SUMSETS ASSOCIATED WITH BEATTY SEQUENCES AND THE FIBONACCI AND LUCAS NUMBERS

By

MISS Sutasinee KAWSUMARNG

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

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Title	Sumsets Associated with Beatty Sequences and the Fibonacci and Lucas Numbers
By	Sutasinee KAWSUMARNG
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Graduate School Silpakorn University in Partial Fulfillment of the Requirements for the Master of Science

.... Dean of graduate school

(Associate Professor Jurairat Nunthanid, Ph.D.)

Approved by

.... Chair person

(Assistant Professor Dr. Wannarut Rungrottheera)

.... Advisor

(Associate Professor Dr. Prapanpong Pongsriiam)

.... Co advisor

(Assistant Professor Dr. Tammatada Khemaratchatakumthorn)

.... Co advisor

(Dr. Passawan Noppakaew)

.... External Examiner

(Assistant Professor Dr. Kantaphon Kuhapatanakul)

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MISS SUTASINEE KAWSUMARNG : SUMSETS ASSOCIATED WITH BEATTY SEQUENCES AND THE FIBONACCI AND LUCAS NUMBERS THESIS ADVISOR : ASSOCIATE PROFESSOR DR. PRAPANPONG PONGSRIIAM

In this thesis, we study the properties of Fibonacci and Lucas numbers and the sumsets associated with Wythoff sequences.

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Sutasinee KAWSUMARNG

TABLE OF CONTENTS

Page

Chapter 1

Introduction

Let G be an additive abelian group, A and B nonempty subsets of G, and $x \in G$. Then the sumset $A+B$ and the translation $x+A$ are defined by

$$A+B=\{a+b \mid a \in A \text{ and } b \in B\} \text{ and } x+A=A+x=\{a+x \mid a \in A\}.$$

Additive number theory and the study of sumsets have a long history dating back at least to Lagrange in 1770 who proved that every natural number can be written as a sum of four squares of integers. Cauchy in 1813 gave a lower bound for the cardinality of the sumset $A+B$ where A and B are nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$. Davenport [3] rediscovered Cauchy's result in 1935 and the results is now known as the Cauchy-Davenport theorem. Several other results on sumsets and in additive number theory have been obtained by various mathematicians, and we refer the reader to the books by Freiman [8], Halberstam and Roth [10], Nathanson [18], Tao and Vu [40], and Vaughan [42] for additional details and references.

On the other hand, Wythoff sequences arise very often in combinatorics and combinatorial game theory, and so many of their combinatorial properties have been extensively studied; see for example in the work of Fraenkel [4, 5, 6, 7], Kimberling [14, 15], Pitman [24], Wythoff [43], and in the online encyclopedia OEIS [39]. However, as far as we are aware, there are no number theoretic results, at least in the spirit of this thesis, concerning the sumsets associated with Wythoff sequences. This motivates us to investigate more on this topic. Note that Pitman's article [24] is closely related to ours but it focuses only on the cardinality of sumsets of certain finite Beatty sequences in connection with Sturmian words and the nearest integer algorithm.

Before proceeding further, let us introduce the notation which will be used throughout this thesis as follows: x is a real number, a, b, m, n are integers, $\alpha = (1+\sqrt{5})/2$ is the golden ratio, $\beta = (1-\sqrt{5})/2$, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\{x\} = x - \lfloor x \rfloor$,

$$B(x) = \{\lfloor nx \rfloor \mid n \in \mathbb{N}\} \quad \text{and} \quad B_0(x) = \{\lfloor nx \rfloor \mid n \in \mathbb{N} \cup \{0\}\}. \tag{1.1}$$

The set $B(x)$ is usually considered as a sequence $(\lfloor nx \rfloor)_{n\geq 1}$ and is called a Beatty sequence. The sets $B(\alpha)$ and $B(\alpha^2)$ are also called lower and upper Wythoff sequences, respectively; but for our purpose, it is more convenient to consider them as sets. In addition, if P is a mathematical statement, then the Iverson notation $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

Recall that a generalized Fibonacci sequence $(f_n)_{n\geq 0}$ is defined by $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ where f_0 and f_1 are arbitrary integers. If $f_0 = 0$ and $f_1 = 1$, then $(f_n)_{n\geq 0} = (F_n)_{n\geq 0}$ is the classical Fibonacci sequence, and if $f_0 = 2$ and $f_1 = 1$, then $(f_n)_{n\geq 0} = (L_n)_{n\geq 0}$ is the classical sequence of Lucas numbers. The roots of the characteristic polynomial $x^2 - x - 1$ for any generalized Fibonacci sequence (f_n) are α and β, but it turns out that the structures of sumsets such as $B(\alpha) + B(\alpha^2)$ and $B(\alpha^2) + B(\alpha^2)$ are best described in terms of the classical Fibonacci numbers F_n. We refer the reader to [12, 13, 20, 21, 22, 25, 26, 27, 34] for some recent results concerning multiplicative properties of F_n, and to [28, 29] for certain Diophantine equations involving additive and multiplicative properties of F_n.

In this thesis, we give a new estimate concerning the fractional part $\{n\alpha\}$ and study the sumsets associated with $B(\alpha)$ and $B(\alpha^2)$. For example, we obtain from Theorems 3.1, 3.8, and 3.5, respectively, that for every $n \geq 4$, $n = \lfloor a\alpha \rfloor + \lfloor b\alpha \rfloor$ for some $a, b \in \mathbb{N}$, for every $n \geq 27$, $n = \lfloor a\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ for some $a, b, c \in \mathbb{N}$, and for every $n \geq 1$, $n = \lfloor a\alpha \rfloor + \lfloor b\alpha^2 \rfloor$ for some $a, b \in \mathbb{N}$ if and only if n is not one less than a Fibonacci number. The structure of $B(\alpha^2) + B(\alpha^2)$ contains some kinds of fractal and palindromic patterns in each interval of the form $[F_n, F_{n+1}]$; see for instance Theorem 3.16, Theorem 3.17 , and Remark 3.19, and so the elements in $(B(\alpha^2) + B(\alpha^2)) \cap [F_{n+1}, F_{n+2}]$ can be completely determined by those of $(B(\alpha^2) + B(\alpha^2)) \cap [F_n, F_{n+1}]$.

For a general result on the sumsets associated with $B(x)$ and $B(x^2)$ where x satisfies the conditions such as $x > 1$ and $x^2 - ax - b = 0$ for some $a, b \in \mathbb{Z}$, we think that the answers may be best described in terms of the Lucas sequence of the first kind. Nevertheless, the calculations even in the case of $B(\alpha)$ and $B(\alpha^2)$ are already complicated, so we postpone this for future research. See also other problems in the last chapter.

We arrange this thesis as follows. In Chapter 2, we give preliminaries and lemmas concerning the floor function, fractional parts, Beatty sequences, and Fibonacci numbers. In Chapter 3, we give our main results concerning various sumsets associated with $B(\alpha)$ and $B(\alpha^2)$. For more information, we invite the reader to visit Pongsriiam's ResearchGate website [38] for some freely downloadable articles [23, 31, 32, 33, 35, 36, 37] in related topics of research.

Chapter 2

Preliminaries and Lemmas

We often use the following fact: $-1 < \beta < 0$, $(|\beta^n|)_{n\geq 1}$ is strictly decreasing, if $a_1 > a_2 > \cdots > a_r$ are even positive integers, then $0 < \beta^{a_1} < \beta^{a_2} < \cdots < \beta^{a_r}$, and if $b_1 > b_2 > \cdots > b_r$ are odd positive integers, then $0 > \beta^{b_1} > \beta^{b_2} > \cdots > \beta^{b_r}$. In addition, α and β are roots of the equation $x^2 - x - 1 = 0$. So, for instance, $\beta^2 = \beta + 1$, $\beta^2 + \beta^4 = 4\beta + 3$, $\alpha\beta = -1$, $\sqrt{5}\beta + \beta = -2$, $\sqrt{5}\beta^2 + 1 = -3\beta$, and $\beta^n + \sqrt{5}\beta^{n-1} + \beta^{n-2} = 0$ for all $n \geq 2$. Moreover, it is useful to have the following numerical approximations: $-0.619 < \beta < -0.618$, $-0.237 < \beta^3 < -0.236$, $0.854 < \sqrt{5}\beta^2 < 0.855$, $-0.528 < \sqrt{5}\beta^3 < -0.527$, $0.326 < \sqrt{5}\beta^4 < 0.327$ and it is convenient to have a list of the first twenty elements of the sequences $B(\alpha)$ and $B(\alpha^2)$ as shown below:

$$B(\alpha) = (1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, \ldots) \text{ and}$$

$$B(\alpha^2) = (2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, 31, 34, 36, 39, 41, 44, 47, 49, 52, \ldots).$$

The following results are also applied throughout this thesis sometimes without reference.

As introduced in the first chapter, for each $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ be the largest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$. Basic properties of $\lfloor x \rfloor$ and $\{x\}$ are as follows.

Lemma 2.1. *For $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, the following statements hold.*

(i) $\lfloor n+x \rfloor = n + \lfloor x \rfloor$.

(ii) $\{n+x\} = \{x\}$.

(iii) $0 \le \{x\} < 1$.

(iv) $\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\}+\{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\}+\{y\} \ge 1. \end{cases}$

Proof. These are well-known and can be proved easily. For more details, see in [9, Chapter 3]. We also refer the reader to [19] and [36, Proof of Lemma 2.6] for a nice application of these properties. □

Lemma 2.2. *The following statements hold for all $n \in \mathbb{N}$.*

(i) (Binet's formula) $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$.

(ii) $\beta^{n+1} = \beta F_{n+1} + F_n$.

(iii) $F_{n+1} = \beta^n + \alpha F_n$.

(iv) $\beta L_{n+1} + L_n = -\sqrt{5}\beta^{n+1}$.

(v) $L_n \alpha = L_{n+1} + \sqrt{5}\beta^n$.

Proof. The proof of (i) and (ii) can be found in [17, pp. 78–79]. The statement (iii) follows from (ii) and the fact that $\alpha\beta = -1$. See also [30] for a result concerning the generating function of the Fibonacci sequence. Since $\alpha\beta = -1$, multiplying (iv) by α, we obtain (v). The formula (iv) follows from (i) and a straightforward calculation:

$$\beta L_{n+1} + L_n = \beta\alpha^{n+1} + \beta^{n+2} + \alpha^n + \beta^n = \beta^{n+2} + \beta^n = \beta^n(-\sqrt{5}\beta) = -\sqrt{5}\beta^{n+1}.$$

□

Lemma 2.3. (Zeckendorf's theorem) For each $n \in \mathbb{N}$, $n = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$ where F_{a_1} is the largest Fibonacci number not exceeding n, $a_\ell \ge 2$, and $a_{i-1} - a_i \ge 2$ for every $i = 2, 3, \ldots, \ell$.

Proof. This is well-known and can be proved by using the greedy algorithm ([41, pp. 108–109] or [44]). See also [16] for a more general result. □

Lemma 2.4. *If $x_1, x_2, \ldots, x_n \in \mathbb{R}$, then*

$$\{x_1 + x_2 + \cdots + x_n\} = \{\{x_1\} + \{x_2\} + \cdots + \{x_n\}\}.$$

Proof. We can write $x_1 + x_2 + \cdots + x_n = m + \{x_1\} + \{x_2\} + \cdots + \{x_n\}$, where $m = \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \cdots + \lfloor x_n \rfloor \in \mathbb{Z}$. So this lemma follows immediately from Lemma 2.1. □

Lemma 2.5. *Let $n \in \mathbb{N}$. Then the following statements hold.*

(i) $\lfloor F_n\alpha \rfloor = F_{n+1} - [n \equiv 0 \pmod 2]$.

(ii) $\lfloor F_n\alpha^2 \rfloor = F_{n+2} - [n \equiv 0 \pmod 2]$.

(iii) $\{F_n\alpha\} = -\beta^n + [n \equiv 0 \pmod 2]$.

(iv) $\{F_n\alpha^2\} = \{F_n\alpha\}$.

(v) $\lfloor L_n\alpha \rfloor = L_{n+1} - [n \equiv 1 \pmod 2]$.

(vi) $\{L_n\alpha\} = \sqrt{5}\beta^n + [n \equiv 1 \pmod 2]$.

(vii) $\lfloor L_n\alpha^2 \rfloor = L_{n+2} - [n \equiv 1 \pmod 2]$.

(viii) $\{L_n\alpha^2\} = \{L_n\alpha\}$.

Proof. By Lemmas 2.2 and 2.1, we obtain $\lfloor F_n\alpha \rfloor = \lfloor F_{n+1} - \beta^n \rfloor = F_{n+1} + \lfloor -\beta^n \rfloor$. If n is even, then $0 < \beta^n < 1$ and so $\lfloor -\beta^n \rfloor = -1$. If n is odd, then $-1 < \beta^n < 0$ and so $\lfloor -\beta^n \rfloor = 0$. Therefore $\lfloor -\beta^n \rfloor = -[n \equiv 0 \pmod 2]$. This implies (i). Then (ii) follows from (i) by writing $\alpha^2 = \alpha + 1$ and $\lfloor F_n\alpha^2 \rfloor = \lfloor F_n\alpha + F_n \rfloor = \lfloor F_n\alpha \rfloor + F_n$. Next, $\{F_n\alpha\} = F_n\alpha - \lfloor F_n\alpha \rfloor$, so (iii) can be obtained from (i) and Lemma 2.2. For (iv), we have $\{F_n\alpha^2\} = \{F_n\alpha + F_n\} = \{F_n\alpha\}$. By Lemma 2.2(v), we obtain $\lfloor L_n\alpha \rfloor = L_{n+1} + \lfloor \sqrt{5}\beta^n \rfloor$. If n is even, then $0 < \sqrt{5}\beta^n \le \sqrt{5}\beta^2 < 1$, and so $\lfloor \sqrt{5}\beta^n \rfloor = 0$. If n is odd, then $-1 <$

$\sqrt{5}\beta^3 \leq \sqrt{5}\beta^n < 0$ and thus $\lfloor \sqrt{5}\beta^n \rfloor = -1$. This implies (v). Then (vi) is a consequence of (v) and Lemma 2.2(v). By writing $\alpha^2 = \alpha + 1$, we obtain (vii) from (v), and (viii) from Lemma 2.1(ii). This completes the proof. $\square$

Lemma 2.6. (Beatty's theorem [1, 2]) Let x and y be irrational numbers such that $x, y > 1$ and $\frac{1}{x} + \frac{1}{y} = 1$. Then $B(x) \cup B(y) = \mathbb{N}$ and $B(x) \cap B(y) = \emptyset$. In particular, $B(\alpha) \cup B(\alpha^2) = \mathbb{N}$ and $B(\alpha) \cap B(\alpha^2) = \emptyset$.

If $A = (a_n)_{n \geq 1}$ is a sequence, then a segment of A is a finite sequence of the form $(a_k, a_{k+1}, \ldots, a_{k+m})$ for some $k, m \in \mathbb{N}$. Then we have the following results.

Lemma 2.7. *The following statements hold.*

(i) *For each* $b \in \mathbb{N}$, $\lfloor (b+1)\,\alpha \rfloor - \lfloor b\alpha \rfloor$ *is either* 1 *or* 2.

(ii) *For each* $b \in \mathbb{N}$, *if* $\lfloor (b+1)\,\alpha \rfloor - \lfloor b\alpha \rfloor = 1$ *then* $\lfloor (b+2)\alpha \rfloor - \lfloor (b+1)\,\alpha \rfloor = 2$.

(iii) *The sequence* $(\lfloor (b+1)\,\alpha \rfloor - \lfloor b\alpha \rfloor)_{b \geq 1}$ *does not contain the segment* $(2, 2, 2)$.

Proof. Let $b \in \mathbb{N}$. By Lemma 2.1, $\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor = \lfloor b\alpha + \alpha \rfloor - \lfloor b\alpha \rfloor = \lfloor \alpha \rfloor$ or $\lfloor \alpha \rfloor + 1 = 1$ or 2. This proves (i). For (ii), suppose that $\lfloor (b+1)\,\alpha \rfloor - \lfloor b\alpha \rfloor = 1 = \lfloor (b+2)\,\alpha \rfloor - \lfloor (b+1)\,\alpha \rfloor$. Then $2 = \lfloor (b+2)\,\alpha \rfloor - \lfloor b\alpha \rfloor \geq \lfloor 2\alpha \rfloor \geq 3$, which is a contradiction. For (iii), suppose that $(2, 2, 2)$ is a segment of the sequence $(\lfloor (b+1)\,\alpha \rfloor - \lfloor b\alpha \rfloor)_{b \geq 1}$, that is, there exists $b \in \mathbb{N}$ such that

$$\lfloor (b+1)\,\alpha \rfloor - \lfloor b\alpha \rfloor = 2, \tag{2.1}$$

$$\lfloor (b+2)\,\alpha \rfloor - \lfloor (b+1)\,\alpha \rfloor = 2, \tag{2.2}$$

$$\lfloor (b+3)\,\alpha \rfloor - \lfloor (b+2)\alpha \rfloor = 2. \tag{2.3}$$

Adding (2.1) to (2.3), we have $6 = \lfloor (b+3)\,\alpha \rfloor - \lfloor b\alpha \rfloor \leq \lfloor 3\alpha \rfloor + 1 = 5$, which is a contradiction. $\square$

Lemma 2.8. *Let* $b \in \mathbb{N}$. *Then the following statements hold.*

(i) $\lfloor (b+1)\,\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor$ *is either* 2 *or* 3.

(ii) *If* $\lfloor (b+1)\,\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor = 2$*, then* $\lfloor (b+2)\,\alpha^2 \rfloor - \lfloor (b+1)\,\alpha^2 \rfloor = 3$.

(iii) *The sequence* $(\lfloor (b+1)\,\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor)_{b\geq 1}$ *does not contain the segment* $(3,3,3)$.

Proof. By Lemma 2.1, $\lfloor (b+1)\,\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor = \lfloor (b+1)\,\alpha \rfloor - \lfloor b\alpha \rfloor + 1$. Therefore this lemma is an immediate consequence of Lemma 2.7. $\square$

Chapter 3

Main Results

In this chapter, we study various sumsets associated with Wythoff sequences. We begin with simple cases such as $B(\alpha) + B(\alpha)$ and $B_0(\alpha) + B(\alpha)$.

Theorem 3.1. *Let $B(\alpha)$ and $B_0(\alpha)$ be the sets as defined in* (1.1)*. Then*

$$B(\alpha) + B(\alpha) = \mathbb{N} \setminus \{1, 3\} \text{ and } B_0(\alpha) + B(\alpha) = \mathbb{N}.$$

Proof. It is easy to check that $1, 3 \notin B(\alpha) + B(\alpha)$ and $2 = \lfloor \alpha \rfloor + \lfloor \alpha \rfloor \in B(\alpha) + B(\alpha)$. So we let $n \geq 4$ and show that $n \in B(\alpha) + B(\alpha)$. Let b be the largest positive integer such that $b\alpha < n$. Then $b \geq 2$ and $\lfloor b\alpha \rfloor < n \leq \lfloor (b+1)\alpha \rfloor$. By Lemma 2.7(i), $n = \lfloor b\alpha \rfloor + k$, where k is either 1 or 2. If $k = 1$, then $n = \lfloor b\alpha \rfloor + \lfloor \alpha \rfloor \in B(\alpha) + B(\alpha)$. So assume that $k = 2$. By Lemma 2.7(i), we can divide the proof into two cases. If $\lfloor b\alpha \rfloor - \lfloor (b-1)\alpha \rfloor = 1$, then $n = \lfloor b\alpha \rfloor + 2 = \lfloor (b-1)\alpha \rfloor + 3 = \lfloor (b-1)\alpha \rfloor + \lfloor 2\alpha \rfloor$. If $\lfloor b\alpha \rfloor - \lfloor (b-1)\alpha \rfloor = 2$, then $n = \lfloor b\alpha \rfloor + 2 = \lfloor (b-1)\alpha \rfloor + 4 = \lfloor (b-1)\alpha \rfloor + \lfloor 3\alpha \rfloor$. In any case, we have $n \in B(\alpha) + B(\alpha)$, as desired. Since 1 and 3 are in $B_0(\alpha) + B(\alpha)$ and $B(\alpha) + B(\alpha) \subseteq B_0(\alpha) + B(\alpha)$, we obtain that $B_0(\alpha) + B(\alpha) = \mathbb{N}$. $\square$

Theorem 3.2. *Let $B(\alpha)$ and $B(\alpha^2)$ be defined as in* (1.1) *and $n \geq 3$. Then the following statements hold.*

(i) *$F_n \in B(\alpha)$ if and only if n is even.*

(ii) *$F_n \in B(\alpha^2)$ if and only if n is odd.*

(iii) $F_n - 1 \in B(\alpha)$ *if and only if* n *is odd.*

(iv) $F_n - 1 \in B(\alpha^2)$ *if and only if* n *is even.*

Proof. By Lemma 2.5, we have

$$F_n - [n \equiv 0 \pmod 2] = \lfloor F_{n-2}\alpha^2 \rfloor \in B(\alpha^2),$$
$$F_n - [n \equiv 1 \pmod 2] = \lfloor F_{n-1}\alpha \rfloor \in B(\alpha).$$

Case 1 n is even. Then by the above equality, we have $F_n - 1 \in B(\alpha^2)$ and $F_n \in B(\alpha)$. Then by Lemma 2.6, $F_n - 1 \notin B(\alpha)$ and $F_n \notin B(\alpha^2)$.
Case 2 n is odd. Then $F_n \in B(\alpha^2)$ and $F_n - 1 \in B(\alpha)$. Then by Lemma 2.6, $F_n \notin B(\alpha)$ and $F_n - 1 \notin B(\alpha^2)$. This implies the desired result. $\square$

The calculation of $B(\alpha) + B(\alpha^2)$ is a bit more complicated than $B(\alpha) + B(\alpha)$ and we need the following theorem.

