{Z}$.
- (v) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\} + \{y\} \geq 1, \end{cases}$
- (vi) $\left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor = \left\lfloor \frac{x}{k} \right\rfloor$ for $k \geq 1$.

Proof. These are well-known and can be proved easily. For more details, see in [10, Chapter 3]. We also refer the reader to [1, 29] for a nice application of these properties. \square

The next three theorems given by Phunphayap and Pongsriiam [31] are important tools for obtaining the main results of this article.

Theorem 4. [31, Theorem 7] Let p be a prime, $a \geq 0$, $\ell \geq 0$, and $m \geq 1$. Assume that $p \equiv \pm 1 \pmod{m}$ and $\delta = [\ell \not\equiv 0 \pmod{m}]$ is the Iverson notation. Then

$$v_p \left(\left\lfloor \frac{\ell p^a}{m} \right\rfloor! \right) = \begin{cases} \frac{\ell(p^a-1)}{m(p-1)} - a \left\{ \frac{\ell}{m} \right\} + v_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv 1 \pmod{m}; \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a}{2} \delta + v_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is even}; \\ \frac{\ell(p^a-1)}{m(p-1)} - \frac{a-1}{2} \delta - \left\{ \frac{\ell}{m} \right\} + v_p \left(\left\lfloor \frac{\ell}{m} \right\rfloor! \right), & \text{if } p \equiv -1 \pmod{m} \text{ and } a \text{ is odd}. \end{cases}$$

Theorem 5. [31, Theorem 11 and Corollary 12] Let $0 \leq k \leq m$ be integers. Then the following statements hold.

(i) Let $A_2 = v_2\left(\left\lfloor \frac{m}{6} \right\rfloor!\right) - v_2\left(\left\lfloor \frac{k}{6} \right\rfloor!\right) - v_2\left(\left\lfloor \frac{m-k}{6} \right\rfloor!\right)$. If $r = m \pmod 6$ and $s = k \pmod 6$, then

$$v_2\left(\binom{m}{k}_F\right) = \begin{cases} A_2, & \text{if } r \geq s \text{ and } (r, s) \neq (3, 1), (3, 2), (4, 2); \\ A_2 + 1, & \text{if } (r, s) = (3, 1), (3, 2), (4, 2); \\ A_2 + 3, & \text{if } r < s \text{ and } (r, s) \neq (0, 3), (1, 3), (2, 3), \\ & (1, 4), (2, 4), (2, 5); \\ A_2 + 2, & \text{if } (r, s) = (0, 3), (1, 3), (2, 3), (1, 4), (2, 4), \\ & (2, 5). \end{cases}$$

(ii) $v_5\left(\binom{m}{k}_F\right) = v_5\left(\binom{m}{k}\right)$.

(iii) Suppose that p is a prime, $p \neq 2$, and $p \neq 5$. If $m' = \left\lfloor \frac{m}{z(p)} \right\rfloor$, $k' = \left\lfloor \frac{k}{z(p)} \right\rfloor$, $r = m \pmod{z(p)}$, and $s = k \pmod{z(p)}$, then

$$v_p\left(\binom{m}{k}_F\right) = v_p\left(\binom{m'}{k'}\right) + [r < s] \left(v_p\left(\left\lfloor \frac{m-k+z(p)}{z(p)} \right\rfloor\right) + v_p(F_{z(p)}) \right).$$

Theorem 6. [31, Theorem 13] Let a, b, ℓ_1 , and ℓ_2 be positive integers and $b \geq a$. For each $p \neq 5$, assume that $\ell_1 p^b > \ell_2 p^a$ and let $m_p = \left\lfloor \frac{\ell_1 p^{b-a}}{z(p)} \right\rfloor$ and $k_p = \left\lfloor \frac{\ell_2}{z(p)} \right\rfloor$. Then the following statements hold.

(i) If $a \equiv b \pmod 2$, then $v_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$ is equal to

$$\begin{cases} v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv \ell_2 \pmod 3 \text{ or } \ell_2 \equiv 0 \pmod 3; \\ a + 2 + v_2(m_2 - k_2) + v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 0 \pmod 3 \text{ and } \ell_2 \not\equiv 0 \pmod 3; \\ \left\lfloor \frac{a}{2} \right\rfloor + 1 + v_2(m_2 - k_2) + v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 1 \pmod 3 \text{ and } \ell_2 \equiv 2 \pmod 3; \\ \left\lfloor \frac{a+1}{2} \right\rfloor + v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 2 \pmod 3 \text{ and } \ell_2 \equiv 1 \pmod 3, \end{cases}$$

and if $a \not\equiv b \pmod 2$, then $v_2\left(\binom{\ell_1 2^b}{\ell_2 2^a}_F\right)$ is equal to

$$\begin{cases} v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv -\ell_2 \pmod 3 \text{ or } \ell_2 \equiv 0 \pmod 3; \\ a + 2 + v_2(m_2 - k_2) + v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 0 \pmod 3 \text{ and } \ell_2 \not\equiv 0 \pmod 3; \\ \left\lfloor \frac{a+1}{2} \right\rfloor + v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 1 \pmod 3 \text{ and } \ell_2 \equiv 1 \pmod 3; \\ \left\lfloor \frac{a}{2} \right\rfloor + 1 + v_2(m_2 - k_2) + v_2\left(\binom{m_2}{k_2}\right), & \text{if } \ell_1 \equiv 2 \pmod 3 \text{ and } \ell_2 \equiv 2 \pmod 3. \end{cases}$$