Theorem 3.3. *Let* $n \geq 3$ *and* $1 \leq b \leq F_{n+1}$. *If* $b \neq F_n$, *then* $0 < \{b\alpha\} + \beta^n < 1$. *If* $b = F_n$, *then* $\{b\alpha\} + \beta^n = [n \equiv 0 \pmod 2]$.

Proof. We use Lemma 2.5 repeatedly without reference. If $b = F_n$, then the result follows immediately. If $b = F_{n+1}$, then $\{b\alpha\} + \beta^n$ is equal to

$$\begin{aligned}-\beta^{n+1} + [n+1 \equiv 0 \pmod 2] + \beta^n &= -\beta^{n-1} + [n-1 \equiv 0 \pmod 2] \\ &= \{F_{n-1}\alpha\} \in (0,1).\end{aligned}$$

Next we consider the case $b = F_k$ for some $k \in \{2, 3, \ldots, n-1\}$. If k is even and n is odd, then

$$1 > \{b\alpha\} > \{b\alpha\} + \beta^n = 1 - \beta^k + \beta^n \geq 1 - \beta^2 + \beta^3 = 1 + \beta > 0.$$

If k and n are even, then $0 < \{b\alpha\} + \beta^n = 1 - \beta^k + \beta^n < 1$. Similarly, if k is odd and n is even, then $0 < \{b\alpha\} + \beta^n = -\beta^k + \beta^n \leq \beta^n - \beta^3 \leq \beta^4 - \beta^3 = \beta^2 < 1$. If k and n are odd, then $1 > \{b\alpha\} + \beta^n = -\beta^k + \beta^n > 0$. Hence this theorem is verified in the case $b = F_k$ for some $k \leq n+1$. Next, we suppose that $F_k < b < F_{k+1}$ for some $k \in \{4, 5, \ldots, n\}$. By Lemma 2.3, we can write

$b = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$ where $\ell \geq 2$, $k = a_1 > a_2 > \cdots > a_\ell \geq 2$, and $a_{i-1} - a_i \geq 2$ for every $i = 2, 3, \ldots, \ell$. Then by Lemma 2.4, we obtain $\{b\alpha\} = \{\{F_{a_1}\alpha\} + \{F_{a_2}\alpha\} + \cdots + \{F_{a_\ell}\alpha\}\}$, which is equal to

$$\left\{\left(1 - \beta^{b_1} + 1 - \beta^{b_2} + \cdots + 1 - \beta^{b_r}\right) + \left(-\beta^{c_1} - \beta^{c_2} - \cdots - \beta^{c_s}\right)\right\},$$

where $\{b_1, b_2, \ldots, b_r\} \cup \{c_1, c_2, \ldots, c_s\} = \{a_1, a_2, \ldots, a_\ell\}$, $b_1, b_2, \ldots, b_r$ are even, and $c_1, c_2, \ldots, c_s$ are odd.

Remark that one of the sets $\{b_1, b_2, \ldots, b_r\}$, $\{c_1, c_2, \ldots, c_s\}$ may be empty. In that case, such the set disappears from the subsequent calculation. Also, for convenience, we let $A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} + \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s}$. Then by Lemma 2.1, $\{b\alpha\} = \{-A\}$.

Case 1 $\{b_1, b_2, \ldots, b_r\}$ is empty. Then $A = \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^3 + \beta^5 + \beta^7 + \cdots = \frac{\beta^3}{1-\beta^2} = -\beta^2$. Therefore $0 < -A < \beta^2 < 1$ and so $\{b\alpha\} = \{-A\} = -A - \lfloor -A \rfloor = -A$. Then

$$\{b\alpha\} + \beta^n < \beta^2 + \beta^n \leq \beta^2 + \beta^4 = 4\beta + 3 < 1.$$

It remains to show that $\{b\alpha\} + \beta^n > 0$. If n is even, then obviously $\{b\alpha\} + \beta^n > 0$. So assume that n is odd. Since $\{b_1, b_2, \ldots, b_r\}$ is empty, we see that a_1 is odd and $-A > -\beta^{a_1}$. Therefore $\{b\alpha\} + \beta^n > -\beta^{a_1} + \beta^n \geq 0$, as required.

Case 2 $\{c_1, c_2, \ldots, c_s\}$ is empty. Then $A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^2 + \beta^4 + \beta^6 + \cdots = \frac{\beta^2}{1-\beta^2} = -\beta$. In addition, a_1 is even and $A > \beta^{a_1}$. Therefore $-\beta^{a_1} > -A > \beta > -1$ and so $\{b\alpha\} = \{-A\} = 1 - A$. Then $\{b\alpha\} + \beta^n < 1 - \beta^{a_1} + \beta^n \leq 1$, and $\{b\alpha\} + \beta^n > 1 + \beta + \beta^n \geq 1 + \beta + \beta^3 = 3\beta + 2 > 0$.

Case 3 $\{b_1, b_2, \ldots, b_r\}$ and $\{c_1, c_2, \ldots, c_s\}$ are not empty. Then there is some cancellation in the sum defining A. Similar to Case 1 and Case 2, we have $A < \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < -\beta$ and $A > \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > -\beta^2$.

Case 3.1 A is positive. Then $-1 < \beta < -A < 0$ and $\{b\alpha\} + \beta^n = 1 - A + \beta^n$. So it suffices to show that $\beta^n < A < 1 + \beta^n$. Since $A < -\beta$, we obtain $A - \beta^n < -\beta - \beta^3 = -3\beta - 1 < 1$, which implies $A < 1 + \beta^n$. So it remains to show that $A > \beta^n$. If n is odd, then $A > 0 > \beta^n$. So suppose that n is even.

Let u be the smallest even number among $\{b_1, b_2, \ldots, b_r\}$ and v the smallest odd number among $\{c_1, c_2, \ldots, c_s\}$. Since $a_i - a_{i-1} \geq 2$ for all $i = 2, 3, \ldots, \ell$ and $a_1 = k \leq n$, we obtain $u \leq n$ and $|v - u| \geq 3$. Then

$$\beta^u < \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^u + \beta^{u+2} + \beta^{u+4} + \cdots = \frac{\beta^u}{1-\beta^2} = -\beta^{u-1}, \tag{3.1}$$

$$\beta^v > \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^v + \beta^{v+2} + \beta^{v+4} + \cdots = \frac{\beta^v}{1-\beta^2} = -\beta^{v-1}. \tag{3.2}$$

By (3.1) and (3.2), we obtain $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v - u| \geq 3$, we see that either $v - u \geq 3$ or $v - u \leq -3$. Suppose for a contradiction that $v - u \leq -3$. Since $v \leq u - 3$ and both v and $u - 3$ are odd, we have $\beta^v \leq \beta^{u-3}$. So $A < \beta^{u-3} - \beta^{u-1} = \beta^{u-3}(1 - \beta^2) = -\beta^{u-2} < 0$, which contradicts the assumption that A is positive. Hence $v - u \geq 3$. Since $v - 1 \geq u + 2$ and both $v - 1$ and $u + 2$ are even, $\beta^{v-1} \leq \beta^{u+2}$. So $A > \beta^u - \beta^{u+2} = \beta^u(1 - \beta^2) = -\beta^{u+1}$. We have $u \leq v - 3 \leq a_1 - 3 \leq n - 3$. Therefore $-\beta^{u+1} = |\beta|^{u+1} > |\beta|^n = \beta^n$. Therefore $A > \beta^n$, as desired.

Case 3.2 A is negative. Then $0 < -A < \beta^2 < 1$ and

$$\{b\alpha\} + \beta^n = \{-A\} + \beta^n = -A + \beta^n < \beta^2 + \beta^n \leq \beta^2 + \beta^4 = 4\beta + 3 < 1.$$

To show that $\{b\alpha\} + \beta^n > 0$, it is enough to show that $\beta^n > A$. If n is even, then obviously $\beta^n > 0 > A$. So assume that n is odd. Let u and v be as in Case 3.1. Then we obtain $u \leq n$, $|v - u| \geq 3$, the inequalities in (3.1) and (3.2) hold, and $\beta^u - \beta^{v-1} < A < -\beta^{u-1} + \beta^v$. Again, we have either $v - u \geq 3$ or $v - u \leq -3$. Suppose for a contradiction that $v - u \geq 3$. Following the argument in Case 3.1, we obtain $A > \beta^u - \beta^{v-1} \geq -\beta^{u+1} > 0$, which contradicts the assumption that A is negative. Therefore $v - u \leq -3$. Then $A < -\beta^{u-1} + \beta^v \leq -\beta^{u-1} + \beta^{u-3} = -\beta^{u-2}$. Since $u - 2 < n$, u is even, and n is odd, we obtain $-\beta^{u-2} = -|\beta|^{u-2} < -|\beta|^n = \beta^n$. Therefore $A < \beta^n$, as desired. Hence the proof is complete. $\square$

Corollary 3.4. *For each $n \geq 3$ and $1 \leq b \leq F_{n+1}$, we have*

$$F_{n+1} = \lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta \text{ and } F_{n+2} = \lfloor (F_n - b)\,\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + 1 - \delta,$$

where $\delta = [n \equiv 1 \pmod 2][b = F_n]$.

Proof. Let $n \geq 3$ and $1 \leq b \leq F_{n+1}$. If $b = F_n$, then we obtain by Lemma 2.5 that $\lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta = F_{n+1} - [n \equiv 0 \pmod 2] + 1 - [n \equiv 1 \pmod 2] = F_{n+1}$. So suppose $b \neq F_n$. Then $\delta = 0$ and we obtain by Lemmas 2.1, 2.2 and Theorem 3.3, respectively, that $\lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta$ is equal to

$$\lfloor F_n\alpha - b\alpha + \lfloor b\alpha \rfloor + 1 \rfloor = \lfloor F_{n+1} - \beta^n - \{b\alpha\} + 1 \rfloor = F_{n+1} + \lfloor 1 - \{b\alpha\} - \beta^n \rfloor = F_{n+1}.$$

This proves the first equality. By writing $\alpha^2 = \alpha + 1$ and applying Lemma 2.1, we see that

$$\lfloor (F_n - b)\,\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + 1 - \delta = \lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta + F_n = F_{n+2}.$$

□

Theorem 3.5. *Let $B(\alpha)$, $B_0(\alpha)$, $B(\alpha^2)$, and $B_0(\alpha^2)$ be the sets as defined in (1.1). Then we have*

(i) $B\,(\alpha) + B\,(\alpha^2) = \mathbb{N} \setminus \{F_n - 1 \mid n \geq 3\}$,

(ii) $B_0\,(\alpha) + B\,(\alpha^2) = \mathbb{N} \setminus \{F_n - 1 \mid n \geq 3$ *and n is odd*$\}$, *and*

(iii) $B\,(\alpha) + B_0\,(\alpha^2) = \mathbb{N} \setminus \{F_n - 1 \mid n \geq 3$ *and n is even*$\}$.

Proof. We first show that $B\,(\alpha) + B\,(\alpha^2) \subseteq \mathbb{N} \setminus \{F_n - 1 \mid n \geq 3\}$. It is easy to check that $F_3 - 1, F_4 - 1 \notin B\,(\alpha) + B\,(\alpha^2)$. So let $n \geq 5$. In order to get a contradiction, suppose $F_n - 1$ is in $B\,(\alpha) + B\,(\alpha^2)$. Then $F_n - 1 = \lfloor b\alpha \rfloor + \lfloor a\alpha^2 \rfloor$ for some $a, b \in \mathbb{N}$. If $b \geq F_{n-1}$, then we obtain by Lemma 2.5 that

$$\lfloor b\alpha \rfloor + \lfloor a\alpha^2 \rfloor \geq \lfloor F_{n-1}\alpha \rfloor + \lfloor \alpha^2 \rfloor = F_n - [n \equiv 1 \pmod 2] + 2 > F_n - 1,$$

which is not in case. So $b < F_{n-1}$. Replacing n by $n-1$ in Corollary 3.4, we have $\lfloor a\alpha^2 \rfloor = F_n - 1 - \lfloor b\alpha \rfloor = \lfloor (F_{n-1} - b)\alpha \rfloor \in B(\alpha)$, so $\lfloor a\alpha^2 \rfloor \in B(\alpha) \cap B(\alpha^2)$, which contradicts Lemma 2.6. Therefore $F_n - 1 \notin B(\alpha) + B(\alpha^2)$ for any $n \geq 3$. This shows that $B(\alpha) + B(\alpha^2)$ is a subset of $\mathbb{N} \setminus \{F_n - 1 \mid n \geq 3\}$. For the other direction, let $m \in \mathbb{N} \setminus \{F_n - 1 \mid n \geq 3\}$. Then there exists $n \in \mathbb{N}$ such that $n \geq 3$ and $F_n - 1 < m < F_{n+1} - 1$. Thus $m = F_n - 1 + b$ where $1 \leq b < F_{n-1}$. By Corollary 3.4, we obtain $m = \lfloor (F_{n-1} - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + b = \lfloor (F_{n-1} - b)\,\alpha \rfloor + \lfloor b\alpha^2 \rfloor \in B\,(\alpha) + B\,(\alpha^2)$. This proves (i). Next $B_0\,(\alpha) + B\,(\alpha^2) = (B(\alpha) + B(\alpha^2)) \cup B(\alpha^2) = \mathbb{N} \setminus \{F_n - 1 \mid n \geq 3 \text{ and } n \text{ is odd}\}$, by (i) and Theorem 3.2. Similarly, (iii) can be obtained by using (i) and Theorem 3.2. This completes the proof. $\square$

Remark 3.6. It follows immediately from Beatty's theorem that $B_0\,(\alpha) + B_0\,(\alpha^2) = \mathbb{N}$.

Theorem 3.7. *Let $B(\alpha)$, $B_0(\alpha)$, $B(\alpha^2)$, and $B_0(\alpha^2)$ be defined as in* (1.1). *Then the following statements hold.*

(i) $B\,(\alpha) + B\,(\alpha^2) + B\,(\alpha^2) = \mathbb{N} \setminus \{1, 2, 3, 4, 6, 9\}$.

(ii) $B_0\,(\alpha) + B\,(\alpha^2) + B\,(\alpha^2) = \mathbb{N} \setminus \{1, 2, 3, 6\}$.

(iii) $B\,(\alpha) + B\,(\alpha^2) + B_0\,(\alpha^2) = \mathbb{N} \setminus \{1, 2, 4\}$.

(iv) $B\,(\alpha) + B_0\,(\alpha^2) + B_0\,(\alpha^2) = \mathbb{N} \setminus \{2\}$.

Proof. We can write Theorem 3.5 in another form as

$$B(\alpha) + B(\alpha^2) = \bigcup_{n=4}^{\infty} ((F_n - 1, F_{n+1} - 1) \cap \mathbb{N}) = \bigcup_{n=4}^{\infty} ([F_n, F_{n+1} - 2] \cap \mathbb{N})\,.$$

Then $B(\alpha) + B(\alpha^2) + \lfloor \alpha^2 \rfloor = \bigcup_{n=4}^{\infty} ([F_n + 2, F_{n+1}] \cap \mathbb{N}) = \mathbb{N} \setminus A$, where $A = \{F_n + 1 \mid n \geq 5\} \cup \{1, 2, 3, 4\}$. Similarly, $B(\alpha) + B(\alpha^2) + \lfloor 2\alpha^2 \rfloor = \mathbb{N} \setminus B$ where $B = \{F_m + 4 \mid m \geq 2\} \cup \{1, 2, 3, 4\}$. Therefore $\mathbb{N} \setminus (A \cap B) = (\mathbb{N} \setminus A) \cup (\mathbb{N} \setminus B) \subseteq B(\alpha) + B(\alpha^2) + B(\alpha^2)$. It is easy to see that

$$\begin{aligned} A \cap B &= (\{F_n + 1 \mid n \geq 5\} \cap \{F_m + 4 \mid m \geq 2\}) \cup \{1, 2, 3, 4\} \\ &= (\{F_n + 1 \mid n \geq 7\} \cap \{F_m + 4 \mid m \geq 6\}) \cup \{1, 2, 3, 4, 6, 9\}. \end{aligned}$$

If $n \geq 7$, $m \geq 6$, and $F_n + 1 = F_m + 4$, then $n > m$ and $3 = F_n - F_m \geq F_n - F_{n-1} = F_{n-2} \geq 5$, which is a contradiction. So $\{F_n + 1 \mid n \geq 7\} \cap \{F_m + 4 \mid m \geq 6\} = \emptyset$. Therefore $A \cap B = \{1, 2, 3, 4, 6, 9\}$ and thus $\mathbb{N} \setminus \{1, 2, 3, 4, 6, 9\} \subseteq B(\alpha) + B(\alpha^2) + B(\alpha^2)$. It is easy to check that $1, 2, 3, 4, 6, 9 \notin B(\alpha) + B(\alpha^2) + B(\alpha^2)$. This proves (i). The other parts follows from (i) and straightforward verification. $\square$

The structure of $B(\alpha^2) + B(\alpha^2)$ seems to be the most complicated among sumsets associated with $B(\alpha)$ and $B(\alpha^2)$. So we first consider a simpler sumset $B(\alpha^2) + B(\alpha^2) + B(\alpha^2)$.

Theorem 3.8. *Let $B(\alpha^2)$ and $B_0(\alpha^2)$ be defined as in* (1.1). *Then we have*

$$\begin{aligned}
B\left(\alpha^2\right) + B\left(\alpha^2\right) + B\left(\alpha^2\right) &= \mathbb{N} \setminus \{1, 2, 3, 4, 5, 7, 8, 10, 13, 18, 26\}, \\
B_0\left(\alpha^2\right) + B\left(\alpha^2\right) + B\left(\alpha^2\right) &= \mathbb{N} \setminus \{1, 2, 3, 5, 8, 13\}, \\
B_0\left(\alpha^2\right) + B_0\left(\alpha^2\right) + B\left(\alpha^2\right) &= \mathbb{N} \setminus \{1, 3, 8\}.
\end{aligned}$$

Proof. Let $A_1 = \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + B(\alpha^2)$, $A_2 = \lfloor 5\alpha^2 \rfloor + \lfloor 5\alpha^2 \rfloor + B(\alpha^2)$, and $A_3 = \lfloor 3\alpha^2 \rfloor + \lfloor 8\alpha^2 \rfloor + B(\alpha^2)$. We first show that $A_1 \cup A_2 \cup A_3 = \{n \in \mathbb{N} \mid n \geq 27\}$. Note that $\lfloor 3\alpha^2 \rfloor$, $\lfloor 4\alpha^2 \rfloor$, $\lfloor 5\alpha^2 \rfloor$, $\lfloor 6\alpha^2 \rfloor$, $\lfloor 8\alpha^2 \rfloor$ are equal to 7, 10, 13, 15, 20, respectively. Then it is easy to see that every element in $A_1 \cup A_2 \cup A_3$ is larger than or equal to 27. Next, let $n \geq 27$. Then there exists $k \in \mathbb{N}$ such that

$$\lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor \leq n < \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor (k+1)\,\alpha^2 \rfloor.$$

By Lemma 2.8, we have $\lfloor (k+1)\,\alpha^2 \rfloor - \lfloor k\alpha^2 \rfloor = 2$ or 3, and so $n = \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor + b$, where $b = 0, 1$ or 2. If $b = 0$, then $n \in A_1$. If $b = 1$, then $n = \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor + 1 = \lfloor 5\alpha^2 \rfloor + \lfloor 5\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor \in A_2$. Similarly, if $b = 2$, then $n = \lfloor 3\alpha^2 \rfloor + \lfloor 8\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor \in A_3$. In any case, $n \in A_1 \cup A_2 \cup A_3$, as required. This implies that $B(\alpha^2) + B(\alpha^2) + B(\alpha^2)$ contains $\mathbb{N} \cap [27, \infty)$. For the integers in $\mathbb{N} \cap [1, 26]$, we can straightforwardly check whether they are in $B(\alpha^2) + B(\alpha^2) + B(\alpha^2)$ or not. For the reader's convenience, we give the integers which are in $B(\alpha^2) + B(\alpha^2) + B(\alpha^2)$ as follows: $6 = 2 + 2 + 2$,

$9 = 5+2+2$, $11 = 7+2+2$, $12 = 5+5+2$, $14 = 10+2+2$, $15 = 5+5+5$, $16 = 7+7+2$, $17 = 7+5+5$, $19 = 15+2+2$, $20 = 10+5+5$, $21 = 7+7+7$, $22 = 18+2+2$, $23 = 13+5+5$, $24 = 20+2+2$, $25 = 15+5+5$. This proves the first part. The other parts follow from the first part and straightforward verification. □

In order to prove Theorem 3.10, it is convenient to use the following observation.