(ii) Let $p \neq 5$ be an odd prime and let $r = \ell_1 p^b \pmod{z(p)}$ and $s = \ell_2 p^a \pmod{z(p)}$. If $p \equiv \pm 1 \pmod 5$, then

$$v_p\left(\binom{\ell_1 p^b}{\ell_2 p^a}_F\right) = [r < s] \left(a + v_p(m_p - k_p) + v_p(F_{z(p)}) \right) + v_p\left(\binom{m_p}{k_p}\right),$$

and if $p \equiv \pm 2 \pmod{5}$, then $v_p \left(\binom{\ell_1 p^b}{\ell_2 p^a}_F \right)$ is equal to

$$\left\{ \begin{array}{ll} v_p \left(\binom{m_p}{k_p} \right), & \text{if } r = s \text{ or } \ell_2 \equiv 0 \pmod{z(p)}; \\ a + v_p(F_{z(p)}) + v_p(m_p - k_p) + v_p \left(\binom{m_p}{k_p} \right), & \text{if } \ell_1 \equiv 0 \pmod{z(p)} \text{ and} \\ & \ell_2 \not\equiv 0 \pmod{z(p)}; \\ \frac{a}{2} + v_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even}; \\ \frac{a}{2} + v_p(F_{z(p)}) + v_p(m_p - k_p) + v_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is even}; \\ \frac{a+1}{2} + v_p(m_p - k_p) + v_p \left(\binom{m_p}{k_p} \right), & \text{if } r > s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd}; \\ \frac{a-1}{2} + v_p(F_{z(p)}) + v_p \left(\binom{m_p}{k_p} \right), & \text{if } r < s, \ell_1, \ell_2 \not\equiv 0 \pmod{z(p)}, \\ & \text{and } a \text{ is odd}. \end{array} \right.$$

In fact, Phunphayap and Pongsriiam [31] obtain other results analogous to Theorems 5 and 6 too but we do not need them in this article.

3. Main results

We begin with the calculation of the 2-adic valuation of $\binom{2^n}{n}_F$ and then use it to determine the integers n such that $\binom{2n}{n}_F, \binom{4n}{n}_F, \binom{8n}{n}_F$ are even. Then we calculate the p -adic valuation of $\binom{p^n}{n}_F$ for all odd primes p . For binomial coefficients, we know that $v_2 \left(\binom{2n}{n} \right) = s_2(n)$. For Fibonomial coefficients, we have the following result.

Theorem 7. Let a and n be positive integers, $\varepsilon = [n \not\equiv 0 \pmod{3}]$, and $A = \left\lfloor \frac{(2^a - 1)n}{3 \cdot 2^{v_2(n)}} \right\rfloor$. Then the following statements hold.

(i) If a is even, then

$$v_2 \left(\binom{2^a n}{n}_F \right) = \delta + A - \frac{a}{2} \varepsilon - v_2(A!) = \delta + s_2(A) - \frac{a}{2} \varepsilon, \quad (3.1)$$

where $\delta = [n \bmod 6 = 3, 5]$. In other words, $\delta = 1$ if $n \equiv 3, 5 \pmod{6}$ and $\delta = 0$ otherwise.

(ii) If a is odd, then

$$v_2 \left(\binom{2^a n}{n}_F \right) = \delta + A - \frac{a-1}{2} \varepsilon - v_2(A!) = \delta + s_2(A) - \frac{a-1}{2} \varepsilon, \quad (3.2)$$

where $\delta = \frac{(n \bmod 6) - 1}{2} [2 \nmid n] + \left\lceil \frac{v_2(n) + 3 - n \bmod 3}{2} \right\rceil [n \bmod 6 = 2, 4]$. In other words, $\delta = \frac{(n \bmod 6) - 1}{2}$ if n is odd, $\delta = 0$ if $n \equiv 0 \pmod{6}$, $\delta = \left\lceil \frac{v_2(n)}{2} \right\rceil + 1$ if $n \equiv 4 \pmod{6}$, and $\delta = \left\lceil \frac{v_2(n) + 1}{2} \right\rceil$ if $n \equiv 2 \pmod{6}$.

Proof. The second equalities in (3.1) and (3.2) follow from Legendre's formula. So it remains to prove the first equalities in (3.1) and (3.2). To prove (i), we suppose that a is even and divide the consideration into two cases.

Case 1. $2 \nmid n$. Let $r = 2^a n \pmod 6$ and $s = n \pmod 6$. Then $s \in \{1, 3, 5\}$, $r \equiv 2^a n \equiv 4n \equiv 4s \pmod 6$, and therefore $(r, s) = (4, 1), (0, 3), (2, 5)$. In addition, $A = \left\lfloor \frac{(2^a-1)n}{3} \right\rfloor = \frac{(2^a-1)n}{3}$ and $\delta = [s = 3, 5]$. By Theorem 5(i), the left-hand side of (3.1) is A_2 if $s = 1$ and $A_2 + 2$ if $s = 3, 5$, where $A_2 = v_2\left(\left\lfloor \frac{2^a n}{6} \right\rfloor!\right) - v_2\left(\left\lfloor \frac{n}{6} \right\rfloor!\right) - v_2\left(\left\lfloor \frac{(2^a-1)n}{6} \right\rfloor!\right)$. We obtain by Theorem 4 that

$$v_2\left(\left\lfloor \frac{2^a n}{6} \right\rfloor!\right) = v_2\left(\left\lfloor \frac{2^{a-1} n}{3} \right\rfloor!\right) = \frac{(2^{a-1} - 1)n}{3} - \frac{a-2}{2}\varepsilon - \left\lfloor \frac{n}{3} \right\rfloor + v_2\left(\left\lfloor \frac{n}{3} \right\rfloor!\right).$$