Lemma 3.9. *Let $n \geq 3$, $a \in \mathbb{Z}$, and $b, c \in \mathbb{N}$. If $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$, then b and c are less than $F_{n-2} + \frac{a}{\alpha^2}$.*

Proof. If b or $c \geq F_{n-2} + \frac{a}{\alpha^2}$, then $\lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ is larger than or equal to

$$\left\lfloor F_{n-2}\alpha^2 + a \right\rfloor + \left\lfloor \alpha^2 \right\rfloor = F_n - [n \equiv 0 \pmod 2] + a + 2 > F_n + a.$$

□

The positive integers in $\mathbb{N} \setminus (B(\alpha^2) + B(\alpha^2))$ are 1, 2, 3, 5, 6, 8, 11, 13, 16, 19, 21, 24, 29, 32, 34, 37, 42, 45, 50, 53, 55, ... From this, we notice the following pattern.

Theorem 3.10. *Let $n \in \mathbb{N}$ and $B(\alpha^2)$ the Beatty set as defined in* (1.1)*. Then the following statements hold.*

(i) $F_n \notin B(\alpha^2) + B(\alpha^2)$.

(ii) *If $n \geq 5$, then $F_n - 1 \in B(\alpha^2) + B(\alpha^2)$.*

(iii) *If $n \neq 1, 2, 3, 5$, then $F_n + 1 \in B(\alpha^2) + B(\alpha^2)$.*

(iv) $F_n - 2 \notin B(\alpha^2) + B(\alpha^2)$.

(v) *If $n \neq 1, 2, 4$, then $F_n + 2 \in B(\alpha^2) + B(\alpha^2)$.*

(vi) *If $n \geq 7$, then $F_n - 3 \in B(\alpha^2) + B(\alpha^2)$.*

(vii) *If $n \geq 3$, then $F_n + 3 \notin B(\alpha^2) + B(\alpha^2)$.*

Proof. For $n \leq 6$, the result is easily checked. So we assume throughout that $n \geq 7$. For (i), suppose for a contradiction that $F_n \in B(\alpha^2) + B(\alpha^2)$. Then $F_n = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ for some $a, b \in \mathbb{N}$. By Lemma 3.9, b and c are less than F_{n-2}. By Corollary 3.4,

$$\lfloor c\alpha^2 \rfloor - \lfloor (F_{n-2} - b)\alpha^2 \rfloor = F_n - \lfloor b\alpha^2 \rfloor - \lfloor (F_{n-2} - b)\alpha^2 \rfloor = 1.$$

But by Lemma 2.8, we know that the difference between the elements in $B(\alpha^2)$ is at least two. So we obtain a contradiction. This proves (i). The statements (ii), (iii), (v), and (vi) also follow from applications of Corollary 3.4 as follows:

$$\begin{aligned}
F_n - 1 &= \lfloor (F_{n-2} - 1)\alpha^2 \rfloor + \lfloor \alpha^2 \rfloor \in B(\alpha^2) + B(\alpha^2), \\
F_n + 1 &= \lfloor (F_{n-2} - 2)\alpha^2 \rfloor + \lfloor 2\alpha^2 \rfloor + 2 \\
&= \lfloor (F_{n-2} - 2)\alpha^2 \rfloor + \lfloor 3\alpha^2 \rfloor \in B(\alpha^2) + B(\alpha^2), \\
F_n + 2 &= \lfloor (F_{n-2} - 1)\alpha^2 \rfloor + \lfloor \alpha^2 \rfloor + 3 \\
&= \lfloor (F_{n-2} - 1)\alpha^2 \rfloor + \lfloor 2\alpha^2 \rfloor \in B(\alpha^2) + B(\alpha^2), \\
F_n - 3 &= \lfloor (F_{n-2} - 3)\alpha^2 \rfloor + \lfloor 3\alpha^2 \rfloor - 2 \\
&= \lfloor (F_{n-2} - 3)\alpha^2 \rfloor + \lfloor 2\alpha^2 \rfloor \in B(\alpha^2) + B(\alpha^2).
\end{aligned}$$

So it remains to prove (iv) and (vii). Similar to the proof of (i), if $F_n - 2 = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$, then we have $b, c < F_{n-2} - \frac{2}{\alpha^2}$, $F_n = \lfloor (F_{n-2} - b)\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + 1$, and $\lfloor (F_{n-2} - b)\alpha^2 \rfloor - \lfloor c\alpha^2 \rfloor = \lfloor (F_{n-2} - b)\alpha^2 \rfloor - (F_n - 2 - \lfloor b\alpha^2 \rfloor) = 1$, which contradicts Lemma 2.8. For (vii), suppose that $F_n + 3 = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ for some $b, c \in \mathbb{N}$. Then by Lemma 3.9, b and $c < F_{n-2} + \frac{3}{\alpha^2} < F_{n-1}$. If b and c $= F_{n-2}$, then $F_n + 3 = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor = 2F_n - 2[n \equiv 0 \pmod 2]$, which leads to $F_7 \leq F_n = 3 + 2[n \equiv 0 \pmod 2] \leq 5$, a contradiction. So one of b, c is not equal to F_{n-2}. Without loss of generality, assume that $b \neq F_{n-2}$. So we can apply Corollary 3.4 and follow the same idea to obtain $\lfloor c\alpha^2 \rfloor - \lfloor (F_{n-2} - b)\alpha^2 \rfloor = 4$. By Lemma 2.8(i), the difference between consecutive terms in $B(\alpha^2)$ is either 2 or 3. So there are $k, r \in \mathbb{N} \cup \{0\}$ such that $4 = 2k + 3r$. If $r \geq 2$, then $2k + 3r > 4$. If $r = 1$, then $2k + 3r = 2k + 3 \neq 4$. So $r = 0$ and $k = 2$. This implies that

$c = F_{n-2} - b + 2$, $\lfloor c\alpha^2 \rfloor - \lfloor (c-1)\alpha^2 \rfloor = \lfloor (c-1)\alpha^2 \rfloor - \lfloor (c-2)\alpha^2 \rfloor = 2$, which contradicts Lemma 2.8(ii). So the proof is complete. □

Our next goal is to determine completely the integers a such that $F_n + a \in B(\alpha^2) + B(\alpha^2)$. The reader will see that there is a recurrence and fractal-like behavior involving those integers.

Lemma 3.11. *Let $n \geq 5$, $a \in \mathbb{Z}$, and $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ where $B(\alpha^2)$ is the set as defined in* (1.1). *Then the following statements hold.*

(i) *For every integer $d \in [-F_{n-3}, 0) \cup (0, F_{n-2})$, we have $a + 1 + \lfloor d\alpha^2 \rfloor \notin B(\alpha^2)$.*

(ii) $a + 1 - [n \equiv 1 \pmod 2] \notin B(\alpha^2)$.

Proof. Let $1 \leq b \leq F_{n-1}$ and $\delta_b = [n \equiv 1 \pmod 2][b = F_{n-2}]$. By Corollary 3.4, we have $F_n + a = \lfloor (F_{n-2} - b)\alpha^2 \rfloor + a + 1 - \delta_b + \lfloor b\alpha^2 \rfloor$. Since $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ and $\lfloor b\alpha^2 \rfloor \in B(\alpha^2)$, we see that

$$\left\lfloor (F_{n-2} - b)\alpha^2 \right\rfloor + a + 1 - \delta_b \notin B(\alpha^2). \tag{3.3}$$

Since (3.3) holds for all $b \leq F_{n-1}$, we can substitute $b = F_{n-2}$ in (3.3) to obtain (ii). Similarly, by running b over the integers in $[1, F_{n-2}) \cup (F_{n-2}, F_{n-1}]$, we obtain (i). □

Suppose $n \geq 5$ and the integers in $[F_n, F_{n+1}] \cap (B(\alpha^2) + B(\alpha^2))$ are given. Then the next theorem gives us some integers in $[F_{n+1}, F_{n+2}] \cap (B(\alpha^2) + B(\alpha^2))$.

Theorem 3.12. *Let $B(\alpha^2)$ be the set as defined in* (1.1). *Let $n \geq 5$, $a \in \mathbb{Z}$, and $1 \leq a < F_n - 2$. If $F_n + a \in B(\alpha^2) + B(\alpha^2)$, then $F_{n+1} + a \in B(\alpha^2) + B(\alpha^2)$.*

Proof. If $n = 5$, the result is easily checked. So assume that $n \geq 6$. Suppose for a contradiction that $F_n + a \in B(\alpha^2) + B(\alpha^2)$ but $F_{n+1} + a \notin B(\alpha^2) + B(\alpha^2)$. By applying Lemma 3.11 to $F_{n+1} + a$, we obtain that

$$a + 1 + \left\lfloor d\alpha^2 \right\rfloor \notin B(\alpha^2) \text{ for } 0 < d < F_{n-1}. \tag{3.4}$$

Since $F_n + a \in B(\alpha^2) + B(\alpha^2)$, there are $b, c \in \mathbb{N}$ such that $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. If $b < F_{n-2}$, then by Corollary 3.4, $a + 1 + \lfloor (F_{n-2} - b)\alpha^2 \rfloor = F_n + a - \lfloor b\alpha^2 \rfloor = \lfloor c\alpha^2 \rfloor \in B(\alpha^2)$, which contradicts (3.4). Therefore $b \geq F_{n-2}$. Similarly, by applying the same argument to c, we obtain $c \geq F_{n-2}$. Then $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor \geq 2\lfloor F_{n-2}\alpha^2 \rfloor = 2(F_n - [n \equiv 0 \pmod 2])$, which implies $a \geq F_n - 2[n \equiv 0 \pmod 2]$ contradicting the assumption that $a < F_n - 2$. Hence the proof is complete. □

Remark 3.13. Let $n \geq 4$. By Lemma 2.5, we have $F_{n-3}\alpha^2 = F_{n-1} - \beta^{n-3}$. So for $a \in \mathbb{Z}$, the condition $a \leq F_{n-1} - [n \equiv 1 \pmod 2]$ is equivalent to $a \leq F_{n-3}\alpha^2$. We will use this observation later.

To obtain the converse of Theorem 3.12, we first prove the following lemma.

Lemma 3.14. *Let $B(\alpha^2)$ be the Beatty set as defined in* (1.1)*, $n \geq 5$, $a, b, c \in \mathbb{N}$, and $1 \leq a \leq F_{n-3}\alpha^2$. Suppose $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ and $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. Then one of b, c is equal to F_{n-1} and the other, say c, satisfying $\lfloor c\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]$.*

Proof. Suppose for a contradiction that both b and c are not equal to F_{n-1}. Since $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$, we obtain by Lemma 3.9 that both b and c are less than $F_{n-1} + \frac{a}{\alpha^2} \leq F_n$. So we can apply Corollary 3.4 to write

$$F_{n+1} + a = \left\lfloor (F_{n-1} - b)\alpha^2 \right\rfloor + \left\lfloor b\alpha^2 \right\rfloor + 1 + a,$$
$$F_{n+1} + a = \left\lfloor (F_{n-1} - c)\alpha^2 \right\rfloor + \left\lfloor c\alpha^2 \right\rfloor + 1 + a.$$

Since $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ and $a + 1 + \lfloor (F_{n-1} - b)\alpha^2 \rfloor = F_{n+1} + a - \lfloor b\alpha^2 \rfloor = \lfloor c\alpha^2 \rfloor \in B(\alpha^2)$, we obtain by Lemma 3.11 that $F_{n-1} - b \notin [-F_{n-3}, 0) \cup (0, F_{n-2})$. Therefore $F_{n-1} - b \geq F_{n-2}$, $F_{n-1} - b = 0$, or $F_{n-1} - b < -F_{n-3}$. Then $b \leq F_{n-3}$ or $b > F_{n-1} + F_{n-3}$. Applying the above argument to $a + 1 + \lfloor (F_{n-1} - c)\alpha^2 \rfloor = \lfloor b\alpha^2 \rfloor \in B(\alpha^2)$, we also obtain $c \leq F_{n-3}$ or $c > F_{n-1} + F_{n-3}$. Recall that b and $c < F_{n-1} + \frac{a}{\alpha^2}$. So if b or $c > F_{n-1} + F_{n+3}$, then we would obtain $F_{n-1} + \frac{a}{\alpha^2} > F_{n-1} + F_{n-3}$, which contradicts the assumption that $a \leq F_{n-3}\alpha^2$. Hence b and

$c \leq F_{n-3}$. Then $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor \leq 2\lfloor F_{n-3}\alpha^2 \rfloor \leq 2F_{n-1} < F_{n+1} + a$, which is a contradiction. Thus b or c is equal to F_{n-1}. If $b = F_{n-1}$, then $F_{n+1} + a = \lfloor F_{n-1}\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor = F_{n+1} - [n \equiv 1 \pmod 2] + \lfloor c\alpha^2 \rfloor$, and so $\lfloor c\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]$. Similarly, if $c = F_{n-1}$, then $\lfloor b\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]$. This completes the proof. □

Lemma 3.15. *Let $B(\alpha^2)$ be the set as defined in* (1.1), *$n \geq 6$ and $1 \leq a \leq F_{n-3}\alpha^2$. If $F_{n-1} + a$ and $F_n + a$ are not in $B(\alpha^2) + B(\alpha^2)$, then $F_{n+1} + a \notin B(\alpha^2) + B(\alpha^2)$.*

Proof. Suppose for a contradiction that $F_{n-1}+a$ and F_n+a are not in $B(\alpha^2)+B(\alpha^2)$ but $F_{n+1} + a \in B(\alpha^2) + B(\alpha^2)$. Then there are $b, c \in \mathbb{N}$ such that $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. By Lemma 3.14, we can assume that $b = F_{n-1}$ and

$$\lfloor c\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]. \tag{3.5}$$

But by applying Lemma 3.11 to the case $F_{n-1}+a \notin B(\alpha^2)+B(\alpha^2)$, we obtain that

$$a + 1 - [n - 1 \equiv 1 \pmod 2] \notin B(\alpha^2), \quad \text{or equivalently,}$$
$$a + [n \equiv 1 \pmod 2] \notin B(\alpha^2),$$

which contradicts (3.5). So the proof is complete. □

Suppose $n \geq 6$ and the integers in $[F_n, F_{n+1}] \cap (B(\alpha^2) + B(\alpha^2))$ are given. Then the next theorem gives us more than a half of the integers in and outside the set

$$[F_{n+1}, F_{n+2}] \cap (B(\alpha^2) + B(\alpha^2)). \tag{3.6}$$

Theorem 3.16. *Let $B(\alpha^2)$ be the Beatty set as defined in* (1.1), *$n \geq 6$, $a \in \mathbb{Z}$, and $0 \leq a < F_{n-1} - 2$. Then*

$$F_n + a \in B(\alpha^2) + B(\alpha^2) \quad \textit{if and only if} \quad F_{n+1} + a \in B(\alpha^2) + B(\alpha^2).$$

Proof. If $a = 0$, the result follows from Theorem 3.10. So assume that $a \geq 1$. By Theorem 3.12, we only need to prove the converse. Assume that $F_n +$

$a \notin B(\alpha^2) + B(\alpha^2)$. Then we obtain by Theorem 3.12 that $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. Then by Lemma 3.15, $F_{n+1} \notin B(\alpha^2) + B(\alpha^2)$. So the proof is complete. □

The next theorem gives the remaining integers in (3.6).

Theorem 3.17. *Let $B(\alpha^2)$ be the set as defined in* (1.1)*, $n \geq 6$, $a \in \mathbb{Z}$, and $0 \leq a \leq F_{n-1} - [n \equiv 1 \pmod 2]$. Then $F_n + a \in B(\alpha^2) + B(\alpha^2)$ if and only if $F_{n+1} - 2 - a \in B(\alpha^2) + B(\alpha^2)$.*

Proof. If $a = 0$, the result follows from Theorem 3.10. So assume that $a \geq 1$. Suppose $F_n + a \in B(\alpha^2) + B(\alpha^2)$ but $F_{n+1} - 2 - a \notin B(\alpha^2) + B(\alpha^2)$. By applying Lemma 3.11 to $F_{n+1} - 2 - a$, we obtain that

$$-a - 1 + \left\lfloor d\alpha^2 \right\rfloor \notin B(\alpha^2) \text{ for } d \in [-F_{n-2}, 0) \cup (0, F_{n-1}). \tag{3.7}$$

Since $F_n + a \in B(\alpha^2) + B(\alpha^2)$, there are $b, c \in \mathbb{N}$ such that $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. Then by Lemma 3.9, b and $c < F_{n-2} + \frac{a}{\alpha^2}$. Recall also from Remark 3.13 that $a \leq F_{n-3}\alpha^2$. If $b < F_{n-2}$, then by Corollary 3.4 and the fact given in (3.7), we obtain, respectively, $-a - 1 + \lfloor c\alpha^2 \rfloor = F_n - 1 - \lfloor b\alpha^2 \rfloor = \lfloor (F_{n-2} - b)\alpha^2 \rfloor \in B(\alpha^2)$, and $c \geq F_{n-1}$, which contradicts the fact that $c < F_{n-2} + \frac{a}{\alpha^2}$. So $b \geq F_{n-2}$. Similarly, applying the above argument to c, we have $c \geq F_{n-2}$. Then $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor \geq 2 \lfloor F_{n-2}\alpha^2 \rfloor \geq 2F_n - 2 > F_n + a$, which is a contradiction. Hence the first part of this theorem is proved.

For the converse, we also suppose for a contradiction that $F_{n+1} - 2 - a \in B(\alpha^2) + B(\alpha^2)$ but $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Then there are $b, c \in \mathbb{N}$ such that $F_{n+1} - 2 - a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ and by Lemma 3.11,

$$a + 1 + \left\lfloor d\alpha^2 \right\rfloor \notin B(\alpha^2) \text{ for } d \in [-F_{n-3}, 0) \cup (0, F_{n-2}). \tag{3.8}$$

By Lemma 3.9, b and $c < F_{n-1} - \frac{a+2}{\alpha^2} < F_{n-1}$. Then by Corollary 3.4, we obtain

$$F_{n+1} - 2 - a = \left\lfloor (F_{n-1} - b)\alpha^2 \right\rfloor + \left\lfloor b\alpha^2 \right\rfloor - a - 1,$$

which implies $a+1+\lfloor c\alpha^2 \rfloor = \lfloor (F_{n-1}-b)\alpha^2 \rfloor \in B(\alpha^2)$. So by (3.8), $c \geq F_{n-2}$. By the same argument, $b \geq F_{n-2}$. Therefore $F_{n+1}-2-a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor \geq 2\lfloor F_{n-2}\alpha^2 \rfloor \geq 2F_n - 2$, which implies $a \leq F_{n-1} - F_n < 0$, a contradiction. Hence the proof is complete. □

Theorems 3.10, 3.16, and 3.17 give a complete description of $B(\alpha^2)+B(\alpha^2)$. We illustrate this in Example 3.18 and Theorem 3.20 as follows.

Example 3.18. For convenience, if $A \subseteq \mathbb{N}$, we write A^c to denote the complement of A in $\mathbb{N}$. That is $A^c = \mathbb{N} \setminus A$. By direct calculation, the elements in $(B(\alpha^2)+B(\alpha^2))^c \cap [1, F_9]$ are 1, 2, 3, 5, 6, 8, 11, 13, 16, 19, 21, 24, 29, 32, 34. To determine the elements in $[F_9, F_{10}] \cap (B(\alpha^2)+B(\alpha^2))^c$, we first observe that for $0 \leq a \leq F_7$,

$$F_8 + a \notin B(\alpha^2)+B(\alpha^2) \text{ if and only if } a \in \{0,3,8,11,13\}.$$

Applying Theorem 3.16 for $n=8$, we obtain that for $0 \leq a < F_7 - 2$,

$$F_9 + a \notin B(\alpha^2)+B(\alpha^2) \text{ if and only if } a \in \{0,3,8\}.$$

Applying Theorem 3.17 for $n=9$, we obtain that for $0 \leq a < F_7 - 2$,

$$F_{10} - 2 - a \notin B(\alpha^2)+B(\alpha^2) \text{ if and only if } a \in \{0,3,8\}.$$

In addition, $F_{10} \notin B(\alpha^2)+B(\alpha^2)$ by Theorem 3.10. The length of the interval $[F_9, F_{10}]$ is $F_{10}-F_9 = F_8$ which is less than $2(F_7-2)$. Therefore the elements in $(B(\alpha^2)+B(\alpha^2))^c \cap [F_9, F_{10}]$ are completely determined. They are F_9, F_9+3, F_9+8, $F_{10}-10$, $F_{10}-5$, $F_{10}-2$, F_{10}, which are $34, 37, 42, 45, 50, 53, 55$. By doing this process repeatedly, we obtain $(B(\alpha^2)+B(\alpha^2))^c \cap A$ where $A = [F_{10}, F_{11}]$, $[F_{11}, F_{12}]$, $[F_{12}, F_{13}]$, and so on. Thus we can find $(B(\alpha^2)+B(\alpha^2)) \cap [1, F_n]$ for any given n.

Remark 3.19. In the abstract and introduction, we mention that the structure of the set $X =: B(\alpha^2)+B(\alpha^2)$ has some kinds of fractal and palindromic patterns. This is not intended to be a precise or mathematically rigorous statement. What we (vaguely) means is that the distribution of the elements of X in the interval $[F_n, F_{n+1}]$ looks like fractal for all $n \geq 6$. Suppose we display the points of $X \cap [F_{n+1}, F_{n+2}]$ on the real line and zoom in for a smaller scale, namely, $X \cap [F_{n+1}, F_{n+1} + F_{n-1} - 3]$. Then, by Theorem 3.16, the picture (in a smaller scale) is the same as that of $X \cap [F_n, F_{n+1}]$. Then by Theorem 3.16 again, the picture (in a smaller scale) of $X \cap [F_{n+2}, F_{n+3}]$ is the same as that of $X \cap [F_{n+1}, F_{n+2}]$. Since Theorem 3.16 holds for all $n \geq 6$, we can continue this process and see the distribution of the elements of X on $[F_n, F_{n+1}]$, $[F_{n+1}, F_{n+2}]$, $[F_{n+2}, F_{n+3}]$, and so on, as fractal-like pattern. See the figure shown below for an illustration.