By Legendre's formula and Lemma 3, we have

$$\begin{aligned} v_2\left(\left\lfloor \frac{n}{6} \right\rfloor!\right) &= v_2\left(\left\lfloor \frac{n}{3} \right\rfloor!\right) - \left\lfloor \frac{n}{6} \right\rfloor, \\ v_2\left(\left\lfloor \frac{(2^a-1)n}{6} \right\rfloor!\right) &= v_2\left(\left\lfloor \frac{(2^a-1)n}{3} \right\rfloor!\right) - \left\lfloor \frac{(2^a-1)n}{6} \right\rfloor = v_2(A!) - \left\lfloor \frac{(2^a-1)n}{6} \right\rfloor, \\ \left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{(2^a-1)n}{6} \right\rfloor &= \frac{n-s}{6} + \frac{2^a n - r}{6} - \frac{n-s}{6} + \left\lfloor \frac{r-s}{6} \right\rfloor = \frac{2^a n - r}{6} - [s \in \{3, 5\}]. \end{aligned}$$

From the above observation, we obtain

$$\begin{aligned} A_2 &= \frac{(2^{a-1} - 1)n}{3} - \frac{a-2}{2}\varepsilon - \left\lfloor \frac{n}{3} \right\rfloor + \frac{2^a n - r}{6} - [s \in \{3, 5\}] - v_2(A!) \\ &= A - \frac{a-2}{2}\varepsilon - \left\lfloor \frac{n}{3} \right\rfloor - \frac{r}{6} - [s \in \{3, 5\}] - v_2(A!) \\ &= \begin{cases} A - \frac{a}{2} - v_2(A!), & \text{if } s = 1; \\ A - v_2(A!) - 1, & \text{if } s = 3; \\ A - \frac{a}{2} - v_2(A!) - 1, & \text{if } s = 5. \end{cases} \end{aligned}$$

It is now easy to check that A_2 (if $s = 1$), $A_2 + 2$ (if $s = 3, 5$) are the same as $\delta + A - \frac{a}{2}\varepsilon - v_2(A!)$ in (3.1). So (3.1) is verified.

Case 2. $2 \mid n$. We write $n = 2^b \ell$ where $2 \nmid \ell$ and let $m = \left\lfloor \frac{2^a \ell}{3} \right\rfloor$, $k = \left\lfloor \frac{\ell}{3} \right\rfloor$, $r = 2^a \ell \pmod 3$, and $s = \ell \pmod 3$. Since a is even, $r = s$. Then we apply Theorem 6(i) to obtain

$$v_2\left(\binom{2^a n}{n}_F\right) = v_2\left(\binom{\ell 2^{a+b}}{\ell 2^b}_F\right) = v_2\left(\binom{m}{k}\right) = v_2(m!) - v_2(k!) - v_2((m-k)!). \quad (3.3)$$

We see that $\ell \not\equiv 0 \pmod 3$ if and only if $n \not\equiv 0 \pmod 3$. In addition, $A = \frac{(2^a-1)\ell}{3}$ and $\delta = 0$. By Theorem 4, we have

$$v_2(m!) = A - \frac{a}{2}\varepsilon + v_2(k!).$$

In addition,

$$m - k = \left\lfloor \frac{2^a \ell}{3} \right\rfloor - \left\lfloor \frac{\ell}{3} \right\rfloor = \frac{2^a \ell - r}{3} - \frac{\ell - s}{3} = \frac{2^a \ell - \ell}{3} = A.$$

So $v_2((m-k)!) = v_2(A!)$. Substituting these in (3.3), we obtain (3.1). This completes the proof of (i).

To prove (ii), we suppose that a is odd and divide the proof into two cases.

Case 1. $2 \nmid n$. This case is similar to Case 1 of the previous part. So we let $r = 2^a n \bmod 6$ and $s = n \bmod 6$. Then $s \in \{1, 3, 5\}$, $r \equiv 2^a n \equiv 2n \equiv 2s \pmod{6}$, $(r, s) = (2, 1), (0, 3), (4, 5)$, $\delta = \frac{s-1}{2}$, and the left-hand side of (3.2) is A_2 if $s = 1$, $A_2 + 2$ if $s = 3$, and $A_2 + 3$ if $s = 5$, where $A_2 = v_2\left(\left\lfloor \frac{2^a n}{6} \right\rfloor!\right) - v_2\left(\left\lfloor \frac{n}{6} \right\rfloor!\right) - v_2\left(\left\lfloor \frac{(2^a-1)n}{6} \right\rfloor!\right)$. In addition, we have

$$v_2\left(\left\lfloor \frac{2^a n}{6} \right\rfloor!\right) = \frac{(2^{a-1}-1)n}{3} - \frac{a-1}{2}\varepsilon + v_2\left(\left\lfloor \frac{n}{3} \right\rfloor!\right),$$

$$v_2\left(\left\lfloor \frac{n}{6} \right\rfloor!\right) = v_2\left(\left\lfloor \frac{n}{3} \right\rfloor!\right) - \left\lfloor \frac{n}{6} \right\rfloor,$$

$$v_2\left(\left\lfloor \frac{(2^a-1)n}{6} \right\rfloor!\right) = v_2(A!) - \left\lfloor \frac{(2^a-1)n}{6} \right\rfloor,$$

$$\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{(2^a-1)n}{6} \right\rfloor = \frac{2^a n - r}{6} - [s \in \{3, 5\}].$$

Therefore

$$A_2 = \frac{(2^{a-1}-1)n}{3} - \frac{a-1}{2}\varepsilon + \frac{2^a n - r}{6} - [s \in \{3, 5\}] - v_2(A!).$$

Furthermore,

$$A = \left\lfloor \frac{(2^a-1)n}{3} \right\rfloor = \frac{2^a n - r}{3} - \frac{n-s}{3} + \left\lfloor \frac{r-s}{3} \right\rfloor = \begin{cases} \frac{(2^a-1)n}{3} - \frac{1}{3}, & \text{if } s = 1; \\ \frac{(2^a-1)n}{3}, & \text{if } s = 3; \\ \frac{(2^a-1)n}{3} - \frac{2}{3}, & \text{if } s = 5, \end{cases}$$

which implies that $A = \frac{(2^a-1)n}{3} - \frac{r}{6}$. Then

$$A_2 = A - \frac{a-1}{2}\varepsilon - [s \in \{3, 5\}] - v_2(A!).$$

It is now easy to check that A_2 (if $s = 1$), $A_2 + 2$ (if $s = 3$), and $A_2 + 3$ (if $s = 5$), are the same as $\delta + A - \frac{a-1}{2}\varepsilon - v_2(A!)$ in (3.2). So (3.2) is verified.

Case 2. $2 \mid n$. This case is similar to Case 2 of the previous part. So we write $n = 2^b \ell$ where $2 \nmid \ell$ and let $m = \left\lfloor \frac{2^a \ell}{3} \right\rfloor$, $k = \left\lfloor \frac{\ell}{3} \right\rfloor$, $r = 2^a \ell \bmod 3$, and $s = \ell \bmod 3$. We obtain by Theorem 6 that $v_2\left(\left(\frac{2^a n}{n}\right)_F\right)$ is equal to

$$v_2\left(\left(\frac{\ell 2^{a+b}}{\ell 2^b}\right)_F\right) = \begin{cases} v_2\left(\binom{m}{k}\right), & \text{if } \ell \equiv 0 \pmod{3}; \\ \left\lfloor \frac{b+1}{2} \right\rfloor + v_2\left(\binom{m}{k}\right), & \text{if } \ell \equiv 1 \pmod{3}; \\ \left\lfloor \frac{b}{2} \right\rfloor + 1 + v_2(m-k) + v_2\left(\binom{m}{k}\right), & \text{if } \ell \equiv 2 \pmod{3}. \end{cases} \quad (3.4)$$

By Theorem 4, we have

$$v_2(m!) = \frac{(2^a-1)\ell}{3} - \frac{a-1}{2}\varepsilon - \left\lfloor \frac{\ell}{3} \right\rfloor + v_2(k!).$$

Since $(2^a - 1)\ell \equiv \ell \pmod{3}$, $\left\{\frac{(2^a-1)\ell}{3}\right\} = \left\{\frac{\ell}{3}\right\}$. This implies that $v_2(m!) = A - \frac{a-1}{2}\varepsilon + v_2(k!)$. In addition, $(r, s) = (0, 0), (2, 1), (1, 2)$, and

$$m - k = \left\lfloor \frac{2^a \ell}{3} \right\rfloor - \left\lfloor \frac{\ell}{3} \right\rfloor = \frac{2^a \ell - r}{3} - \frac{\ell - s}{3} = \frac{(2^a - 1)\ell - (r - s)}{3} = A + [s = 2].$$

From the above observation, we obtain

$$v_2\left(\binom{m}{k}\right) = v_2(m!) - v_2(k!) - v_2((m-k)!) = \begin{cases} A - \frac{a-1}{2}\varepsilon - v_2(A!), & \text{if } s = 0, 1; \\ A - \frac{a-1}{2}\varepsilon - v_2((A+1)!), & \text{if } s = 2. \end{cases}$$

Substituting this in (3.4), we see that

$$v_2\left(\binom{2^a n}{n}_F\right) = \begin{cases} A - v_2(A!), & \text{if } \ell \equiv 0 \pmod{3}; \\ \left\lfloor \frac{b+1}{2} \right\rfloor + A - \frac{a-1}{2} - v_2(A!), & \text{if } \ell \equiv 1 \pmod{3}; \\ \left\lfloor \frac{b}{2} \right\rfloor + 1 + A - \frac{a-1}{2} - v_2(A!), & \text{if } \ell \equiv 2 \pmod{3}. \end{cases} \quad (3.5)$$

Recall that $n = 2^b \ell \equiv (-1)^b \ell \pmod{3}$. So (3.5) implies that

$$v_2\left(\binom{2^a n}{n}_F\right) = \begin{cases} A - v_2(A!), & \text{if } n \equiv 0 \pmod{3}; \\ \frac{b}{2} + 1 + A - \frac{a-1}{2} - v_2(A!), & \text{if } n \equiv 1 \pmod{3} \text{ and } b \text{ is even}; \\ \frac{b+1}{2} + 1 + A - \frac{a-1}{2} - v_2(A!), & \text{if } n \equiv 1 \pmod{3} \text{ and } b \text{ is odd}; \\ \frac{b}{2} + 1 + A - \frac{a-1}{2} - v_2(A!), & \text{if } n \equiv 2 \pmod{3} \text{ and } b \text{ is even}; \\ \frac{b+1}{2} + A - \frac{a-1}{2} - v_2(A!), & \text{if } n \equiv 2 \pmod{3} \text{ and } b \text{ is odd}, \end{cases}$$

which is the same as (3.2). This completes the proof. \square

We can obtain the main result of Marques and Trojovský [25] as a corollary.

Corollary 8. (Marques and Trojovský [25]) $\binom{2n}{n}_F$ is even for all $n \geq 2$.