For the palindromicity, recall that a positive integer n can be written uniquely in the decimal expansion as

$$n = (a_k a_{k-1} \cdots a_0)_{10} = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_0,$$

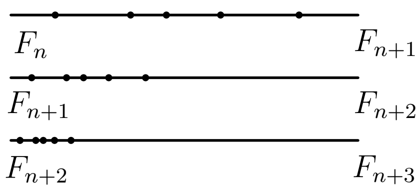
where $a_k \neq 0$ and $0 \leq a_i \leq 9$ for all i, and n is called a palindrome or a palindromic number if $a_{k-i} = a_i$ for $0 \leq i \leq \lfloor k/2 \rfloor$. So if n is a palindrome and we know the values of a_{k-i} only for $0 \leq i \leq \lfloor k/2 \rfloor$, then we can completely find all the decimal digits of n. Now suppose $n \geq 6$ and the elements of $X \cap [F_n, F_{n+1}]$ are known. We can divide $[F_{n+1}, F_{n+2}]$ into two overlapped intervals:

$$\text{the left-hand interval } L =: [F_{n+1}, F_{n+1} + F_{n-1} - 3]$$

$$\text{the right-hand interval } R =: [F_{n+2} - F_{n-1} - 1, F_{n+2}].$$

By Theorem 3.16, $X \cap L$ is completely determined by $X \cap [F_n, F_{n+1}]$. Theorem 3.17 gives us the palindromic pattern which helps us obtain all elements in R from L. Hence we can basically say that for all $n \geq 6$, the distribution of points in $X \cap [F_{n+1}, F_{n+2}]$ are completely determined by that of $X \cap [F_n, F_{n+1}]$ by the fractal-like and palindromic patterns.

In general, we have the following result.



Theorem 3.20. *Let $B(\alpha^2)$ be the set as defined in* (1.1)*. For each $n \in \mathbb{N}$, let $A_n = \{a \in \mathbb{Z} \mid 0 \leq a \leq F_{n-1} \text{ and } F_n + a \notin B(\alpha^2) + B(\alpha^2)\}$. Then $A_1 = \{0\}$, $A_2 = A_3 = \{0, 1\}$, $A_4 = \{0, 2\}$, $A_5 = \{0, 1, 3\}$, $A_6 = \{0, 3, 5\}$, $A_7 = \{0, 3, 6, 8\}$, and for $n \geq 8$, the set A_n is the disjoint union*

$$\begin{aligned} A_n = {} & (A_{n-1} \setminus \{F_{n-2}, F_{n-2} - 2\}) \\ & \cup \{F_{n-1} - 2 - a \mid a \in A_{n-1} \text{ and } 0 \leq a \leq F_{n-3}\} \cup \{F_{n-1}\}. \end{aligned}$$

Proof. The sets $A_1, A_2, \ldots, A_7$ can be obtained by direct calculation. So assume that $n \geq 8$. Since $F_n + F_{n-1} = F_{n+1} \notin B(\alpha^2) + B(\alpha^2)$, $F_{n-1} \in A_n$. Then we write $A_n = C \cup B \cup \{F_{n-1}\}$, where $C = A_n \cap [0, F_{n-2} - 2)$ and $B = A_n \cap [F_{n-2} - 2, F_{n-1})$. Obviously, the sets C, B, and $\{F_{n-1}\}$ are disjoint. So it remains to show that

$$C = A_{n-1} \setminus \{F_{n-2}, F_{n-2} - 2\} \quad \text{and} \tag{3.9}$$

$$B = \{F_{n-1} - 2 - a \mid a \in A_{n-1} \text{ and } 0 \leq a \leq F_{n-3}\}. \tag{3.10}$$

To prove (3.9), let $a \in C$. Then $0 \leq a < F_{n-2} - 2$ and $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Applying Theorem 3.16, we obtain $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. So $a \in A_{n-1} \setminus \{F_{n-2}, F_{n-2} - 2\}$. Conversely, suppose that $a \in A_{n-1} \setminus \{F_{n-2}, F_{n-2} - 2\}$. Then $a \in [0, F_{n-2} - 2) \cup \{F_{n-2} - 1\}$ and $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. If $a = F_{n-2} - 1$, then we obtain by Theorem 3.10 that $F_{n-1} + a = F_n - 1 \in B(\alpha^2) + B(\alpha^2)$, which is not the case. So $0 \leq a < F_{n-2} - 2$. In addition, by Theorem 3.16, $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Hence $a \in A_n \cap [0, F_{n-2} - 2) = C$. This proves (3.9).

Next, let $b \in B$. Then $F_{n-2} - 2 \leq b < F_{n-1}$ and $F_n + b \notin B(\alpha^2) + B(\alpha^2)$. If $b = F_{n-1} - 1$, then we obtain by Theorem 3.10 that $F_n + b = F_{n+1} - 1 \in$

$B(\alpha^2)+B(\alpha^2)$, which is not the case. So $b \le F_{n-1}-2$. Let $a = F_{n-1}-2-b$. Then $b = F_{n-1}-2-a$ and $0 \le a \le F_{n-3}$. So it remains to show that $a \in A_{n-1}$. Since $F_n + b \notin B(\alpha^2)+B(\alpha^2)$, we obtain by Theorem 3.17 that $F_n+a = F_{n+1}-2-(F_{n-1}-2-a) = F_{n+1}-2-b \notin B(\alpha^2)+B(\alpha^2)$. Since $F_n+a \notin B(\alpha^2)+B(\alpha^2)$, we obtain by Theorem 3.16 that $F_{n-1}+a \notin B(\alpha^2)+B(\alpha^2)$. So $a \in A_{n-1}$, as required.

Finally, suppose $b = F_{n-1}-2-a$ where $a \in A_{n-1}$ and $0 \le a \le F_{n-3}$. Then $F_{n-2}-2 \le b < F_{n-1}$. Since $a \in A_{n-1}$, $F_{n-1}+a \notin B(\alpha^2)+B(\alpha^2)$. Then by Theorem 3.16, $F_n + a \notin B(\alpha^2)+B(\alpha^2)$. Applying Theorem 3.17, we obtain $F_n+b = F_{n+1}-2-a \notin B(\alpha^2)+B(\alpha^2)$. So $b \in A_n \cap [F_{n-2}-2, F_{n-1}) = B$, as desired. This completes the proof. $\square$

Questions

Q1 Let (f_n) be a kth order linear recurrence sequence defined by

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k} \quad \text{for } n \ge 2$$

with the initial values $f_{-(k-2)}, f_{-(k-3)}, \ldots, f_0, f_1 \in \mathbb{Z}$. Let α be the root of the characteristic polynomial $x^k - x^{k-1} - x^{k-2} - \cdots - 1$ with maximal absolute value. Can we described the structure of the sumsets associated with $B(\alpha), B(\alpha^2), \ldots, B(\alpha^k)$? Is the structure best described in terms of the k-step Fibonacci sequence $(F_n^{(k)})$ defined by the same recurrence as (f_n) but with the initial values

$$F^{(k)}_{-(k-2)} = F^{(k)}_{-(k-3)} = \cdots = F^{(k)}_{-1} = F^{(k)}_0 = 0 \text{ and } F^{(k)}_1 = 1?$$

Q2 Let $\alpha = (1+\sqrt{5})/2$. Since $\alpha^2-\alpha-1=0$, the set $\{\alpha^2, \alpha, 1\}$ is not linearly independent over $\mathbb{Q}$. Suppose $\{\alpha^k, \alpha^{k-1}, \ldots, \alpha, 1\}$ is linearly independent over $\mathbb{Q}$, for example, α is an algebraic number of degree larger than k, $\alpha = e$, or $\alpha = \pi$, can we describe the structure of the sumsets associated with $B(\alpha^k), B(\alpha^{k-1}), \ldots, B(\alpha)$?

Q3 Let $a, b \in \mathbb{Z}$, $(a, b) = 1$, $b \neq 0$, and let (u_n) be the Lucas sequence of the first kind defined by $u_n = au_{n-1} + bu_{n-2}$ for $n \geq 2$ with $u_0 = 0$ and $u_1 = 1$. Let α be the root of the characteristic polynomial $x^2 - ax - b$. Is the structure of the sumsets associated with $B(\alpha)$ and $B(\alpha^2)$ connected to (u_n)?

The proof of the following theorem is similar to that of Theorem 3.3. In fact, applying Theorem 3.3 leads to Theorem 3.21 but with a smaller range of b, which may not be enough in some applications.

Theorem 3.21. *Let $n \geq 5$ and $1 \leq b \leq F_{n+1}$. Then the following statements hold.*

(i) *If $b = L_{n-1}$, then $\sqrt{5}\beta^{n-1} - \{b\alpha\} = -[n \equiv 0 \pmod 2]$.*

(ii) *If $b \in \{F_{n-2}, F_n\}$, then $0 < \sqrt{5}\beta^{n-1} - \{b\alpha\} + 2[n \equiv 0 \pmod 2] < 1$.*

(iii) *If $b \notin \{F_{n-2}, F_n, L_{n-1}\}$, then $-1 < \sqrt{5}\beta^{n-1} - \{b\alpha\} < 0$.*

Proof. The statement (i) follows immediately from Lemma 2.5(vi). For (ii), let $b \in \{F_{n-2}, F_n\}$ and $A = \sqrt{5}\beta^{n-1} - \{b\alpha\} + 2[n \equiv 0 \pmod 2]$. Since $\beta^n + \sqrt{5}\beta^{n-1} + \beta^{n-2} = 0$, we obtain by Lemma 2.5(iii) that
if $b = F_n$, then

$$A = \sqrt{5}\beta^{n-1} + \beta^n + [n \equiv 0 \pmod 2] = -\beta^{n-2} + [n \equiv 0 \pmod 2]$$

if $b = F_{n-2}$, then

$$A = \sqrt{5}\beta^{n-1} + \beta^{n-2} + [n \equiv 0 \pmod 2] = -\beta^n + [n \equiv 0 \pmod 2].$$

By calculating A according to the parity of n, it is not difficult to see that $0 < A < 1$. This proves (ii). For (iii), if $b = F_{n+1}$, then we apply Lemma 2.5(iii) to obtain

$$\sqrt{5}\beta^{n-1} - \{b\alpha\} = \sqrt{5}\beta^{n-1} + \beta^{n+1} - [n \equiv 1 \pmod 2] = \beta^{n-3} - [n \equiv 1 \pmod 2],$$

which is in the interval $(-1,0)$. Next, let $B=\sqrt{5}\beta^{n-1}-\{b\alpha\}+1$, where b is not equal to any of F_{n-2}, F_n, L_{n-1}, F_{n+1}. We need to show that $0<B<1$.

Case 1 $b=F_k$ where $2\leq k\leq n-3$ or $k=n-1$.

Case 1.1 $b=F_2$. Then by Lemma 2.5, $B=\sqrt{5}\beta^{n-1}+\beta^2$. Therefore, $B\leq\sqrt{5}\beta^4+\beta^2=\beta^2(-3\beta)=-3\beta^3<1$. If n is odd, then it is obvious that $B>0$. If n is even, then $n\geq 6$, and $B\geq\sqrt{5}\beta^5+\beta^2=\beta^2(\sqrt{5}\beta^3+1)>0$.

Case 1.2 $b=F_{n-1}$. Then by Lemma 2.5, $B=\sqrt{5}\beta^{n-1}+\beta^{n-1}-[n\equiv 1 \pmod 2]+1$. If n is even, then $B<1$ and $B\geq 1+\beta^5+\sqrt{5}\beta^5=1-2\beta^4>0$. If n is odd, then $B>0$ and $B\leq\sqrt{5}\beta^4+\beta^4=-2\beta^3<1$.

Case 1.3 $b=F_k$ and $3\leq k\leq n-3$. This case occurs only when $n\geq 6$. By Lemma 2.5,

$$B=\sqrt{5}\beta^{n-1}+\beta^k-[k\equiv 0 \pmod 2]+1.$$

We first consider the case that k is even. Then $B=\sqrt{5}\beta^{n-1}+\beta^k$. If n is odd, then $B>0$ and $B\leq\sqrt{5}\beta^4+\beta^4=-2\beta^3<1$. If n is even, then $B<\beta^k\leq\beta^4<1$, $k\leq n-4$, and $B\geq\sqrt{5}\beta^{n-1}+\beta^{n-4}=\beta^{n-4}(\sqrt{5}\beta^3+1)>0$. Next, suppose k is odd. Then $B=\sqrt{5}\beta^{n-1}+\beta^k+1$. If n is even, then $B<1$ and $B\geq\sqrt{5}\beta^5+\beta^3+1=1-3\beta^4>0$. If n is odd, then $k\leq n-4$, $B>\sqrt{5}\beta^{n-1}>0$ and $B\leq\sqrt{5}\beta^{n-1}+\beta^{n-4}+1<1$.

Case 2 $F_k<b<F_{k+1}$ for some $k\in\{4,5,\ldots,n\}$. We apply Lemma 2.5 without further reference. By Zeckendorf's theorem, we can write $b=F_{a_1}+F_{a_2}+\cdots+F_{a_\ell}$ where $\ell\geq 2$, $k=a_1>a_2>\cdots>a_\ell\geq 2$ and $a_{i-1}-a_i\geq 2$ for every $i=2,3,\ldots,\ell$. Then by Lemma 2.4, we obtain $\{b\alpha\}=\{\{F_{a_1}\alpha\}+\{F_{a_2}\alpha\}+\cdots+\{F_{a_\ell}\alpha\}\}$ which is equal to

$$\{(1-\beta^{b_1}+1-\beta^{b_2}+\cdots+1-\beta^{b_r})+(-\beta^{c_1}-\beta^{c_2}-\cdots-\beta^{c_s})\},$$

where $\{b_1,b_2,\ldots,b_r\}\cup\{c_1,c_2,\ldots,c_s\}=\{a_1,a_2,\ldots,a_\ell\}$, $b_1>b_2>\cdots>b_r$ are even numbers, and $c_1>c_2>\cdots>c_s$ are odd numbers. Remark that one of the sets $\{b_1,b_2,\ldots,b_r\}$ and $\{c_1,c_2,\ldots,c_s\}$ may be empty. In this case, such the set disappears from the subsequent calculation. Also, for convenience, we let $A=\beta^{b_1}+\beta^{b_2}+\cdots+\beta^{b_r}+\beta^{c_1}+\beta^{c_2}+\cdots+\beta^{c_s}$. Then by Lemma 2.1,

$\{b\alpha\} = \{-A\}$. To show that $0 < B < 1$, it is enough to prove

$$\sqrt{5}\beta^{n-1} < \{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}.$$

Case 2.1 $\{b_1, b_2, \dots, b_r\}$ is empty. Then

$$A = \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^3 + \beta^5 + \cdots = \frac{\beta^3}{1-\beta^2} = -\beta^2.$$

Therefore $0 < -A < \beta^2 < 1$ and so $\{b\alpha\} = \{-A\} = -A$. If n is even, then obviously $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = -A < \beta^2 < 1 + \sqrt{5}\beta^3 < 1 + \sqrt{5}\beta^{n-1}$. So assume that n is odd. Then $\{b\alpha\} = -A < \beta^2 < 1 + \sqrt{5}\beta^{n-1}$, and $\{b\alpha\} = -A = |\beta|^{c_1} + |\beta|^{c_2} + \cdots + |\beta|^{c_s}$. If $\ell \geq 3$, then $s \geq 3$, and so

$$\{b\alpha\} \geq |\beta|^{c_1} + |\beta|^{c_2} + |\beta|^{c_3} \geq |\beta|^n + |\beta|^{n-2} + |\beta|^{n-4} > |\beta|^n + |\beta|^{n-2} = \sqrt{5}\beta^{n-1}.$$

Suppose $\ell = 2$. Then $s = 2$ and $\{b\alpha\} = |\beta|^{c_1} + |\beta|^{c_2}$. If $c_1 \neq n$, then

$$|\beta|^{c_1} + |\beta|^{c_2} \geq |\beta|^{n-2} + |\beta|^{n-4} > |\beta|^{n-2} + |\beta|^n = -(\beta^n + \beta^{n-2}) = \sqrt{5}\beta^{n-1}.$$

Since $L_{n-1} = F_n + F_{n-2}$ and $b \neq L_{n-1}$, we see that $\{c_1, c_2\} \neq \{n, n-2\}$. Therefore, if $c_1 = n$, then $c_2 \neq n-2$, and so $c_2 \leq n-4$

$$|\beta|^{c_1} + |\beta|^{c_2} \geq |\beta|^n + |\beta|^{n-4} > |\beta|^n + |\beta|^{n-2} = \sqrt{5}\beta^{n-1}.$$

In any case, $\{b\alpha\} > \sqrt{5}\beta^{n-1}$, as required.

Case 2.2 $\{c_1, c_2, \dots, c_s\}$ is empty. Then

$$A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^2 + \beta^4 + \cdots = \frac{\beta^2}{1-\beta^2} = -\beta.$$

Therefore $-1 < \beta < -A < 0$ and $\{b\alpha\} = \{-A\} = 1 - A$. Suppose n is even. Then $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = 1 - A = 1 - \beta^{b_1} - \beta^{b_2} - \cdots - \beta^{b_r}$. Similar to the proof of Case 2.1, if $\ell \geq 3$, then $r \geq 3$ and

$$\{b\alpha\} \leq 1 - \beta^{b_1} - \beta^{b_2} - \beta^{b_3} \leq 1 - \beta^n - \beta^{n-2} - \beta^{n-4} < 1 - \beta^n - \beta^{n-2} = 1 + \sqrt{5}\beta^{n-1}.$$

If $\ell = 2$ and $b_1 \neq n$, then

$$\{b\alpha\} = 1 - \beta^{b_1} - \beta^{b_2} \leq 1 - \beta^{n-2} - \beta^{n-4} < 1 - \beta^{n-2} - \beta^n = 1 + \sqrt{5}\beta^{n-1}.$$

If $\ell = 2$ and $b_1 = n$, then $b_2 \le n-4$ and

$$\{b\alpha\} = 1 - \beta^{b_1} - \beta^{b_2} \le 1 - \beta^n - \beta^{n-4} < 1 - \beta^n - \beta^{n-2} = 1 + \sqrt{5}\beta^{n-1}.$$

If n is odd, then $\{b\alpha\} < 1 < 1 + \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = 1 - A > 1 + \beta = \beta^2 \ge \beta^{n-3} \ge \sqrt{5}\beta^{n-1}$.

Case 2.3 $\{b_1, b_2, \ldots, b_r\}$ and $\{c_1, c_2, \ldots, c_s\}$ are not empty. Then there is some cancellation in the sum defining A. Similar to Case 2.1 and Case 2.2, we have $A < \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < -\beta$ and $A > \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > -\beta^2$.

Case 2.3.1 A is positive. Then $-1 < \beta < -A < 0$, and so $\{b\alpha\} = \{-A\} = 1 - A$. If n is odd, then $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = 1 - A > 1 + \beta > \sqrt{5}\beta^4 \ge \sqrt{5}\beta^{n-1}$. Assume that n is even. Then $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$. It remains to show that $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$. Let $u = \min\{b_1, b_2, \ldots, b_r\}$ and $v = \min\{c_1, c_2, \ldots, c_s\}$. Since $a_{i-1} - a_i \ge 2$ for all $i = 2, 3, \ldots, \ell$ and $a_1 = k \le n$, we obtain that $u \le n$ and $|v - u| \ge 3$. Then

$$\beta^u \le \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^u + \beta^{u+2} + \beta^{u+4} + \cdots = \frac{\beta^u}{1-\beta^2} = -\beta^{u-1}, \tag{3.11}$$

$$\beta^v \ge \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^v + \beta^{v+2} + \beta^{v+4} + \cdots = \frac{\beta^v}{1-\beta^2} = -\beta^{v-1}. \tag{3.12}$$

By (3.11) and (3.12), we obtain $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v-u| \ge 3$, we see that either $v - u \ge 3$ or $v - u \le -3$. Suppose for a contradiction that $v - u \le -3$. Since $v \le u - 3$ and both v and $u - 3$ are odd, we have $\beta^v \le \beta^{u-3}$. Thus $A < \beta^v - \beta^{u-1} \le \beta^{u-3} - \beta^{u-1} = \beta^{u-3}(1-\beta^2) = -\beta^{u-2} < 0$, which contradicts the assumption that A is positive. Hence $v - u \ge 3$. Since $v - 1 \ge u + 2$ and both $v - 1$ and $u + 2$ are even, $\beta^{v-1} \le \beta^{u+2}$. So $A > \beta^u - \beta^{u+2} = \beta^u(1-\beta^2) = -\beta^{u+1}$. We have $u \le v - 3 \le n - 3$. Thus $u + 1 \le n - 2$. Since $n - 2$ is even and $u + 1$ is odd, we have $u + 1 \le n - 3$. Then $\{b\alpha\} = 1 - A < 1 + \beta^{u+1} \le 1 + \beta^{n-3}$. Since $\sqrt{5}\beta^2 < 1$ and $n - 3$ is odd, $\sqrt{5}\beta^{n-1} > \beta^{n-3}$. Therefore $\{b\alpha\} < 1 + \beta^{n-3} < 1 + \sqrt{5}\beta^{n-1}$, as required.