Proof. Let $n \geq 2$ and apply Theorem 7 with $a = 1$ to obtain $v_2\left(\binom{2n}{n}_F\right) = \delta + s_2(A)$. If $n \not\equiv 0, 1 \pmod{6}$, then $\delta > 0$. If $n \equiv 0 \pmod{6}$, then $n \geq 3 \cdot 2^{\nu_2(n)}$, and so $A \geq 1$ and $s_2(A) > 0$. If $n \equiv 1 \pmod{6}$, then $A = \left\lfloor \frac{n}{3} \right\rfloor > 1$ and so $s_2(A) > 0$. In any case, $v_2\left(\binom{2n}{n}_F\right) > 0$. So $\binom{2n}{n}_F$ is even. \square

Corollary 9. Let $n \geq 2$. Then $\binom{4n}{n}_F$ is even if and only if n is not a power of 2. In other words, for each $n \in \mathbb{N}$, $\binom{4n}{n}_F$ is odd if and only if $n = 2^k$ for some $k \geq 0$.

Proof. Let δ , ε , and A be as in Theorem 7. If $n = 2^k$ for some $k \geq 1$, then we apply Theorem 7 with $a = 2$, $\delta = 0$, $\varepsilon = 1$, $A = 1$ leading to $v_2\left(\binom{4n}{n}_F\right) = 0$, which implies that $\binom{4n}{n}_F$ is odd.

Suppose n is not a power of 2. By Theorem 7, $v_2\left(\binom{4n}{n}_F\right) = \delta + s_2(A) - \varepsilon \geq s_2(A) - 1$. Since n is not a power of 2, the sum $s_2(n) \geq 2$. It is easy to see that $s_2(m) = s_2(2^c m)$ for any $c, m \in \mathbb{N}$. Therefore $s_2(A) = s_2\left(\frac{n}{2^{\nu_2(m)}}\right) = s_2\left(2^{\nu_2(n)} \cdot \frac{n}{2^{\nu_2(m)}}\right) = s_2(n) \geq 2$, which implies $v_2\left(\binom{4n}{n}_F\right) \geq 1$, as required. \square

Observe that $2, 2^2, 2^3$ are congruent to $2, 4, 1 \pmod{7}$, respectively. This implies that if $k \geq 1$ and $k \equiv 1 \pmod{3}$, then $(1 + 3 \cdot 2^k)/7$ is an integer. We can determine the integers n such that $\binom{8n}{n}_F$ is odd as follows.

Corollary 10. $\binom{8n}{n}_F$ is odd if and only if $n = \frac{1+3 \cdot 2^k}{7}$ for some $k \equiv 1 \pmod{3}$.

Proof. Let $a, \delta, A, \varepsilon$ be as in Theorem 7. We first suppose $n = (1 + 3 \cdot 2^k)/7$ where $k \geq 1$ and $k \equiv 1 \pmod{3}$. Then $n \equiv 7n \equiv 1 + 3 \cdot 2^k \equiv 1 \pmod{6}$. Then $a = 3, \varepsilon = 1, \delta = 0, A = 2^k$, and so $v_2\left(\binom{8n}{n}_F\right) = 0$. Therefore $\binom{8n}{n}_F$ is odd. Next, assume that $\binom{8n}{n}_F$ is odd. Observe that $A \geq 2$ and $s_2(A) > 0$. If $n \equiv 0 \pmod{3}$, then $\varepsilon = 0$ and $v_2\left(\binom{8n}{n}_F\right) = \delta + s_2(A) > 0$, which is not the case. Therefore $n \equiv 1, 2 \pmod{3}$, and so $\varepsilon = 1$. If $n \equiv 0 \pmod{2}$, then $\delta = \left\lceil \frac{v_2(n) + 3 - n \bmod 3}{2} \right\rceil \geq 1$, and so $\left(\binom{8n}{n}_F\right) \geq s_2(A) > 0$, which is a contradiction. So $n \equiv 1 \pmod{2}$. This implies $n \equiv 1, 5 \pmod{6}$. But if $n \equiv 5 \pmod{6}$, then $\delta \geq 2$ and $v_2\left(\binom{8n}{n}_F\right) > 0$, a contradiction. Hence $n \equiv 1 \pmod{6}$. Then $\delta = 0$. Since $s_2(A) - 1 = v_2\left(\binom{8n}{n}_F\right) = 0$, we see that $A = 2^k$ for some $k \geq 1$. Then $\frac{7n-1}{3} = \left\lfloor \frac{7n}{3} \right\rfloor = A = 2^k$, which implies $n = \frac{1+3 \cdot 2^k}{7}$, as required. \square

Theorem 11. For each $a, n \in \mathbb{N}$, $v_5\left(\binom{5^a n}{n}_F\right) = v_5\left(\binom{5^a n}{n}\right) = \frac{s_5((5^a - 1)n)}{4}$. In particular, $\binom{5^a n}{n}_F$ is divisible by 5 for every $a, n \in \mathbb{N}$.

Proof. The first equality follows immediately from Theorem 5(ii). By Legendre's formula, $v_5\left(\binom{n}{k}\right) = \frac{s_5(k) + s_5(n-k) - s_5(n)}{4}$ for all $n \geq k \geq 1$. So $v_5\left(\binom{5^a n}{n}_F\right)$ is

$$\frac{s_5(n) + s_5(5^a n - n) - s_5(5^a n)}{4} = \frac{s_5((5^a - 1)n)}{4}. \quad \square$$

Theorem 12. Let $p \neq 2, 5, a, n \in \mathbb{N}$, $r = p^a n \bmod z(p)$, $s = n \bmod z(p)$, and $A = \left\lfloor \frac{n(p^a - 1)}{p^{v_p(n)} z(p)} \right\rfloor$. Then the following statements hold.