Case 2.3.2 A is negative. Then $0 < -A < \beta^2 < 1$. Then $\{b\alpha\} = \{-A\} = -A$. We first show that $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$. If n is odd, then $\{b\alpha\} < 1 < 1 + \sqrt{5}\beta^{n-1}$. If n is even, then $\{b\alpha\} = -A < \beta^2 < 1 + \sqrt{5}\beta^3 < 1 + \sqrt{5}\beta^{n-1}$. Next, we show that $\{b\alpha\} > \sqrt{5}\beta^{n-1}$. If n is even, then $\sqrt{5}\beta^{n-1} < 0 < \{b\alpha\}$. So assume that n is odd. Let $u = \min\{b_1, b_2, \ldots, b_r\}$ and $v = \min\{c_1, c_2, \ldots, c_s\}$. Similar to Case 2.3.1, we have $u \leq n$, $|v - u| \geq 3$, the equalities (3.11) and (3.12) hold, and $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v - u| \geq 3$, we see that either $v - u \geq 3$ or $v - u \leq -3$. If $v - u \geq 3$, then $\beta^{v-1} \leq \beta^{u+2}$ and $A > \beta^u - \beta^{v-1} \geq \beta^u - \beta^{u+2} = -\beta^{u+1} > 0$, which contradicts the assumption that $A < 0$. Thus $v - u \leq -3$, and so $A < \beta^{u-3} - \beta^{u-1} = -\beta^{u-2}$. Since $u \leq n$, u is even and n is odd, we have $u - 2 \leq n - 3$. Then $-A > \beta^{u-2} \geq \beta^{n-3} > \sqrt{5}\beta^{n-1}$. Therefore $\{b\alpha\} = -A > \sqrt{5}\beta^{n-1}$ as desired. This completes the proof. □

We can apply Theorem 3.21 to give a short proof the key result in [11, Theorem 3.3].

Corollary 3.22. [11, Theorem 3.3] Let $n \geq 5$, $1 \leq b \leq F_{n+1}$, and $b \neq F_n$. Then

$$0 < \{b\alpha\} + \beta^n < 1.$$

Proof. If $b = F_{n-2}$ or $b = L_{n-1}$, we can apply Lemma 2.5 to obtain the desired result. So suppose that $b \neq F_{n-2}$ and $b \neq L_{n-1}$. We first consider the case n is odd. Then it is obvious that $\{b\alpha\} + \beta^n < 1$. For the other inequality, we apply Theorem 3.21 to obtain $\{b\alpha\} > \sqrt{5}\beta^{n-1} > -\beta^n$. Similarly, if n is even, then it is immediate that $\{b\alpha\} + \beta^n > 0$ and by using Theorem 3.21, we obtain $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1} < 1 - \beta^n$. This completes the proof. □

It is possible to extend the range of b in Theorem 3.21 and Corollary 3.22 but the results are not nice and we do not need them in our application. Therefore, we only give some special cases as an example and leave the general case to the interested readers.

Example 3.23. Let $n \geq 5$, $k \geq n + 2$, $b = F_k$, and $B = \{b\alpha\} + \beta^n$. Then the following statements hold.

(i) If k and n are odd, then $-1 < B < 0$.

(ii) If $k \not\equiv n \pmod 2$, then $0 < B < 1$.

(iii) If k and n are even, then $1 < B < 2$.

Proof. By Lemma 2.5, $B = \beta^n - \beta^k + [k \equiv 0 \pmod 2]$.

Case 1 k is odd. Then $B = \beta^n - \beta^k$. If n is odd, then $-1 < \beta^n < \beta^{n+2} \leq \beta^k < 0$, and so $-1 < B < 0$. If n is even, then $k \geq n+3$, and $0 < B \leq \beta^n - \beta^{n+3} = -2\beta^{n+1} < 1$.

Case 2 k is even. Then $B = \beta^n - \beta^k + 1$. If n is odd, then $B < 1$, $k \geq n+3$, and $B \geq 1 + \beta^n - \beta^{n+3} = 1 - 2\beta^{n+1} > 0$. If n is even, then $0 < \beta^k \leq \beta^{n+2} < \beta^n < 1$, and so $1 < B < 2$. This completes the proof. $\square$

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Appendix

Sumsets associated with Wythoff sequences and Fibonacci numbers

Sutasinee Kawsumarng, Tammatada Khemaratchatakumthorn, Passawan Noppakaew & Prapanpong Pongsriiam

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Sumsets associated with Wythoff sequences and Fibonacci numbers

Sutasinee Kawsumarng[1] · Tammatada Khemaratchatakumthorn[1] · Passawan Noppakaew[1] · Prapanpong Pongsriiam[1]

Abstract
Let $\alpha = (1+\sqrt{5})/2$ be the golden ratio, and let $B(\alpha) = (\lfloor n\alpha \rfloor)_{n\ge1}$ and $B(\alpha^2) = \left(\lfloor n\alpha^2 \rfloor\right)_{n\ge1}$ be the lower and upper Wythoff sequences, respectively. In this article, we obtain a new estimate concerning the fractional part $\{n\alpha\}$ and study the sumsets associated with Wythoff sequences. For example, we show that every $n \ge 4$ can be written as a sum of two terms in $B(\alpha)$ and a positive integer n can be written as the sum $\lfloor a\alpha \rfloor + \lfloor b\alpha^2 \rfloor$ for some $a, b \in \mathbb{N}$ if and only if n is not one less than a Fibonacci number. The structure of the set $B(\alpha^2) + B(\alpha^2)$ contains some kinds of fractal and palindromic patterns and is more complicated than the other sets, but we can also give a complete description of this set.

Keywords Wythoff sequence · Sumset · Fibonacci number · Golden ratio · Beatty sequence

Mathematics Subject Classification 11B13 · 11B39

1 Introduction

Let G be an additive abelian group, A and B nonempty subsets of G, and $x \in G$. Then the sumset $A + B$ and the translation $x + A$ are defined by

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\} \quad \text{and} \quad x + A = A + x = \{a + x \mid a \in A\}.$$

Additive number theory and the study of sumsets have a long history dating back at least to Lagrange in 1770 who proved that every natural number can be written as a sum of four

✉ Prapanpong Pongsriiam
prapanpong@gmail.com; pongsriiam_p@silpakorn.edu

Sutasinee Kawsumarng
Kawsumarng@gmail.com

Tammatada Khemaratchatakumthorn
tammatada@gmail.com

Passawan Noppakaew
noppakaew_p@silpakorn.edu

1 Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom 73000, Thailand

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squares of integers. Cauchy in 1813 gave a lower bound for the cardinality of the sumset $A+B$ where A and B are nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$. Davenport [3] rediscovered Cauchy's result in 1935 and the results is now known as the Cauchy-Davenport theorem. Several other results on sumsets and in additive number theory have been obtained by various mathematicians, and we refer the reader to the books by Freiman [8], Halberstam and Roth [10], Nathanson [17], Tao and Vu [39], and Vaughan [41] for additional details and references.

On the other hand, Wythoff sequences arise very often in combinatorics and combinatorial game theory, and so many of their combinatorial properties have been extensively studied; see for example in the work of Fraenkel [4–7], Kimberling [13,14], Pitman [23], Wythoff [42], and in the online encyclopedia OEIS [38]. However, as far as we are aware, there are no number theoretic results, at least in the spirit of this paper, concerning the sumsets associated with Wythoff sequences. This motivates us to investigate more on this topic. Note that Pitman's article [23] is closely related to ours but it focuses only on the cardinality of sumsets of certain finite Beatty sequences in connection with Sturmian words and the nearest integer algorithm.

Before proceeding further, let us introduce the notation which will be used throughout this article as follows: x is a real number, a, b, m, n are integers, $\alpha = (1+\sqrt{5})/2$ is the golden ratio, $\beta = (1-\sqrt{5})/2$, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\{x\} = x - \lfloor x \rfloor$,

$$B(x) = \{\lfloor nx \rfloor \mid n \in \mathbb{N}\} \quad \text{and} \quad B_0(x) = \{\lfloor nx \rfloor \mid n \geq 0\}. \tag{1.1}$$

The set $B(x)$ is usually considered as a sequence $(\lfloor nx \rfloor)_{n\geq 1}$ and is called a Beatty sequence. The sets $B(\alpha)$ and $B(\alpha^2)$ are also called lower and upper Wythoff sequences, respectively; but for our purpose, it is more convenient to consider them as sets. In addition, if P is a mathematical statement, then the Iverson notation $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

Recall that a generalized Fibonacci sequence $(f_n)_{n\geq 0}$ is defined by $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ where f_0 and f_1 are arbitrary integers. If $f_0 = 0$ and $f_1 = 1$, then $(f_n)_{n\geq 0} = (F_n)_{n\geq 0}$ is the classical Fibonacci sequence, and if $f_0 = 2$ and $f_1 = 1$, then $(f_n)_{n\geq 0} = (L_n)_{n\geq 0}$ is the classical sequence of Lucas numbers. The roots of the characteristic polynomial $x^2 - x - 1$ for any generalized Fibonacci sequence (f_n) are α and β, but it turns out that the structures of sumsets such as $B(\alpha)+B(\alpha^2)$ and $B(\alpha^2)+B(\alpha^2)$ are best described in terms of the classical Fibonacci numbers F_n. We refer the reader to [11,12,19–21,24–26,33] for some recent results concerning multiplicative properties of F_n, and to [27,28] for certain Diophantine equations involving additive and multiplicative properties of F_n.

In this article, we give a new estimate concerning the fractional part $\{n\alpha\}$ and study the sumsets associated with $B(\alpha)$ and $B(\alpha^2)$. For example, we obtain from Theorems 3.1, 3.5, and 3.8, respectively, that for every $n \geq 4$, $n = \lfloor a\alpha \rfloor + \lfloor b\alpha \rfloor$ for some $a, b \in \mathbb{N}$, for every $n \geq 27$, $n = \lfloor a\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ for some $a, b, c \in \mathbb{N}$, and for every $n \geq 1$, $n = \lfloor a\alpha \rfloor + \lfloor b\alpha^2 \rfloor$ for some $a, b \in \mathbb{N}$ if and only if n is not one less than a Fibonacci number. The structure of $B(\alpha^2) + B(\alpha^2)$ contains some kinds of fractal and palindromic patterns in each interval of the form $[F_n, F_{n+1}]$; see for instance Theorems 3.16, 3.17, and Remark 3.19, and so the elements in $(B(\alpha^2) + B(\alpha^2)) \cap [F_{n+1}, F_{n+2}]$ can be completely determined by those of $(B(\alpha^2) + B(\alpha^2)) \cap [F_n, F_{n+1}]$.

For a general result on the sumsets associated with $B(x)$ and $B(x^2)$ where x satisfies the conditions such as $x > 1$ and $x^2 - ax - b = 0$ for some $a, b \in \mathbb{Z}$, we think that the answers may be best described in terms of the Lucas sequence of the first kind. Nevertheless, the calculations even in the case of $B(\alpha)$ and $B(\alpha^2)$ are already complicated, so we postpone this for future research. See also other problems in the last section.

We arrange this article as follows. In Sect. 2, we give preliminaries and lemmas concerning the floor function, fractional parts, Beatty sequences, and Fibonacci numbers. In Sect. 3, we give our main results concerning various sumsets associated with $B(\alpha)$ and $B(\alpha^2)$. For more information, we invite the reader to visit the fourth author's ResearchGate website [37] for some freely downloadable articles [22,30–32,34–36] in related topics of research.

2 Preliminaries and lemmas

We often use the following fact: $-1 < \beta < 0$, $(|\beta^n|)_{n\geq 1}$ is strictly decreasing, if $a_1 > a_2 > \cdots > a_r$ are even positive integers, then $0 < \beta^{a_1} < \beta^{a_2} < \cdots < \beta^{a_r}$, and if $b_1 > b_2 > \cdots > b_r$ are odd positive integers, then $0 > \beta^{b_1} > \beta^{b_2} > \cdots > \beta^{b_r}$. In addition, α and β are roots of the equation $x^2 - x - 1 = 0$. So, for instance, $\beta^2 = \beta + 1$ and $\beta^2 + \beta^4 = 4\beta + 3$. Moreover, it is convenient to have a list of the first twenty elements of the sequences $B(\alpha)$ and $B(\alpha^2)$ as shown below:

$$B(\alpha) = (1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, 29, 30, 32, \ldots) \text{ and}$$
$$B(\alpha^2) = (2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, 31, 34, 36, 39, 41, 44, 47, 49, 52, \ldots).$$

The following results are also applied throughout this article sometimes without reference.

Lemma 2.1 *For $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, the following statements hold.*

(i) $\lfloor n + x\rfloor = n + \lfloor x\rfloor$.
(ii) $\{n + x\} = \{x\}$.
(iii) $0 \leq \{x\} < 1$.
(iv) $\lfloor x + y\rfloor = \begin{cases} \lfloor x\rfloor + \lfloor y\rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x\rfloor + \lfloor y\rfloor + 1, & \text{if } \{x\} + \{y\} \geq 1. \end{cases}$

Proof These are well-known and can be proved easily. For more details, see in [9, Chapter 3]. We also refer the reader to [18] and [35, Proof of Lemma 2.6] for a nice application of these properties. □

Lemma 2.2 *The following statements hold for all $n \in \mathbb{N}$.*

(i) *(Binet's formula)* $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.
(ii) $\beta^{n+1} = \beta F_{n+1} + F_n$.
(iii) $F_{n+1} = \beta^n + \alpha F_n$.

Proof The proof of (i) and (ii) can be found in [16, pp. 78–79]. The statement (iii) follows from (ii) and the fact that $\alpha\beta = -1$. See also [29] for a result concerning the generating function of the Fibonacci sequence. □

Lemma 2.3 (Zeckendorf's theorem) *For each $n \in \mathbb{N}$, $n = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$ where F_{a_1} is the largest Fibonacci number not exceeding n, $a_\ell \geq 2$, and $a_{i-1} - a_i \geq 2$ for every $i = 2, 3, \ldots, \ell$.*

Proof This is well-known and can be proved by using the greedy algorithm ([40, pp. 108–109] or [43]). See also [15] for a more general result. □

Lemma 2.4 *If $x_1, x_2, \ldots, x_n \in \mathbb{R}$, then*

$$\{x_1 + x_2 + \cdots + x_n\} = \{\{x_1\} + \{x_2\} + \cdots + \{x_n\}\}.$$

Proof We can write $x_1 + x_2 + \cdots + x_n = m + \{x_1\} + \{x_2\} + \cdots + \{x_n\}$, where $m = \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \cdots + \lfloor x_n \rfloor \in \mathbb{Z}$. So this lemma follows immediately from Lemma 2.1(ii). □

Lemma 2.5 *Let $n \in \mathbb{N}$. Then the following statements hold.*

(i) $\lfloor F_n\alpha \rfloor = F_{n+1} - [n \equiv 0 \pmod 2]$.
(ii) $\lfloor F_n\alpha^2 \rfloor = F_{n+2} - [n \equiv 0 \pmod 2]$.
(iii) $\{F_n\alpha\} = -\beta^n + [n \equiv 0 \pmod 2]$.
(iv) $\{F_n\alpha^2\} = \{F_n\alpha\}$.

Proof By Lemmas 2.2 and 2.1, we obtain $\lfloor F_n\alpha \rfloor = \lfloor F_{n+1} - \beta^n \rfloor = F_{n+1} + \lfloor -\beta^n \rfloor$. If n is even, then $0 < \beta^n < 1$ and so $\lfloor -\beta^n \rfloor = -1$. If n is odd, then $-1 < \beta^n < 0$ and so $\lfloor -\beta^n \rfloor = 0$. Therefore $\lfloor -\beta^n \rfloor = -[n \equiv 0 \pmod 2]$. This implies (i). Then (ii) follows from (i) by writing $\alpha^2 = \alpha + 1$ and $\lfloor F_n\alpha^2 \rfloor = \lfloor F_n\alpha + F_n \rfloor = \lfloor F_n\alpha \rfloor + F_n$. Next, $\{F_n\alpha\} = F_n\alpha - \lfloor F_n\alpha \rfloor$, so (iii) can be obtained from (i) and Lemma 2.2. For (iv), we have $\{F_n\alpha^2\} = \{F_n\alpha + F_n\} = \{F_n\alpha\}$. □

Lemma 2.6 (Beatty's theorem [1,2]) *Let x and y be irrational numbers such that $x, y > 1$ and $\frac{1}{x} + \frac{1}{y} = 1$. Then $B(x) \cup B(y) = \mathbb{N}$ and $B(x) \cap B(y) = \emptyset$. In particular, $B(\alpha) \cup B(\alpha^2) = \mathbb{N}$ and $B(\alpha) \cap B(\alpha^2) = \emptyset$.*

If $A = (a_n)_{n\geq 1}$ is a sequence, then a segment of A is a finite sequence of the form $(a_k, a_{k+1}, \ldots, a_{k+m})$ for some $k, m \in \mathbb{N}$. Then we have the following results.

Lemma 2.7 *The following statements hold.*

(i) *For each $b \in \mathbb{N}$, $\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor$ is either 1 or 2.*
(ii) *For each $b \in \mathbb{N}$, if $\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor = 1$ then $\lfloor (b+2)\alpha \rfloor - \lfloor (b+1)\alpha \rfloor = 2$.*
(iii) *The sequence $(\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor)_{b\geq 1}$ does not contain the segment $(2, 2, 2)$.*

Proof Let $b \in \mathbb{N}$. By Lemma 2.1, $\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor = \lfloor b\alpha + \alpha \rfloor - \lfloor b\alpha \rfloor = \lfloor \alpha \rfloor$ or $\lfloor \alpha \rfloor + 1 = 1$ or 2. This proves (i). For (ii), suppose that $\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor = 1 = \lfloor (b+2)\alpha \rfloor - \lfloor (b+1)\alpha \rfloor$. Then $2 = \lfloor (b+2)\alpha \rfloor - \lfloor b\alpha \rfloor \geq \lfloor 2\alpha \rfloor \geq 3$, which is a contradiction. For (iii), suppose that $(2, 2, 2)$ is a segment of the sequence $(\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor)_{b\geq 1}$, that is, there exists $b \in \mathbb{N}$ such that

$$\lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor = 2, \tag{2.1}$$
$$\lfloor (b+2)\alpha \rfloor - \lfloor (b+1)\alpha \rfloor = 2, \tag{2.2}$$
$$\lfloor (b+3)\alpha \rfloor - \lfloor (b+2)\alpha \rfloor = 2. \tag{2.3}$$

Adding (2.1)–(2.3), we have $6 = \lfloor (b+3)\alpha \rfloor - \lfloor b\alpha \rfloor \leq \lfloor 3\alpha \rfloor + 1 = 5$, which is a contradiction. □

Lemma 2.8 *Let $b \in \mathbb{N}$. Then the following statements hold.*

(i) *$\lfloor (b+1)\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor$ is either 2 or 3.*
(ii) *If $\lfloor (b+1)\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor = 2$, then $\lfloor (b+2)\alpha^2 \rfloor - \lfloor (b+1)\alpha^2 \rfloor = 3$.*
(iii) *The sequence $(\lfloor (b+1)\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor)_{b\geq 1}$ does not contain the segment $(3, 3, 3)$.*

Proof By Lemma 2.1, $\lfloor (b+1)\alpha^2 \rfloor - \lfloor b\alpha^2 \rfloor = \lfloor (b+1)\alpha \rfloor - \lfloor b\alpha \rfloor + 1$. Therefore this lemma is an immediate consequence of Lemma 2.7. □

3 Main results

In this section, we study various sumsets associated with Wythoff sequences. We begin with simple cases such as $B(\alpha) + B(\alpha)$ and $B_0(\alpha) + B(\alpha)$.