(i) If $p \equiv \pm 1 \pmod{5}$, then $v_p\left(\binom{p^a n}{n}_F\right)$ is equal to

$$\frac{A}{p-1} - a \left\{ \frac{n}{p^{v_p(n)} z(p)} \right\} - v_p(A!) = \frac{s_p(A)}{p-1} - a \left\{ \frac{n}{p^{v_p(n)} z(p)} \right\}. \quad (3.6)$$

(ii) If $p \equiv \pm 2 \pmod{5}$ and a is even, then $v_p\left(\binom{p^a n}{n}_F\right)$ is equal to

$$\frac{A}{p-1} - \frac{a}{2} [s \neq 0] - v_p(A!) = \frac{s_p(A)}{p-1} - \frac{a}{2} [s \neq 0]. \quad (3.7)$$

(iii) If $p \equiv \pm 2 \pmod{5}$ and a is odd, then $v_p\left(\binom{p^a n}{n}_F\right)$ is equal to

$$\left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!) + \delta, \quad (3.8)$$

where $\delta = \left(\left\lfloor \frac{v_p(n)}{2} \right\rfloor + [2 \nmid v_p(n)][r > s] + [r < s] v_p(F_{z(p)}) \right) [r \neq s]$, or equivalently, $\delta = 0$ if $r = s$, $\delta = \left\lfloor \frac{v_p(n)}{2} \right\rfloor + v_p(F_{z(p)})$ if $r < s$, and $\delta = \left\lfloor \frac{v_p(n)}{2} \right\rfloor$ if $r > s$.

Proof. We first prove (i) and (ii). So we suppose that the hypothesis of (i) or (ii) is true. By writing $v_p(A!) = \frac{A - s_p(A)}{p-1}$, we obtain the equalities in (3.6) and (3.7). By Lemma 1(ii), $p^a \equiv 1 \pmod{z(p)}$. Then $r = s$.

Case 1. $p \nmid n$. Let $m = \left\lfloor \frac{p^a n}{z(p)} \right\rfloor$ and $k = \left\lfloor \frac{n}{z(p)} \right\rfloor$. Then we obtain by Theorem 5(iii) that

$$v_p \left(\binom{p^a n}{n}_F \right) = v_p \left(\binom{m}{k} \right) = v_p(m!) - v_p(k!) - v_p((m-k)!). \quad (3.9)$$

By Lemma 1(ii) and Theorem 4, we see that if $p \equiv \pm 1 \pmod{5}$, then $p \equiv 1 \pmod{z(p)}$ and

$$v_p(m!) = v_p \left(\left\lfloor \frac{np^a}{z(p)} \right\rfloor! \right) = \frac{n(p^a - 1)}{z(p)(p-1)} - a \left\{ \frac{n}{z(p)} \right\} + v_p(k!), \quad (3.10)$$

and if $p \equiv \pm 2 \pmod{5}$ and a is even, then $p \equiv -1 \pmod{z(p)}$ and

$$v_p(m!) = \frac{n(p^a - 1)}{z(p)(p-1)} - \frac{a}{2}[s \neq 0] + v_p(k!). \quad (3.11)$$

Since $z(p) \mid p^a - 1$ and $p \nmid n$, $A = \frac{n(p^a - 1)}{z(p)}$. Therefore

$$m - k = \left\lfloor \frac{p^a n}{z(p)} \right\rfloor - \left\lfloor \frac{n}{z(p)} \right\rfloor = \frac{p^a n - r}{z(p)} - \frac{n - s}{z(p)} = \frac{n(p^a - 1)}{z(p)} = A. \quad (3.12)$$

Substituting (3.10), (3.11), and (3.12) in (3.9), we obtain (3.6) and (3.7).

Case 2. $p \mid n$. Let $n = p^b \ell$ where $p \nmid \ell$, $m = \left\lfloor \frac{\ell p^a}{z(p)} \right\rfloor$, and $k = \left\lfloor \frac{\ell}{z(p)} \right\rfloor$. Since $r = s$, we obtain by Theorem 6 that $v_p \left(\binom{p^a n}{n}_F \right)$ is equal to

$$v_p \left(\binom{\ell p^{a+b}}{\ell p^b}_F \right) = v_p \left(\binom{m}{k} \right) = v_p(m!) - v_p(k!) - v_p((m-k)!). \quad (3.13)$$

Since $\gcd(p, z(p)) = 1$, we see that $\ell \equiv 0 \pmod{z(p)} \Leftrightarrow n \equiv 0 \pmod{z(p)} \Leftrightarrow s = 0$. Similar to Case 1, we have $v_p(m!) = \frac{\ell(p^a - 1)}{z(p)(p-1)} - a \left\{ \frac{\ell}{z(p)} \right\} + v_p(k!)$ if $p \equiv \pm 1 \pmod{5}$, $v_p(m!) = \frac{\ell(p^a - 1)}{z(p)(p-1)} - \frac{a}{2}[s \neq 0] + v_p(k!)$ if $p \equiv \pm 2 \pmod{5}$ and a is even, $\ell p^a \equiv \ell \pmod{z(p)}$, $A = \frac{\ell(p^a - 1)}{z(p)}$, and $m - k = A$. So (3.13) leads to (3.6) and (3.7). This proves (i) and (ii).

To prove (iii), suppose that $p \equiv \pm 2 \pmod{5}$ and a is odd. By Lemma 1(ii), $p \equiv -1 \pmod{z(p)}$. In addition, $\frac{p^a - 1}{p-1} = p^{a-1} + p^{a-2} + \dots + 1 \equiv 1 \pmod{z(p)}$. We divide the consideration into two cases.

Case 1. $p \nmid n$. This case is similar to Case 1 of the previous part. So we apply Theorems 4 and 5(iii). Let $m = \left\lfloor \frac{p^a n}{z(p)} \right\rfloor$ and $k = \left\lfloor \frac{n}{z(p)} \right\rfloor$. Then

$$v_p(m!) = \frac{n(p^a - 1)}{z(p)(p-1)} - \frac{a-1}{2}[s \neq 0] - \left\{ \frac{n}{z(p)} \right\} + v_p(k!),$$

$$m - k = \frac{p^a n - r}{z(p)} - \frac{n - s}{z(p)} = \frac{n(p^a - 1) - (r - s)}{z(p)},$$

$$A = \left\lfloor \frac{np^a - r}{z(p)} - \frac{n - s}{z(p)} + \frac{r - s}{z(p)} \right\rfloor = m - k + \left\lfloor \frac{r - s}{z(p)} \right\rfloor.$$

Since $\frac{p^a - 1}{p-1} \equiv 1 \pmod{z(p)}$, $\frac{n(p^a - 1)}{p-1} \equiv n \pmod{z(p)}$. This implies that $\left\lfloor \frac{n(p^a - 1)}{z(p)(p-1)} \right\rfloor = \left\lfloor \frac{n}{z(p)} \right\rfloor$. Therefore

$$v_p(m!) = \left\lfloor \frac{n(p^a - 1)}{z(p)(p-1)} \right\rfloor - \frac{a-1}{2}[s \neq 0] + v_p(k!) = \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2}[s \neq 0] + v_p(k!).$$

From the above observation, if $r \geq s$, then $A = m - k$ and

$$v_p \left(\binom{p^a n}{n}_F \right) = v_p \left(\binom{m}{k} \right) = \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!),$$

which leads to (3.8). If $r < s$, then $A = m - k - 1$, $\left\lfloor \frac{p^a n - n + z(p)}{z(p)} \right\rfloor = A + 1$, and $v_p \left(\binom{p^a n}{n}_F \right)$ is equal to

$$\begin{aligned} & \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p((A+1)!) + v_p(A+1) + v_p(F_{z(p)}) \\ &= \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!) + v_p(F_{z(p)}), \end{aligned}$$

which is the same as (3.8).

Case 2. $p \mid n$. Let $n = p^b \ell$ where $p \nmid \ell$, $m = \left\lfloor \frac{\ell p^a}{z(p)} \right\rfloor$, and $k = \left\lfloor \frac{\ell}{z(p)} \right\rfloor$. Similar to Case 1, $s = 0 \Leftrightarrow \ell \equiv 0 \pmod{z(p)}$. In addition, $\frac{\ell(p^a-1)}{p-1} \equiv \ell \pmod{z(p)}$, and so we obtain by Theorem 4 that $v_p(m!) = \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] + v_p(k!)$. The calculation of $v_p \left(\binom{p^a n}{n}_F \right) = v_p \left(\binom{\ell p^{a+b}}{\ell p^b}_F \right)$ is done by the applications of Theorem 6 and is divided into several cases. Suppose $r = s$. Then $p^{a+b} \ell \equiv p^a n \equiv r \equiv s \equiv n \equiv p^b \ell \pmod{z(p)}$. Since $(p, z(p)) = 1$, this implies $\ell p^a \equiv \ell \pmod{z(p)}$. Therefore $A = \left\lfloor \frac{\ell p^a - \ell}{z(p)} \right\rfloor = \frac{\ell p^a - \ell}{z(p)} = m - k$ and

$$v_p \left(\binom{p^a n}{n}_F \right) = v_p \left(\binom{m}{k} \right) = v_p(m!) - v_p(k!) - v_p((m-k)!),$$

which is (3.8). Obviously, if $\ell \equiv 0 \pmod{z(p)}$, then $r = s$, which is already done. So from this point on, we assume that $r \neq s$ and $\ell \not\equiv 0 \pmod{z(p)}$. Recall that $p \equiv -1 \pmod{z(p)}$ and a is odd. So if b is odd, then

$$r \equiv np^a \equiv -n \equiv -p^b \ell \equiv \ell \pmod{z(p)}, \quad s \equiv n \equiv p^b \ell \equiv -\ell \equiv \ell p^a \pmod{z(p)}, \quad \text{and}$$

$$A = \left\lfloor \frac{\ell p^a - s}{z(p)} - \frac{\ell - r}{z(p)} + \frac{s - r}{z(p)} \right\rfloor = \frac{\ell p^a - s}{z(p)} - \frac{\ell - r}{z(p)} + \left\lfloor \frac{s - r}{z(p)} \right\rfloor = m - k + \left\lfloor \frac{s - r}{z(p)} \right\rfloor.$$

Similarly, if b is even, then $r = \ell p^a \pmod{z(p)}$, $s = \ell \pmod{z(p)}$, and $A = m - k + \left\lfloor \frac{r-s}{z(p)} \right\rfloor$. Let $R = \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!) + \delta$ be the quantity in (3.8). From the above observation and the application of Theorem 6, we obtain $v_p \left(\binom{p^a n}{n}_F \right)$ as follows. If $r > s$ and b is even, then $A = m - k$ and

$$v_p \left(\binom{p^a n}{n}_F \right) = \frac{b}{2} + v_p \left(\binom{m}{k} \right) = \frac{b}{2} + \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!) = R.$$

If $r > s$ and b is odd, then $A = m - k - 1$ and

$$\begin{aligned} v_p \left(\binom{p^a n}{n}_F \right) &= \frac{b+1}{2} + v_p(A+1) + v_p \left(\binom{m}{k} \right) \\ &= \frac{b+1}{2} + \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!) = R. \end{aligned}$$

If $r < s$ and b is even, then $A = m - k - 1$ and

$$\begin{aligned} v_p \left(\binom{p^a n}{n}_F \right) &= \frac{b}{2} + v_p(F_{z(p)}) + v_p(A + 1) + v_p \left(\binom{m}{k} \right) \\ &= \frac{b}{2} + v_p(F_{z(p)}) + \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!) = R. \end{aligned}$$

If $r < s$ and b is odd, then $A = m - k$ and

$$\begin{aligned} v_p \left(\binom{p^a n}{n}_F \right) &= \frac{b-1}{2} + v_p(F_{z(p)}) + v_p \left(\binom{m}{k} \right) \\ &= \frac{b-1}{2} + v_p(F_{z(p)}) + \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} [s \neq 0] - v_p(A!) = R. \end{aligned}$$

This completes the proof. \square

In the next two corollaries, we give some characterizations of the integers n such that $\binom{p^a n}{n}_F$ is divisible by p .