Theorem 3.1 *Let $B(\alpha)$ and $B_0(\alpha)$ be the sets as defined in (1.1). Then*

$$B(\alpha) + B(\alpha) = \mathbb{N}\backslash\{1, 3\} \text{ and } B_0(\alpha) + B(\alpha) = \mathbb{N}.$$

Proof It is easy to check that $1, 3 \notin B(\alpha) + B(\alpha)$ and $2 = \lfloor\alpha\rfloor + \lfloor\alpha\rfloor \in B(\alpha) + B(\alpha)$. So we let $n \geq 4$ and show that $n \in B(\alpha) + B(\alpha)$. Let b be the largest positive integer such that $b\alpha < n$. Then $b \geq 2$ and $\lfloor b\alpha\rfloor < n \leq \lfloor(b + 1)\alpha\rfloor$. By Lemma 2.7(i), $n = \lfloor b\alpha\rfloor + k$, where k is either 1 or 2. If $k = 1$, then $n = \lfloor b\alpha\rfloor + \lfloor\alpha\rfloor \in B(\alpha) + B(\alpha)$. So assume that $k = 2$. By Lemma 2.7(i), we can divide the proof into two cases. If $\lfloor b\alpha\rfloor - \lfloor(b - 1)\alpha\rfloor = 1$, then $n = \lfloor b\alpha\rfloor + 2 = \lfloor(b - 1)\alpha\rfloor + 3 = \lfloor(b - 1)\alpha\rfloor + \lfloor 2\alpha\rfloor$. If $\lfloor b\alpha\rfloor - \lfloor(b - 1)\alpha\rfloor = 2$, then $n = \lfloor b\alpha\rfloor + 2 = \lfloor(b - 1)\alpha\rfloor + 4 = \lfloor(b - 1)\alpha\rfloor + \lfloor 3\alpha\rfloor$. In any case, we have $n \in B(\alpha) + B(\alpha)$, as desired. Since 1 and 3 are in $B_0(\alpha) + B(\alpha)$ and $B(\alpha) + B(\alpha) \subseteq B_0(\alpha) + B(\alpha)$, we obtain that $B_0(\alpha) + B(\alpha) = \mathbb{N}$. □

Theorem 3.2 *Let $B(\alpha)$ and $B(\alpha^2)$ be defined as in (1.1) and $n \geq 3$. Then the following statements hold.*

(i) *$F_n \in B(\alpha)$ if and only if n is even.*
(ii) *$F_n \in B(\alpha^2)$ if and only if n is odd.*
(iii) *$F_n - 1 \in B(\alpha)$ if and only if n is odd.*
(iv) *$F_n - 1 \in B(\alpha^2)$ if and only if n is even.*

Proof By Lemma 2.5, we have

$$\begin{aligned} F_n - [n \equiv 0 \pmod 2] &= \lfloor F_{n-2}\alpha^2\rfloor \in B(\alpha^2), \\ F_n - [n \equiv 1 \pmod 2] &= \lfloor F_{n-1}\alpha\rfloor \in B(\alpha). \end{aligned}$$

Case 1: n is even. Then by the above equality, we have $F_n - 1 \in B(\alpha^2)$ and $F_n \in B(\alpha)$. Then by Lemma 2.6, $F_n - 1 \notin B(\alpha)$ and $F_n \notin B(\alpha^2)$.

Case 2: n is odd. Then $F_n \in B(\alpha^2)$ and $F_n - 1 \in B(\alpha)$. Then by Lemma 2.6, $F_n \notin B(\alpha)$ and $F_n - 1 \notin B(\alpha^2)$. This implies the desired result. □

The calculation of $B(\alpha) + B(\alpha^2)$ is a bit more complicated than $B(\alpha) + B(\alpha)$ and we need the following theorem.

Theorem 3.3 *Let $n \geq 3$ and $1 \leq b \leq F_{n+1}$. If $b \neq F_n$, then $0 < \{b\alpha\} + \beta^n < 1$. If $b = F_n$, then $\{b\alpha\} + \beta^n = [n \equiv 0 \pmod 2]$.*

Proof We use Lemma 2.5 repeatedly without reference. If $b = F_n$, then the result follows immediately. If $b = F_{n+1}$, then $\{b\alpha\} + \beta^n$ is equal to

$$\begin{aligned} -\beta^{n+1} + [n + 1 \equiv 0 \pmod 2] + \beta^n &= -\beta^{n-1} + [n - 1 \equiv 0 \pmod 2] \\ &= \{F_{n-1}\alpha\} \in (0, 1). \end{aligned}$$

Next we consider the case $b = F_k$ for some $k \in \{2, 3, \ldots, n - 1\}$. If k is even and n is odd, then

$$1 > \{b\alpha\} > \{b\alpha\} + \beta^n = 1 - \beta^k + \beta^n \geq 1 - \beta^2 + \beta^3 = 1 + \beta > 0.$$

If k and n are even, then $0 < \{b\alpha\} + \beta^n = 1 - \beta^k + \beta^n < 1$. Similarly, if k is odd and n is even, then $0 < \{b\alpha\} + \beta^n = -\beta^k + \beta^n \leq \beta^n - \beta^3 \leq \beta^4 - \beta^3 = \beta^2 < 1$. If k and n are odd, then $1 > \{b\alpha\} + \beta^n = -\beta^k + \beta^n > 0$. Hence this theorem is verified in the case $b = F_k$ for some $k \leq n+1$. Next, we suppose that $F_k < b < F_{k+1}$ for some $k \in \{4, 5, \ldots, n\}$. By Lemma 2.3, we can write $b = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$ where $\ell \geq 2$, $k = a_1 > a_2 > \cdots > a_\ell \geq 2$, and $a_{i-1} - a_i \geq 2$ for every $i = 2, 3, \ldots, \ell$. Then by Lemma 2.4, we obtain $\{b\alpha\} = \left\{\{F_{a_1}\alpha\} + \{F_{a_2}\alpha\} + \cdots + \{F_{a_\ell}\alpha\}\right\}$, which is equal to

$$\left\{\left(1 - \beta^{b_1} + 1 - \beta^{b_2} + \cdots + 1 - \beta^{b_r}\right) + \left(-\beta^{c_1} - \beta^{c_2} - \cdots - \beta^{c_s}\right)\right\},$$

where $\{b_1, b_2, \ldots, b_r\} \cup \{c_1, c_2, \ldots, c_s\} = \{a_1, a_2, \ldots, a_\ell\}$, $b_1, b_2, \ldots, b_r$ are even, and $c_1, c_2, \ldots, c_s$ are odd.

Remark that one of the sets $\{b_1, b_2, \ldots, b_r\}$, $\{c_1, c_2, \ldots, c_s\}$ may be empty. In that case, such the set disappears from the subsequent calculation. Also, for convenience, we let $A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} + \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s}$. Then by Lemma 2.1, $\{b\alpha\} = \{-A\}$.

Case 1: $\{b_1, b_2, \ldots, b_r\}$ is empty. Then $A = \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^3 + \beta^5 + \beta^7 + \cdots = \frac{\beta^3}{1-\beta^2} = -\beta^2$. Therefore $0 < -A < \beta^2 < 1$ and so $\{b\alpha\} = \{-A\} = -A - \lfloor -A \rfloor = -A$. Then

$$\{b\alpha\} + \beta^n < \beta^2 + \beta^n \leq \beta^2 + \beta^4 = 4\beta + 3 < 1.$$

It remains to show that $\{b\alpha\} + \beta^n > 0$. If n is even, then obviously $\{b\alpha\} + \beta^n > 0$. So assume that n is odd. Since $\{b_1, b_2, \ldots, b_r\}$ is empty, we see that a_1 is odd and $-A > -\beta^{a_1}$. Therefore $\{b\alpha\} + \beta^n > -\beta^{a_1} + \beta^n \geq 0$, as required.

Case 2: $\{c_1, c_2, \ldots, c_s\}$ is empty. Then $A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^2 + \beta^4 + \beta^6 + \cdots = \frac{\beta^2}{1-\beta^2} = -\beta$. In addition, a_1 is even and $A > \beta^{a_1}$. Therefore $-\beta^{a_1} > -A > \beta > -1$ and so $\{b\alpha\} = \{-A\} = 1 - A$. Then $\{b\alpha\} + \beta^n < 1 - \beta^{a_1} + \beta^n \leq 1$, and $\{b\alpha\} + \beta^n > 1 + \beta + \beta^n \geq 1 + \beta + \beta^3 = 3\beta + 2 > 0$.

Case 3: $\{b_1, b_2, \ldots, b_r\}$ and $\{c_1, c_2, \ldots, c_s\}$ are not empty. Then there is some cancellation in the sum defining A. Similar to Case 1 and Case 2, we have $A < \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < -\beta$ and $A > \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > -\beta^2$.

Case 3.1: A is positive. Then $-1 < \beta < -A < 0$ and $\{b\alpha\} + \beta^n = 1 - A + \beta^n$. So it suffices to show that $\beta^n < A < 1 + \beta^n$. Since $A < -\beta$, we obtain $A - \beta^n < -\beta - \beta^3 = -3\beta - 1 < 1$, which implies $A < 1 + \beta^n$. So it remains to show that $A > \beta^n$. If n is odd, then $A > 0 > \beta^n$. So suppose that n is even. Let u be the smallest even number among $\{b_1, b_2, \ldots, b_r\}$ and v the smallest odd number among $\{c_1, c_2, \ldots, c_s\}$. Since $a_i - a_{i-1} \geq 2$ for all $i = 2, 3, \ldots, \ell$ and $a_1 = k \leq n$, we obtain $u \leq n$ and $|v - u| \geq 3$. Then

$$\beta^u < \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^u + \beta^{u+2} + \beta^{u+4} + \cdots = \frac{\beta^u}{1-\beta^2} = -\beta^{u-1}, \tag{3.1}$$

$$\beta^v > \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^v + \beta^{v+2} + \beta^{v+4} + \cdots = \frac{\beta^v}{1-\beta^2} = -\beta^{v-1}. \tag{3.2}$$

By (3.1) and (3.2), we obtain $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v - u| \geq 3$, we see that either $v - u \geq 3$ or $v - u \leq -3$. Suppose for a contradiction that $v - u \leq -3$. Since $v \leq u - 3$ and both v and $u - 3$ are odd, we have $\beta^v \leq \beta^{u-3}$. So $A < \beta^{u-3} - \beta^{u-1} = \beta^{u-3}(1 - \beta^2) = -\beta^{u-2} < 0$, which contradicts the assumption that A is positive. Hence $v - u \geq 3$. Since $v - 1 \geq u + 2$ and both $v - 1$ and $u + 2$ are even, $\beta^{v-1} \leq \beta^{u+2}$. So

$A > \beta^u - \beta^{u+2} = \beta^u(1-\beta^2) = -\beta^{u+1}$. We have $u \le v-3 \le a_1 - 3 \le n-3$. Therefore $-\beta^{u+1} = |\beta|^{u+1} > |\beta|^n = \beta^n$. Therefore $A > \beta^n$, as desired.

Case 3.2: A is negative. Then $0 < -A < \beta^2 < 1$ and

$$\{b\alpha\} + \beta^n = \{-A\} + \beta^n = -A + \beta^n < \beta^2 + \beta^n \le \beta^2 + \beta^4 = 4\beta + 3 < 1.$$

To show that $\{b\alpha\} + \beta^n > 0$, it is enough to show that $\beta^n > A$. If n is even, then obviously $\beta^n > 0 > A$. So assume that n is odd. Let u and v be as in Case 3.1. Then we obtain $u \le n$, $|v-u| \ge 3$, the inequalities in (3.1) and (3.2) hold, and $\beta^u - \beta^{v-1} < A < -\beta^{u-1} + \beta^v$. Again, we have either $v-u \ge 3$ or $v-u \le -3$. Suppose for a contradiction that $v-u \ge 3$. Following the argument in Case 3.1, we obtain $A > \beta^u - \beta^{v-1} \ge -\beta^{u+1} > 0$, which contradicts the assumption that A is negative. Therefore $v-u \le -3$. Then $A < -\beta^{u-1} + \beta^v \le -\beta^{u-1} + \beta^{u-3} = -\beta^{u-2}$. Since $u-2 < n$, u is even, and n is odd, we obtain $-\beta^{u-2} = -|\beta|^{u-2} < -|\beta|^n = \beta^n$. Therefore $A < \beta^n$, as desired. Hence the proof is complete. □

Corollary 3.4 *For each $n \ge 3$ and $1 \le b \le F_{n+1}$, we have*

$$F_{n+1} = \lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta \text{ and } F_{n+2} = \lfloor (F_n - b)\,\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + 1 - \delta,$$

where $\delta = [n \equiv 1 \pmod 2][b = F_n]$.

Proof Let $n \ge 3$ and $1 \le b \le F_{n+1}$. If $b = F_n$, then we obtain by Lemma 2.5 that $\lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta = F_{n+1} - [n \equiv 0 \pmod 2] + 1 - [n \equiv 1 \pmod 2] = F_{n+1}$. So suppose $b \ne F_n$. Then $\delta = 0$ and we obtain by Lemmas 2.1, 2.2 and Theorem 3.3, respectively, that $\lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta$ is equal to

$$\lfloor F_n\alpha - b\alpha + \lfloor b\alpha \rfloor + 1 \rfloor = \lfloor F_{n+1} - \beta^n - \{b\alpha\} + 1 \rfloor = F_{n+1} + \lfloor 1 - \{b\alpha\} - \beta^n \rfloor = F_{n+1}.$$

This proves the first equality. By writing $\alpha^2 = \alpha + 1$ and applying Lemma 2.1, we see that

$$\lfloor (F_n - b)\,\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + 1 - \delta = \lfloor (F_n - b)\,\alpha \rfloor + \lfloor b\alpha \rfloor + 1 - \delta + F_n = F_{n+2}.$$

□

Theorem 3.5 *Let $B(\alpha)$, $B_0(\alpha)$, $B(\alpha^2)$, and $B_0(\alpha^2)$ be the sets as defined in (1.1). Then we have*

(i) $B(\alpha) + B\left(\alpha^2\right) = \mathbb{N} \setminus \{F_n - 1 \mid n \ge 3\}$,
(ii) $B_0(\alpha) + B\left(\alpha^2\right) = \mathbb{N} \setminus \{F_n - 1 \mid n \ge 3$ *and n is odd*$\}$, *and*
(iii) $B(\alpha) + B_0\left(\alpha^2\right) = \mathbb{N} \setminus \{F_n - 1 \mid n \ge 3$ *and n is even*$\}$.

Proof We first show that $B(\alpha) + B\left(\alpha^2\right) \subseteq \mathbb{N} \setminus \{F_n - 1 \mid n \ge 3\}$. It is easy to check that $F_3 - 1, F_4 - 1 \notin B(\alpha) + B\left(\alpha^2\right)$. So let $n \ge 5$. In order to get a contradiction, suppose $F_n - 1$ is in $B(\alpha) + B\left(\alpha^2\right)$. Then $F_n - 1 = \lfloor b\alpha \rfloor + \left\lfloor a\alpha^2 \right\rfloor$ for some $a, b \in \mathbb{N}$. If $b \ge F_{n-1}$, then we obtain by Lemma 2.5 that

$$\lfloor b\alpha \rfloor + \left\lfloor a\alpha^2 \right\rfloor \ge \lfloor F_{n-1}\alpha \rfloor + \left\lfloor \alpha^2 \right\rfloor = F_n - [n \equiv 1 \pmod 2] + 2 > F_n - 1,$$

which is not in case. So $b < F_{n-1}$. Replacing n by $n-1$ in Corollary 3.4, we have $\left\lfloor a\alpha^2 \right\rfloor = F_n - 1 - \lfloor b\alpha \rfloor = \lfloor (F_{n-1} - b)\alpha \rfloor \in B(\alpha)$, so $\left\lfloor a\alpha^2 \right\rfloor \in B(\alpha) \cap B(\alpha^2)$, which contradicts Lemma 2.6. Therefore $F_n - 1 \notin B(\alpha) + B(\alpha^2)$ for any $n \ge 3$. This shows that $B(\alpha) + B(\alpha^2)$ is a subset of $\mathbb{N} \setminus \{F_n - 1 \mid n \ge 3\}$. For the other direction, let $m \in \mathbb{N} \setminus \{F_n - 1 \mid n \ge 3\}$. Then there exists $n \in \mathbb{N}$ such that $n \geq 3$ and $F_n - 1 < m < F_{n+1} - 1$. Thus $m = F_n - 1 + b$ where $1 \leq b < F_{n-1}$. By Corollary 3.4, we obtain $m = \lfloor (F_{n-1} - b)\alpha \rfloor + \lfloor b\alpha \rfloor + b = \lfloor (F_{n-1} - b)\alpha \rfloor + \lfloor b\alpha^2 \rfloor \in B(\alpha) + B(\alpha^2)$. This proves (i). Next $B_0(\alpha) + B(\alpha^2) = (B(\alpha) + B(\alpha^2)) \cup B(\alpha^2) = \mathbb{N} \backslash \{F_n - 1 \mid n \geq 3 \text{ and } n \text{ is odd}\}$, by (i) and Theorem 3.2. Similarly, (iii) can be obtained by using (i) and Theorem 3.2. This completes the proof. $\square$

Remark 3.6 It follows immediately from Beatty's theorem that $B_0(\alpha) + B_0(\alpha^2) = \mathbb{N}$.

Theorem 3.7 *Let $B(\alpha)$, $B_0(\alpha)$, $B(\alpha^2)$, and $B_0(\alpha^2)$ be defined as in (1.1). Then the following statements hold.*

(i) $B(\alpha) + B(\alpha^2) + B(\alpha^2) = \mathbb{N} \backslash \{1, 2, 3, 4, 6, 9\}$.
(ii) $B_0(\alpha) + B(\alpha^2) + B(\alpha^2) = \mathbb{N} \backslash \{1, 2, 3, 6\}$.
(iii) $B(\alpha) + B(\alpha^2) + B_0(\alpha^2) = \mathbb{N} \backslash \{1, 2, 4\}$.
(iv) $B(\alpha) + B_0(\alpha^2) + B_0(\alpha^2) = \mathbb{N} \backslash \{2\}$.

Proof We can write Theorem 3.5 in another form as

$$B(\alpha) + B(\alpha^2) = \bigcup_{n=4}^{\infty} ((F_n - 1, F_{n+1} - 1) \cap \mathbb{N}) = \bigcup_{n=4}^{\infty} ([F_n, F_{n+1} - 2] \cap \mathbb{N}).$$

Then $B(\alpha) + B(\alpha^2) + \lfloor \alpha^2 \rfloor = \bigcup_{n=4}^{\infty} ([F_n + 2, F_{n+1}] \cap \mathbb{N}) = \mathbb{N} \backslash A$, where $A = \{F_n + 1 \mid n \geq 5\} \cup \{1, 2, 3, 4\}$. Similarly, $B(\alpha) + B(\alpha^2) + \lfloor 2\alpha^2 \rfloor = \mathbb{N} \backslash B$ where $B = \{F_m + 4 \mid m \geq 2\} \cup \{1, 2, 3, 4\}$. Therefore $\mathbb{N} \backslash (A \cap B) = (\mathbb{N} \backslash A) \cup (\mathbb{N} \backslash B) \subseteq B(\alpha) + B(\alpha^2) + B(\alpha^2)$. It is easy to see that

$$\begin{aligned} A \cap B &= (\{F_n + 1 \mid n \geq 5\} \cap \{F_m + 4 \mid m \geq 2\}) \cup \{1, 2, 3, 4\} \\ &= (\{F_n + 1 \mid n \geq 7\} \cap \{F_m + 4 \mid m \geq 6\}) \cup \{1, 2, 3, 4, 6, 9\}. \end{aligned}$$

If $n \geq 7$, $m \geq 6$, and $F_n + 1 = F_m + 4$, then $n > m$ and $3 = F_n - F_m \geq F_n - F_{n-1} = F_{n-2} \geq 5$, which is a contradiction. So $\{F_n + 1 \mid n \geq 7\} \cap \{F_m + 4 \mid m \geq 6\} = \emptyset$. Therefore $A \cap B = \{1, 2, 3, 4, 6, 9\}$ and thus $\mathbb{N} \backslash \{1, 2, 3, 4, 6, 9\} \subseteq B(\alpha) + B(\alpha^2) + B(\alpha^2)$. It is easy to check that $1, 2, 3, 4, 6, 9 \notin B(\alpha) + B(\alpha^2) + B(\alpha^2)$. This proves (i). The other parts follows from (i) and a straightforward verification. $\square$

The structure of $B(\alpha^2) + B(\alpha^2)$ seems to be the most complicated among sumsets associated with $B(\alpha)$ and $B(\alpha^2)$. So we first consider a simpler sumset $B(\alpha^2) + B(\alpha^2) + B(\alpha^2)$.

Theorem 3.8 *Let $B(\alpha^2)$ and $B_0(\alpha^2)$ be defined as in (1.1). Then we have*

$$\begin{aligned} B(\alpha^2) + B(\alpha^2) + B(\alpha^2) &= \mathbb{N} \backslash \{1, 2, 3, 4, 5, 7, 8, 10, 13, 18, 26\}, \\ B_0(\alpha^2) + B(\alpha^2) + B(\alpha^2) &= \mathbb{N} \backslash \{1, 2, 3, 5, 8, 13\}, \\ B_0(\alpha^2) + B_0(\alpha^2) + B(\alpha^2) &= \mathbb{N} \backslash \{1, 3, 8\}. \end{aligned}$$

Proof Let $A_1 = \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + B(\alpha^2)$, $A_2 = \lfloor 5\alpha^2 \rfloor + \lfloor 5\alpha^2 \rfloor + B(\alpha^2)$, and $A_3 = \lfloor 3\alpha^2 \rfloor + \lfloor 8\alpha^2 \rfloor + B(\alpha^2)$. We first show that $A_1 \cup A_2 \cup A_3 = \{n \in \mathbb{N} \mid n \geq 27\}$. Note that $\lfloor 3\alpha^2 \rfloor, \lfloor 4\alpha^2 \rfloor, \lfloor 5\alpha^2 \rfloor, \lfloor 6\alpha^2 \rfloor, \lfloor 8\alpha^2 \rfloor$ are equal to 7, 10, 13, 15, 20, respectively. Then it is easy to see that every element in $A_1 \cup A_2 \cup A_3$ is larger than or equal to 27. Next, let $n \geq 27$. Then there exists $k \in \mathbb{N}$ such that

$$\lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor \leq n < \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor (k+1)\alpha^2 \rfloor.$$

By Lemma 2.8, we have $\lfloor (k+1)\alpha^2 \rfloor - \lfloor k\alpha^2 \rfloor = 2$ or 3, and so $n = \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor + b$, where $b = 0, 1$ or 2. If $b = 0$, then $n \in A_1$. If $b = 1$, then $n = \lfloor 4\alpha^2 \rfloor + \lfloor 6\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor + 1 = \lfloor 5\alpha^2 \rfloor + \lfloor 5\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor \in A_2$. Similarly, if $b = 2$, then $n = \lfloor 3\alpha^2 \rfloor + \lfloor 8\alpha^2 \rfloor + \lfloor k\alpha^2 \rfloor \in A_3$. In any case, $n \in A_1 \cup A_2 \cup A_3$, as required. This implies that $B\left(\alpha^2\right) + B\left(\alpha^2\right) + B\left(\alpha^2\right)$ contains $\mathbb{N} \cap [27, \infty)$. For the integers in $\mathbb{N} \cap [1, 26]$, we can straightforwardly check whether they are in $B\left(\alpha^2\right) + B\left(\alpha^2\right) + B\left(\alpha^2\right)$ or not. For the reader's convenience, we give the integers which are in $B\left(\alpha^2\right) + B\left(\alpha^2\right) + B\left(\alpha^2\right)$ as follows: $6 = 2 + 2 + 2$, $9 = 5 + 2 + 2$, $11 = 7 + 2 + 2$, $12 = 5 + 5 + 2$, $14 = 10 + 2 + 2$, $15 = 5 + 5 + 5$, $16 = 7 + 7 + 2$, $17 = 7 + 5 + 5$, $19 = 15 + 2 + 2$, $20 = 10 + 5 + 5$, $21 = 7 + 7 + 7$, $22 = 18 + 2 + 2$, $23 = 13 + 5 + 5$, $24 = 20 + 2 + 2$, $25 = 15 + 5 + 5$. This proves the first part. The other parts follow from the first part and straightforward verification. □

In order to prove Theorem 3.10, it is convenient to use the following observation.

Lemma 3.9 *Let $n \geq 3$, $a \in \mathbb{Z}$, and $b, c \in \mathbb{N}$. If $F_n + a = \left\lfloor b\alpha^2 \right\rfloor + \left\lfloor c\alpha^2 \right\rfloor$, then b and c are less than $F_{n-2} + \frac{a}{\alpha^2}$.*

Proof If b or $c \geq F_{n-2} + \frac{a}{\alpha^2}$, then $\left\lfloor b\alpha^2 \right\rfloor + \left\lfloor c\alpha^2 \right\rfloor$ is larger than or equal to

$$\left\lfloor F_{n-2}\alpha^2 + a \right\rfloor + \left\lfloor \alpha^2 \right\rfloor = F_n - [n \equiv 0 \pmod 2] + a + 2 > F_n + a.$$

□

The positive integers in $\mathbb{N} \setminus \left(B\left(\alpha^2\right) + B\left(\alpha^2\right)\right)$ are 1, 2, 3, 5, 6, 8, 11, 13, 16, 19, 21, 24, 29, 32, 34, 37, 42, 45, 50, 53, 55, ... From this, we notice the following pattern.

Theorem 3.10 *Let $n \in \mathbb{N}$ and $B(\alpha^2)$ the Beatty set as defined in (1.1). Then the following statements hold.*

(i) $F_n \notin B(\alpha^2) + B(\alpha^2)$.
(ii) *If $n \geq 5$, then $F_n - 1 \in B(\alpha^2) + B(\alpha^2)$.*
(iii) *If $n \neq 1, 2, 3, 5$, then $F_n + 1 \in B(\alpha^2) + B(\alpha^2)$.*
(iv) $F_n - 2 \notin B(\alpha^2) + B(\alpha^2)$.
(v) *If $n \neq 1, 2, 4$, then $F_n + 2 \in B(\alpha^2) + B(\alpha^2)$.*
(vi) *If $n \geq 7$, then $F_n - 3 \in B(\alpha^2) + B(\alpha^2)$.*
(vii) *If $n \geq 3$, then $F_n + 3 \notin B(\alpha^2) + B(\alpha^2)$.*

Proof For $n \leq 6$, the result is easily checked. So we assume throughout that $n \geq 7$. For (i), suppose for a contradiction that $F_n \in B(\alpha^2) + B(\alpha^2)$. Then $F_n = \left\lfloor b\alpha^2 \right\rfloor + \left\lfloor c\alpha^2 \right\rfloor$ for some $a, b \in \mathbb{N}$. By Lemma 3.9, b and c are less than F_{n-2}. By Corollary 3.4,

$$\left\lfloor c\alpha^2 \right\rfloor - \left\lfloor (F_{n-2} - b)\alpha^2 \right\rfloor = F_n - \left\lfloor b\alpha^2 \right\rfloor - \left\lfloor (F_{n-2} - b)\alpha^2 \right\rfloor = 1.$$

But by Lemma 2.8, we know that the difference between the elements in $B(\alpha^2)$ is at least two. So we obtain a contradiction. This proves (i). The statements (ii), (iii), (v), and (vi) also follow from applications of Corollary 3.4 as follows:

$$\begin{aligned} F_n - 1 &= \left\lfloor (F_{n-2} - 1)\alpha^2 \right\rfloor + \left\lfloor \alpha^2 \right\rfloor \in B(\alpha^2) + B(\alpha^2), \\ F_n + 1 &= \left\lfloor (F_{n-2} - 2)\alpha^2 \right\rfloor + \left\lfloor 2\alpha^2 \right\rfloor + 2 \\ &= \left\lfloor (F_{n-2} - 2)\alpha^2 \right\rfloor + \left\lfloor 3\alpha^2 \right\rfloor \in B(\alpha^2) + B(\alpha^2), \\ F_n + 2 &= \left\lfloor (F_{n-2} - 1)\alpha^2 \right\rfloor + \left\lfloor \alpha^2 \right\rfloor + 3 \end{aligned}$$

$$
\begin{aligned}
&= \lfloor (F_{n-2}-1)\alpha^2 \rfloor + \lfloor 2\alpha^2 \rfloor \in B(\alpha^2)+B(\alpha^2),\\
F_n - 3 &= \lfloor (F_{n-2}-3)\alpha^2 \rfloor + \lfloor 3\alpha^2 \rfloor - 2\\
&= \lfloor (F_{n-2}-3)\alpha^2 \rfloor + \lfloor 2\alpha^2 \rfloor \in B(\alpha^2)+B(\alpha^2).
\end{aligned}
$$

So it remains to prove (iv) and (vii). Similar to the proof of (i), if $F_n - 2 = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$, then we have $b, c < F_{n-2} - \frac{2}{\alpha^2}$, $F_n = \lfloor (F_{n-2}-b)\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + 1$, and $\lfloor (F_{n-2}-b)\alpha^2 \rfloor - \lfloor c\alpha^2 \rfloor = \lfloor (F_{n-2}-b)\alpha^2 \rfloor - (F_n - 2 - \lfloor b\alpha^2 \rfloor) = 1$, which contradicts Lemma 2.8. For (vii), suppose that $F_n + 3 = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ for some $b, c \in \mathbb{N}$. Then by Lemma 3.9, b and $c < F_{n-2} + \frac{3}{\alpha^2} < F_{n-1}$. If b and $c = F_{n-2}$, then $F_n + 3 = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor = 2F_n - 2[n \equiv 0 \pmod 2]$, which leads to $F_7 \leq F_n = 3 + 2[n \equiv 0 \pmod 2] \leq 5$, a contradiction. So one of b, c is not equal to F_{n-2}. Without loss of generality, assume that $b \neq F_{n-2}$. So we can apply Corollary 3.4 and follow the same idea to obtain $\lfloor c\alpha^2 \rfloor - \lfloor (F_{n-2}-b)\alpha^2 \rfloor = 4$. By Lemma 2.8(i), the difference between consecutive terms in $B(\alpha^2)$ is either 2 or 3. So there are $k, r \in \mathbb{N} \cup \{0\}$ such that $4 = 2k + 3r$. If $r \geq 2$, then $2k + 3r > 4$. If $r = 1$, then $2k + 3r = 2k + 3 \neq 4$. So $r = 0$ and $k = 2$. This implies that $c = F_{n-2} - b + 2$, $\lfloor c\alpha^2 \rfloor - \lfloor (c-1)\alpha^2 \rfloor = \lfloor (c-1)\alpha^2 \rfloor - \lfloor (c-2)\alpha^2 \rfloor = 2$, which contradicts Lemma 2.8(ii). So the proof is complete. □

Our next goal is to determine completely the integers a such that $F_n + a \in B(\alpha^2) + B(\alpha^2)$. The reader will see that there is a recurrence and fractal-like behavior involving those integers.

Lemma 3.11 *Let $n \geq 5$, $a \in \mathbb{Z}$, and $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ where $B(\alpha^2)$ is the set as defined in (1.1). Then the following statements hold.*

(i) *For every integer $d \in [-F_{n-3}, 0) \cup (0, F_{n-2})$, we have $a + 1 + \lfloor d\alpha^2 \rfloor \notin B(\alpha^2)$.*
(ii) *$a + 1 - [n \equiv 1 \pmod 2] \notin B(\alpha^2)$.*

Proof Let $1 \leq b \leq F_{n-1}$ and $\delta_b = [n \equiv 1 \pmod 2][b = F_{n-2}]$. By Corollary 3.4, we have $F_n + a = \lfloor (F_{n-2}-b)\alpha^2 \rfloor + a + 1 - \delta_b + \lfloor b\alpha^2 \rfloor$. Since $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ and $\lfloor b\alpha^2 \rfloor \in B(\alpha^2)$, we see that

$$
\lfloor (F_{n-2}-b)\alpha^2 \rfloor + a + 1 - \delta_b \notin B(\alpha^2). \tag{3.3}
$$

Since (3.3) holds for all $b \leq F_{n-1}$, we can substitute $b = F_{n-2}$ in (3.3) to obtain (ii). Similarly, by running b over the integers in $[1, F_{n-2}) \cup (F_{n-2}, F_{n-1}]$, we obtain (i). □

Suppose $n \geq 5$ and the integers in $[F_n, F_{n+1}] \cap (B(\alpha^2) + B(\alpha^2))$ are given. Then the next theorem gives us some integers in $[F_{n+1}, F_{n+2}] \cap (B(\alpha^2) + B(\alpha^2))$.

Theorem 3.12 *Let $B(\alpha^2)$ be the set as defined in (1.1). Let $n \geq 5$, $a \in \mathbb{Z}$, and $1 \leq a < F_n - 2$. If $F_n + a \in B(\alpha^2) + B(\alpha^2)$, then $F_{n+1} + a \in B(\alpha^2) + B(\alpha^2)$.*

Proof If $n = 5$, the result is easily checked. So assume that $n \geq 6$. Suppose for a contradiction that $F_n + a \in B(\alpha^2) + B(\alpha^2)$ but $F_{n+1} + a \notin B(\alpha^2) + B(\alpha^2)$. By applying Lemma 3.11 to $F_{n+1} + a$, we obtain that

$$
a + 1 + \lfloor d\alpha^2 \rfloor \notin B(\alpha^2) \text{ for } 0 < d < F_{n-1}. \tag{3.4}
$$

Since $F_n + a \in B(\alpha^2) + B(\alpha^2)$, there are $b, c \in \mathbb{N}$ such that $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. If $b < F_{n-2}$, then by Corollary 3.4, $a + 1 + \lfloor (F_{n-2}-b)\alpha^2 \rfloor = F_n + a - \lfloor b\alpha^2 \rfloor = \lfloor c\alpha^2 \rfloor \in B(\alpha^2)$, which contradicts (3.4). Therefore $b \geq F_{n-2}$. Similarly, by applying the same argument to

c, we obtain $c \geq F_{n-2}$. Then $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor \geq 2\lfloor F_{n-2}\alpha^2 \rfloor = 2(F_n - [n \equiv 0 \pmod 2])$, which implies $a \geq F_n - 2[n \equiv 0 \pmod 2]$ contradicting the assumption that $a < F_n - 2$. Hence the proof is complete. □

Remark 3.13 Let $n \geq 4$. By Lemma 2.5, we have $F_{n-3}\alpha^2 = F_{n-1} - \beta^{n-3}$. So for $a \in \mathbb{Z}$, the condition $a \leq F_{n-1} - [n \equiv 1 \pmod 2]$ is equivalent to $a \leq F_{n-3}\alpha^2$. We will use this observation later.

To obtain the converse of Theorem 3.12, we first prove the following lemma.

Lemma 3.14 *Let $B(\alpha^2)$ be the Beatty set as defined in (1.1), $n \geq 5$, $a, b, c \in \mathbb{N}$, and $1 \leq a \leq F_{n-3}\alpha^2$. Suppose $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ and $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. Then one of b, c is equal to F_{n-1} and the other, say c, satisfying $\lfloor c\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]$.*

Proof Suppose for a contradiction that both b and c are not equal to F_{n-1}. Since $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$, we obtain by Lemma 3.9 that both b and c are less than $F_{n-1} + \frac{a}{\alpha^2} \leq F_n$. So we can apply Corollary 3.4 to write

$$F_{n+1} + a = \lfloor (F_{n-1} - b)\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor + 1 + a,$$
$$F_{n+1} + a = \lfloor (F_{n-1} - c)\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor + 1 + a.$$

Since $F_n + a \notin B(\alpha^2) + B(\alpha^2)$ and $a + 1 + \lfloor (F_{n-1} - b)\alpha^2 \rfloor = F_{n+1} + a - \lfloor b\alpha^2 \rfloor = \lfloor c\alpha^2 \rfloor \in B(\alpha^2)$, we obtain by Lemma 3.11 that $F_{n-1} - b \notin [-F_{n-3}, 0) \cup (0, F_{n-2})$. Therefore $F_{n-1} - b \geq F_{n-2}$, $F_{n-1} - b = 0$, or $F_{n-1} - b < -F_{n-3}$. Then $b \leq F_{n-3}$ or $b > F_{n-1} + F_{n-3}$. Applying the above argument to $a + 1 + \lfloor (F_{n-1} - c)\alpha^2 \rfloor = \lfloor b\alpha^2 \rfloor \in B(\alpha^2)$, we also obtain $c \leq F_{n-3}$ or $c > F_{n-1} + F_{n-3}$. Recall that b and $c < F_{n-1} + \frac{a}{\alpha^2}$. So if b or $c > F_{n-1} + F_{n+3}$, then we would obtain $F_{n-1} + \frac{a}{\alpha^2} > F_{n-1} + F_{n-3}$, which contradicts the assumption that $a \leq F_{n-3}\alpha^2$. Hence b and $c \leq F_{n-3}$. Then $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor \leq 2\lfloor F_{n-3}\alpha^2 \rfloor \leq 2F_{n-1} < F_{n+1} + a$, which is a contradiction. Thus b or c is equal to F_{n-1}. If $b = F_{n-1}$, then $F_{n+1} + a = \lfloor F_{n-1}\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor = F_{n+1} - [n \equiv 1 \pmod 2] + \lfloor c\alpha^2 \rfloor$, and so $\lfloor c\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]$. Similarly, if $c = F_{n-1}$, then $\lfloor b\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]$. This completes the proof. □

Lemma 3.15 *Let $B(\alpha^2)$ be the set as defined in (1.1), $n \geq 6$ and $1 \leq a \leq F_{n-3}\alpha^2$. If $F_{n-1} + a$ and $F_n + a$ are not in $B(\alpha^2) + B(\alpha^2)$, then $F_{n+1} + a \notin B(\alpha^2) + B(\alpha^2)$.*

Proof Suppose for a contradiction that $F_{n-1} + a$ and $F_n + a$ are not in $B(\alpha^2) + B(\alpha^2)$ but $F_{n+1} + a \in B(\alpha^2) + B(\alpha^2)$. Then there are $b, c \in \mathbb{N}$ such that $F_{n+1} + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. By Lemma 3.14, we can assume that $b = F_{n-1}$ and

$$\lfloor c\alpha^2 \rfloor = a + [n \equiv 1 \pmod 2]. \tag{3.5}$$

But by applying Lemma 3.11 to the case $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$, we obtain that

$$a + 1 - [n - 1 \equiv 1 \pmod 2] \notin B(\alpha^2), \quad \text{or equivalently,}$$
$$a + [n \equiv 1 \pmod 2] \notin B(\alpha^2),$$

which contradicts (3.5). So the proof is complete. □

Suppose $n \geq 6$ and the integers in $[F_n, F_{n+1}] \cap (B(\alpha^2) + B(\alpha^2))$ are given. Then the next theorem gives us more than a half of the integers in and outside the set

$$[F_{n+1}, F_{n+2}] \cap (B(\alpha^2) + B(\alpha^2)). \tag{3.6}$$

Theorem 3.16 *Let $B(\alpha^2)$ be the Beatty set as defined in (1.1), $n \geq 6$, $a \in \mathbb{Z}$, and $0 \leq a < F_{n-1} - 2$. Then*

$$F_n + a \in B(\alpha^2) + B(\alpha^2) \quad \textit{if and only if} \quad F_{n+1} + a \in B(\alpha^2) + B(\alpha^2).$$

Proof If $a = 0$, the result follows from Theorem 3.10. So assume that $a \geq 1$. By Theorem 3.12, we only need to prove the converse. Assume that $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Then we obtain by Theorem 3.12 that $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. Then by Lemma 3.15, $F_{n+1} \notin B(\alpha^2) + B(\alpha^2)$. So the proof is complete. □

The next theorem gives the remaining integers in (3.6).

Theorem 3.17 *Let $B(\alpha^2)$ be the set as defined in (1.1), $n \geq 6$, $a \in \mathbb{Z}$, and $0 \leq a \leq F_{n-1} - [n \equiv 1 \pmod 2]$. Then $F_n + a \in B(\alpha^2) + B(\alpha^2)$ if and only if $F_{n+1} - 2 - a \in B(\alpha^2) + B(\alpha^2)$.*

Proof If $a = 0$, the result follows from Theorem 3.10. So assume that $a \geq 1$. Suppose $F_n + a \in B(\alpha^2) + B(\alpha^2)$ but $F_{n+1} - 2 - a \notin B(\alpha^2) + B(\alpha^2)$. By applying Lemma 3.11 to $F_{n+1} - 2 - a$, we obtain that

$$-a - 1 + \lfloor d\alpha^2 \rfloor \notin B(\alpha^2) \text{ for } d \in [-F_{n-2}, 0) \cup (0, F_{n-1}). \tag{3.7}$$

Since $F_n + a \in B(\alpha^2) + B(\alpha^2)$, there are $b, c \in \mathbb{N}$ such that $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$. Then by Lemma 3.9, b and $c < F_{n-2} + \frac{a}{\alpha^2}$. Recall also from Remark 3.13 that $a \leq F_{n-3}\alpha^2$. If $b < F_{n-2}$, then by Corollary 3.4 and the fact given in (3.7), we obtain, respectively, $-a - 1 + \lfloor c\alpha^2 \rfloor = F_n - 1 - \lfloor b\alpha^2 \rfloor = \lfloor (F_{n-2} - b)\alpha^2 \rfloor \in B(\alpha^2)$, and $c \geq F_{n-1}$, which contradicts the fact that $c < F_{n-2} + \frac{a}{\alpha^2}$. So $b \geq F_{n-2}$. Similarly, applying the above argument to c, we have $c \geq F_{n-2}$. Then $F_n + a = \lfloor b\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor \geq 2\lfloor F_{n-2}\alpha^2 \rfloor \geq 2F_n - 2 > F_n + a$, which is a contradiction. Hence the first part of this theorem is proved.

For the converse, we also suppose for a contradiction that $F_{n+1} - 2 - a \in B(\alpha^2) + B(\alpha^2)$ but $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Then there are $b, c \in \mathbb{N}$ such that $F_{n+1} - 2 - a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor$ and by Lemma 3.11,

$$a + 1 + \lfloor d\alpha^2 \rfloor \notin B(\alpha^2) \text{ for } d \in [-F_{n-3}, 0) \cup (0, F_{n-2}). \tag{3.8}$$

By Lemma 3.9, b and $c < F_{n-1} - \frac{a+2}{\alpha^2} < F_{n-1}$. Then by Corollary 3.4, we obtain

$$F_{n+1} - 2 - a = \lfloor (F_{n-1} - b)\alpha^2 \rfloor + \lfloor b\alpha^2 \rfloor - a - 1,$$

which implies $a + 1 + \lfloor c\alpha^2 \rfloor = \lfloor (F_{n-1} - b)\alpha^2 \rfloor \in B(\alpha^2)$. So by (3.8), $c \geq F_{n-2}$. By the same argument, $b \geq F_{n-2}$. Therefore $F_{n+1} - 2 - a = \lfloor b\alpha^2 \rfloor + \lfloor c\alpha^2 \rfloor \geq 2\lfloor F_{n-2}\alpha^2 \rfloor \geq 2F_n - 2$, which implies $a \leq F_{n-1} - F_n < 0$, a contradiction. Hence the proof is complete. □

Theorems 3.10, 3.16, and 3.17 give a complete description of $B(\alpha^2) + B(\alpha^2)$. We illustrate this in Example 3.18 and Theorem 3.20 as follows.