Corollary 13. *Let p be a prime and let a and n be positive integers. If $n \equiv 0 \pmod{z(p)}$, then $p \mid \binom{p^a n}{n}_F$.*

Proof. We first consider the case $p \neq 2, 5$. Assume that $n \equiv 0 \pmod{z(p)}$ and r, s, A , and δ are as in Theorem 12. Then $\frac{n}{p^{v_p(n)z(p)}}, \frac{A}{p-1} \in \mathbb{Z}$, $r = s = 0$, and $\delta = 0$. Every case in Theorem 12 leads to $v_p \left(\binom{p^a n}{n}_F \right) = \frac{s_p(A)}{p-1} > 0$, which implies $p \mid \binom{p^a n}{n}_F$. If $p = 5$, then the result follows immediately from Theorem 11. If $p = 2$, then every case of Theorem 7 leads to $v_2 \left(\binom{2^a n}{n}_F \right) \geq s_2(A) > 0$, which implies the desired result. \square

Corollary 14. *Let $p \neq 2, 5$ be a prime and let a, n, r, s , and A be as in Theorem 12. Assume that $p \equiv \pm 2 \pmod{5}$ and $n \not\equiv 0 \pmod{z(p)}$. Then the following statements hold.*

- (i) *Assume that a is even. Then $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) > \frac{a}{2}(p-1)$.*
- (ii) *Assume that a is odd and $p \nmid n$. If $r < s$, then $p \mid \binom{p^a n}{n}_F$. If $r \geq s$, then $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) \geq \frac{a+1}{2}(p-1)$.*
- (iii) *Assume that a is odd and $p \mid n$. If $r \neq s$, then $p \mid \binom{p^a n}{n}_F$. If $r = s$, then $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) \geq \frac{a+1}{2}(p-1)$.*

Proof. We use Lemmas 2 and 3 repeatedly without reference. For (i), we obtain by (3.7) that

$$v_p \left(\binom{p^a n}{n}_F \right) = \frac{s_p(A)}{p-1} - \frac{a}{2}, \text{ which is positive if and only if } s_p(A) > \frac{a}{2}(p-1).$$

This proves (i). To prove (ii) and (iii), we let δ be as in Theorem 12 and divide the consideration into two cases.

Case 1. $p \nmid n$. If $r < s$, then we obtain by Theorem 5(iii) that $v_p \left(\binom{p^a n}{n}_F \right) \geq v_p(F_{z(p)}) \geq 1$. Suppose $r \geq s$. Then $\delta = 0$ and (3.8) is

$$\left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} - v_p(A!) = \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2} - \frac{A - s_p(A)}{p-1} = \frac{s_p(A)}{p-1} - \left\lfloor \frac{A}{p-1} \right\rfloor - \frac{a-1}{2}.$$

If $s_p(A) \geq \frac{a+1}{2}(p-1)$, then (3.8) implies that

$$v_p\left(\binom{p^a n}{n}_F\right) \geq 1 - \left\{\frac{A}{p-1}\right\} > 0.$$

Similarly, if $s_p(A) < \frac{a+1}{2}(p-1)$, then $v_p\left(\binom{p^a n}{n}_F\right) < 1 - \left\{\frac{A}{p-1}\right\} \leq 1$. This proves (ii).

Case 2. $p \mid n$. We write $n = p^b \ell$ where $p \nmid \ell$. Then $b \geq 1$. Recall that $v_p(F_{z(p)}) \geq 1$. If $r \neq s$, then Theorem 6 implies that $v_p\left(\binom{p^a n}{n}\right) \geq \frac{b}{2}$ if b is even and it is $\geq \frac{b+1}{2}$ if b is odd. In any case, $v_p\left(\binom{p^a n}{n}_F\right) \geq 1$. So $p \mid \binom{p^a n}{n}_F$. If $r = s$, then $\delta = 0$ and we obtain as in Case 1 that $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) \geq \frac{a+1}{2}(p-1)$. This proves (iii). \square

Corollary 15. Let $p \neq 2, 5$ be a prime and let $A = \frac{n(p-1)}{p^{v_p(n)z(p)}}$. Assume that $p \equiv \pm 1 \pmod{5}$. Then $p \mid \binom{pn}{n}_F$ if and only if $s_p(A) \geq p-1$.

Proof. We remark that by Lemma 1(ii), A is an integer. Let $x = \frac{n}{p^{v_p(n)z(p)}}$. We apply Theorem 12(i) with $a = 1$. If $s_p(A) \geq p-1$, then (3.6) implies that $v_p\left(\binom{pn}{n}_F\right) \geq 1 - \{x\} > 0$. If $s_p(A) < p-1$, then $v_p\left(\binom{pn}{n}_F\right) < 1 - \{x\} \leq 1$. This completes the proof. \square

4. Conclusions

We give exact formulas for the p -adic valuations of Fibonomial coefficients of the form $\binom{p^a n}{n}_F$ for all primes p and $a, n \in \mathbb{N}$. Then we use it to characterize the integers n such that $\binom{p^a n}{n}_F$ is divisible by p .

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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