Example 3.18 For convenience, if $A \subseteq \mathbb{N}$, we write A^c to denote the complement of A in $\mathbb{N}$. That is $A^c = \mathbb{N} \backslash A$. By direct calculation, the elements in $(B(\alpha^2) + B(\alpha^2))^c \cap [1, F_9]$ are 1, 2, 3, 5, 6, 8, 11, 13, 16, 19, 21, 24, 29, 32, 34. To determine the elements in $[F_9, F_{10}] \cap (B(\alpha^2) + B(\alpha^2))^c$, we first observe that for $0 \leq a \leq F_7$,

$$F_8 + a \notin B(\alpha^2) + B(\alpha^2) \text{ if and only if } a \in \{0, 3, 8, 11, 13\}.$$

Applying Theorem 3.16 for $n = 8$, we obtain that for $0 \leq a < F_7 - 2$,

$$F_9 + a \notin B(\alpha^2) + B(\alpha^2) \text{ if and only if } a \in \{0, 3, 8\}.$$

Applying Theorem 3.17 for $n = 9$, we obtain that for $0 \leq a < F_7 - 2$,

$$F_{10} - 2 - a \notin B(\alpha^2) + B(\alpha^2) \text{ if and only if } a \in \{0, 3, 8\}.$$

In addition, $F_{10} \notin B(\alpha^2) + B(\alpha^2)$ by Theorem 3.10. The length of the interval $[F_9, F_{10}]$ is $F_{10} - F_9 = F_8$ which is less than $2(F_7 - 2)$. Therefore the elements in $(B(\alpha^2) + B(\alpha^2))^c \cap [F_9, F_{10}]$ are completely determined. They are F_9, $F_9 + 3$, $F_9 + 8$, $F_{10} - 10$, $F_{10} - 5$, $F_{10} - 2$, F_{10}, which are $34, 37, 42, 45, 50, 53, 55$. By doing this process repeatedly, we obtain $(B(\alpha^2) + B(\alpha^2))^c \cap A$ where $A = [F_{10}, F_{11}]$, $[F_{11}, F_{12}]$, $[F_{12}, F_{13}]$, and so on. Thus we can find $(B(\alpha^2) + B(\alpha^2)) \cap [1, F_n]$ for any given n.

Remark 3.19 In the abstract and introduction, we mention that the structure of the set $X =: B(\alpha^2) + B(\alpha^2)$ has some kinds of fractal and palindromic patterns. This is not intended to be a precise or mathematically rigorous statement. What we (vaguely) means is that the distribution of the elements of X in the interval $[F_n, F_{n+1}]$ looks like fractal for all $n \geq 6$. Suppose we display the points of $X \cap [F_{n+1}, F_{n+2}]$ on the real line and zoom in for a smaller scale, namely, $X \cap [F_{n+1}, F_{n+1} + F_{n-1} - 3]$. Then, by Theorem 3.16, the picture (in a smaller scale) is the same as that of $X \cap [F_n, F_{n+1}]$. Then by Theorem 3.16 again, the picture (in a smaller scale) of $X \cap [F_{n+2}, F_{n+3}]$ is the same as that of $X \cap [F_{n+1}, F_{n+2}]$. Since Theorem 3.16 holds for all $n \geq 6$, we can continue this process and see the distribution of the elements of X on $[F_n, F_{n+1}]$, $[F_{n+1}, F_{n+2}]$, $[F_{n+2}, F_{n+3}]$, and so on, as fractal-like pattern. See the figure shown below for an illustration.

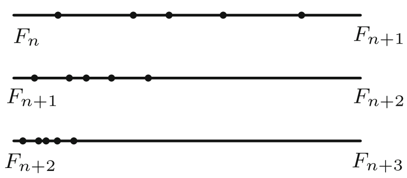
For the palindromicity, recall that a positive integer n can be written uniquely in the decimal expansion as

$$n = (a_k a_{k-1} \dots a_0)_{10} = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_0,$$

where $a_k \neq 0$ and $0 \leq a_i \leq 9$ for all i, and n is called a palindrome or a palindromic number if $a_{k-i} = a_i$ for $0 \leq i \leq \lfloor k/2 \rfloor$. So if n is a palindrome and we know the values of a_{k-i} only for $0 \leq i \leq \lfloor k/2 \rfloor$, then we can completely find all the decimal digits of n. Now suppose $n \geq 6$ and the elements of $X \cap [F_n, F_{n+1}]$ are known. We can divide $[F_{n+1}, F_{n+2}]$ into two overlapped intervals:

$$\text{the left-hand interval } L =: [F_{n+1}, F_{n+1} + F_{n-1} - 3]$$
$$\text{the right-hand interval } R =: [F_{n+2} - F_{n-1} - 1, F_{n+2}].$$

By Theorem 3.16, $X \cap L$ is completely determined by $X \cap [F_n, F_{n+1}]$. Theorem 3.17 gives us the palindromic pattern which helps us obtain all elements in R from L. Hence we can basically says that for all $n \geq 6$, the distribution of points in $X \cap [F_{n+1}, F_{n+2}]$ are completely determined by that of $X \cap [F_n, F_{n+1}]$ by the fractal-like and palindromic patterns.



In general, we have the following result.

Theorem 3.20 *Let $B(\alpha^2)$ be the set as defined in (1.1). For each $n \in \mathbb{N}$, let $A_n = \{a \in \mathbb{Z} \mid 0 \leq a \leq F_{n-1}$ and $F_n + a \notin B(\alpha^2) + B(\alpha^2)\}$. Then $A_1 = \{0\}$, $A_2 = A_3 = \{0, 1\}$,*

$A_4 = \{0, 2\}$, $A_5 = \{0, 1, 3\}$, $A_6 = \{0, 3, 5\}$, $A_7 = \{0, 3, 6, 8\}$, and for $n \geq 8$, the set A_n is the disjoint union

$$A_n = (A_{n-1} \backslash \{F_{n-2}, F_{n-2} - 2\}) \\ \cup \{F_{n-1} - 2 - a \mid a \in A_{n-1} \text{ and } 0 \leq a \leq F_{n-3}\} \cup \{F_{n-1}\}.$$

Proof The set $A_1, A_2, \ldots, A_7$ can be obtained by direct calculation. So assume that $n \geq 8$. Since $F_n + F_{n-1} = F_{n+1} \notin B(\alpha^2) + B(\alpha^2)$, $F_{n-1} \in A_n$. Then we write $A_n = C \cup B \cup \{F_{n-1}\}$, where $C = A_n \cap [0, F_{n-2} - 2)$ and $B = A_n \cap [F_{n-2} - 2, F_{n-1})$. Obviously, the sets C, B, and $\{F_{n-1}\}$ are disjoint. So it remains to show that

$$C = A_{n-1} \backslash \{F_{n-2}, F_{n-2} - 2\} \quad \text{and} \tag{3.9}$$

$$B = \{F_{n-1} - 2 - a \mid a \in A_{n-1} \text{ and } 0 \leq a \leq F_{n-3}\}. \tag{3.10}$$

To prove (3.9), let $a \in C$. Then $0 \leq a < F_{n-2} - 2$ and $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Applying Theorem 3.16, we obtain $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. So $a \in A_{n-1} \backslash \{F_{n-2}, F_{n-2} - 2\}$. Conversely, suppose that $a \in A_{n-1} \backslash \{F_{n-2}, F_{n-2} - 2\}$. Then $a \in [0, F_{n-2} - 2) \cup \{F_{n-2} - 1\}$ and $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. If $a = F_{n-2} - 1$, then we obtain by Theorem 3.10 that $F_{n-1} + a = F_n - 1 \in B(\alpha^2) + B(\alpha^2)$, which is not the case. So $0 \leq a < F_{n-2} - 2$. In addition, by Theorem 3.16, $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Hence $a \in A_n \cap [0, F_{n-2} - 2) = C$. This proves (3.9).

Next, let $b \in B$. Then $F_{n-2} - 2 \leq b < F_{n-1}$ and $F_n + b \notin B(\alpha^2) + B(\alpha^2)$. If $b = F_{n-1} - 1$, then we obtain by Theorem 3.10 that $F_n + b = F_{n+1} - 1 \in B(\alpha^2) + B(\alpha^2)$, which is not the case. So $b \leq F_{n-1} - 2$. Let $a = F_{n-1} - 2 - b$. Then $b = F_{n-1} - 2 - a$ and $0 \leq a \leq F_{n-3}$. So it remains to show that $a \in A_{n-1}$. Since $F_n + b \notin B(\alpha^2) + B(\alpha^2)$, we obtain by Theorem 3.17 that $F_n + a = F_{n+1} - 2 - (F_{n-1} - 2 - a) = F_{n+1} - 2 - b \notin B(\alpha^2) + B(\alpha^2)$. Since $F_n + a \notin B(\alpha^2) + B(\alpha^2)$, we obtain by Theorem 3.16 that $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. So $a \in A_{n-1}$, as required.

Finally, suppose $b = F_{n-1} - 2 - a$ where $a \in A_{n-1}$ and $0 \leq a \leq F_{n-3}$. Then $F_{n-2} - 2 \leq b < F_{n-1}$. Since $a \in A_{n-1}$, $F_{n-1} + a \notin B(\alpha^2) + B(\alpha^2)$. Then by Theorem 3.16, $F_n + a \notin B(\alpha^2) + B(\alpha^2)$. Applying Theorem 3.17, we obtain $F_n + b = F_{n+1} - 2 - a \notin B(\alpha^2) + B(\alpha^2)$. So $b \in A_n \cap [F_{n-2} - 2, F_{n-1}) = B$, as desired. This completes the proof. □

Some possible questions for future research are as follows.

Questions Q1 Let (f_n) be a kth order linear recurrence sequence defined by

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k} \quad \text{for } n \geq 2$$

with the initial values $f_{-(k-2)}, f_{-(k-3)}, \ldots, f_0, f_1 \in \mathbb{Z}$. Let α be the root of the characteristic polynomial $x^k - x^{k-1} - x^{k-2} - \cdots - 1$ with maximal absolute value. Can we described the structure of the sumsets associated with $B(\alpha), B(\alpha^2), \ldots, B(\alpha^k)$? Is the structure best described in terms of the k-step Fibonacci sequence $(F_n^{(k)})$ defined by the same recurrence as (f_n) but with the initial values

$$F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_{-1}^{(k)} = F_0^{(k)} = 0 \text{ and } F_1^{(k)} = 1?$$

Q2 Let $\alpha = (1 + \sqrt{5})/2$. Since $\alpha^2 - \alpha - 1 = 0$, the set $\{\alpha^2, \alpha, 1\}$ is not linearly independent over $\mathbb{Q}$. Suppose $\{\alpha^k, \alpha^{k-1}, \ldots, \alpha, 1\}$ is linearly independent over $\mathbb{Q}$, for example, α is an algebraic number of degree larger than k, $\alpha = e$, or $\alpha = \pi$, can we describe the structure of the sumsets associated with $B(\alpha^k), B(\alpha^{k-1}), \ldots, B(\alpha)$?

Q3 Let $a, b \in \mathbb{Z}$, $(a, b) = 1$, $b \neq 0$, and let (u_n) be the Lucas sequence of the first kind defined by $u_n = au_{n-1} + bu_{n-2}$ for $n \geq 2$ with $u_0 = 0$ and $u_1 = 1$. Let α be the root of the characteristic polynomial $x^2 - ax - b$. Is the structure of the sumsets associated with $B(\alpha)$ and $B(\alpha^2)$ connected to (u_n)?

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Distribution of Wythoff Sequences Modulo One

Sutasinee Kawsumarng, Tammatada Khemaratchatakumthorn, Passawan Noppakaew, Prapanpong Pongsriiam

Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom, 73000, Thailand

email: Kawsumarng@gmail.com, tammatada@gmail.com,
noppakaew_p@silpakorn.edu, prapanpong@gmail.com,
pongsriiam_p@silpakorn.edu

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Abstract

Let α be the golden ratio and $\beta\alpha = -1$. In the study of sumsets associated with Wythoff sequences, it is important to prove the inequality $0 < \{b\alpha\} + \beta^n < 1$ for integers b and n in a certain range. In this article, we continue the investigation by replacing $\{b\alpha\} + \beta^n$ by $\sqrt{5}\beta^{n-1} - \{b\alpha\}$.

1 Introduction

Wythoff sequences arise very often in combinatorics and combinatorial game theory. As a result, many of their combinatorial properties have been extensively studied (see, for example, the works of Fraenkel [1, 2], Kimberling [5], Pitman [8], and Wythoff [10]). However, as far as we know, there are only a few number theoretic results concerning the sumsets associated with Wythoff sequences. In order to describe the structure of such sumsets, it is

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Prapanpong Pongsriiam is the corresponding author.
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important to prove the inequality [4, Theorem 3.3]:

$$0 < \{b\alpha\} + \beta^n < 1 \text{ for all integers } n \geq 5 \text{ and } 1 \leq b \leq F_{n+1} \text{ with } b \neq F_n. \tag{1.1}$$

Here and throughout this article, $\alpha = (1+\sqrt{5})/2$ is the golden ratio, $\beta\alpha = -1$, x is a real number, a, b, m, n are integers, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\{x\} = x - \lfloor x \rfloor$, F_n and L_n are the nth Fibonacci number and the nth Lucas number which are defined by $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with the initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$. Moreover, if P is a mathematical statement, then the Iverson notation $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

In this article, we replace $\{b\alpha\} + \beta^n$ in (1.1) by $\sqrt{5}\beta^{n-1} - \{b\alpha\}$. Our interest is that we have an application in mind. Indeed, it is useful in the study of sumsets associated with Wythoff sequences and Lucas numbers. For a short discussion on the sumsets associated with some Beatty sequences generated by a real number $x > 1$ with $x^2 - ax - b = 0$ for some $a, b \in \mathbb{Z}$ see the last section of [4].

2 Preliminaries and Lemmas

We often use the following facts:
Let $-1 < \beta < 0$ and $(|\beta^n|)_{n\geq 1}$ is strictly decreasing. If $a_1 > a_2 > \cdots > a_r$ are even positive integers, then $0 < \beta^{a_1} < \beta^{a_2} < \cdots < \beta^{a_r}$. If $b_1 > b_2 > \cdots > b_r$ are odd positive integers, then $0 > \beta^{b_1} > \beta^{b_2} > \cdots > \beta^{b_r}$.
In addition, let α and β are roots of the equation $x^2 - x - 1 = 0$. So, for instance, $\alpha\beta = -1$, $\beta^2 = \beta + 1$, $\sqrt{5}\beta + \beta = -2$, $\sqrt{5}\beta^2 + 1 = -3\beta$, and $\beta^n + \sqrt{5}\beta^{n-1} + \beta^{n-2} = 0$ for all $n \geq 2$.
Moreover, it is useful to have the following numerical approximations:
$-0.619 < \beta < -0.618$, $-0.237 < \beta^3 < -0.236$, $0.854 < \sqrt{5}\beta^2 < 0.855$, $-0.528 < \sqrt{5}\beta^3 < -0.527$, $0.326 < \sqrt{5}\beta^4 < 0.327$.
The following results are also applied throughout this article sometimes without reference.

Lemma 2.1. *For $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, the following statements hold:*

(i) $\lfloor n + x \rfloor = n + \lfloor x \rfloor$.

(ii) $\{n+x\} = \{x\}$.

(iii) $0 \le \{x\} < 1$.

(iv) $\lfloor x+y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\}+\{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\}+\{y\} \ge 1. \end{cases}$

Proof. These are well-known and can be proved easily. For more details, see [3, Chapter 3]. We also refer the reader to [7] and [9, Proof of Lemma 2.6] for a nice application of these properties. □

Lemma 2.2. *The following statements hold for all $n \in \mathbb{N} \cup \{0\}$:*

(i) (Binet's formula) $L_n = \alpha^n + \beta^n$.

(ii) $\beta L_{n+1} + L_n = -\sqrt{5}\beta^{n+1}$.

(iii) $L_n \alpha = L_{n+1} + \sqrt{5}\beta^n$.

Proof. The formula (i) is well-known. Multiplying (ii) by α, we obtain (iii). The formula (ii) follows a straightforward calculation:
$\beta L_{n+1} + L_n$ is equal to

$$\beta\alpha^{n+1} + \beta^{n+2} + \alpha^n + \beta^n = \beta^{n+2} + \beta^n = \beta^n(-\sqrt{5}\beta) = -\sqrt{5}\beta^{n+1}.$$

□

Lemma 2.3. (Zeckendorf's theorem [11]) *For each $n \in \mathbb{N}$, $n = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell}$ where F_{a_1} is the largest Fibonacci number not exceeding n, $a_{i-1} - a_i \ge 2$ for every $i = 2, 3, \ldots, \ell$, and $a_\ell \ge 2$.*

Proof. This is well-known and can be proved by using the greedy algorithm. See also [6] for a more general result. □

Lemma 2.4. [4, Lemma 2.4] *If $x_1, x_2, \ldots, x_n \in \mathbb{R}$, then*

$$\{x_1 + x_2 + \cdots + x_n\} = \{\{x_1\} + \{x_2\} + \cdots + \{x_n\}\}.$$

Lemma 2.5. *Let $n \ge 2$. Then the following statements hold:*

(i) $\lfloor F_n \alpha \rfloor = F_{n+1} - [n \equiv 0 \pmod 2]$.

(ii) $\lfloor F_n \alpha^2 \rfloor = F_{n+2} - [n \equiv 0 \pmod 2]$.

(iii) $\{F_n \alpha\} = -\beta^n + [n \equiv 0 \pmod 2]$.

(iv) $\{F_n\alpha^2\} = \{F_n\alpha\}$.

(v) $\lfloor L_n\alpha \rfloor = L_{n+1} - [n \equiv 1 \pmod 2]$.

(vi) $\{L_n\alpha\} = \sqrt{5}\beta^n + [n \equiv 1 \pmod 2]$.

(vii) $\lfloor L_n\alpha^2 \rfloor = L_{n+2} - [n \equiv 1 \pmod 2]$.

(viii) $\{L_n\alpha^2\} = \{L_n\alpha\}$.

Proof. The proofs of (i) to (iv) can be found in [4, Lemma 2.5]. By Lemma 2.2(iii), we obtain $\lfloor L_n\alpha \rfloor = L_{n+1} + \lfloor \sqrt{5}\beta^n \rfloor$. If n is even, then $0 < \sqrt{5}\beta^n \leq \sqrt{5}\beta^2 < 1$, and so $\lfloor \sqrt{5}\beta^n \rfloor = 0$. If n is odd, then $-1 < \sqrt{5}\beta^3 \leq \sqrt{5}\beta^n < 0$ and thus $\lfloor \sqrt{5}\beta^n \rfloor = -1$. This implies (v). Then (vi) is a consequence of (v) and Lemma 2.2(iii). By writing $\alpha^2 = \alpha + 1$, we obtain (vii) from (v), and (viii) from Lemma 2.1(ii). This completes the proof. □

3 Main results

The proof of the following theorem is similar to that of [4, Theroem 3.3]. In fact, applying Theorem 3.3 of [4] leads to our main theorem but with a smaller range of b, which is not enough in our application. Therefore, we still need to adjust the proof from [4] to obtain the following theorem:

Theorem 3.1. *Let $n \geq 5$ and $1 \leq b \leq F_{n+1}$. Then the following statements hold:*

(i) *If $b = L_{n-1}$, then $\sqrt{5}\beta^{n-1} - \{b\alpha\} = -[n \equiv 0 \pmod 2]$.*

(ii) *If $b \in \{F_{n-2}, F_n\}$, then $0 < \sqrt{5}\beta^{n-1} - \{b\alpha\} + 2[n \equiv 0 \pmod 2] < 1$.*

(iii) *If $b \notin \{F_{n-2}, F_n, L_{n-1}\}$, then $-1 < \sqrt{5}\beta^{n-1} - \{b\alpha\} < 0$.*

Proof. The statement (i) follows immediately from Lemma 2.5(vi). For (ii), let $b \in \{F_{n-2}, F_n\}$ and $A = \sqrt{5}\beta^{n-1} - \{b\alpha\} + 2[n \equiv 0 \pmod 2]$. Since $\beta^n + \sqrt{5}\beta^{n-1} + \beta^{n-2} = 0$, we obtain by Lemma 2.5(iii) that if $b = F_n$, then

$$A = \sqrt{5}\beta^{n-1} + \beta^n + [n \equiv 0 \pmod 2] = -\beta^{n-2} + [n \equiv 0 \pmod 2],$$

if $b = F_{n-2}$, then

$$A = \sqrt{5}\beta^{n-1} + \beta^{n-2} + [n \equiv 0 \pmod 2] = -\beta^n + [n \equiv 0 \pmod 2].$$

By calculating A according to the parity of n, it is not difficult to see that $0 < A < 1$. This proves (ii). For (iii), if $b = F_{n+1}$, then we apply Lemma 2.5(iii) to obtain

$$\begin{aligned}\sqrt{5}\beta^{n-1} - \{b\alpha\} &= \sqrt{5}\beta^{n-1} + \beta^{n+1} - [n \equiv 1 \pmod 2]\\ &= \beta^{n-3} - [n \equiv 1 \pmod 2],\end{aligned}$$

which is in the interval $(-1, 0)$. Next, let $B = \sqrt{5}\beta^{n-1} - \{b\alpha\} + 1$, where b is not equal to any of F_{n-2}, F_n, L_{n-1}, F_{n+1}. We need to show that $0 < B < 1$.

Case 1 $b = F_k$ where $2 \le k \le n-3$ or $k = n-1$.

Case 1.1 $b = F_2$. Then by Lemma 2.5, $B = \sqrt{5}\beta^{n-1} + \beta^2$. Therefore, $B \le \sqrt{5}\beta^4 + \beta^2 = \beta^2(-3\beta) = -3\beta^3 < 1$. If n is odd, then it is obvious that $B > 0$. If n is even, then $n \ge 6$, and $B \ge \sqrt{5}\beta^5 + \beta^2 = \beta^2(\sqrt{5}\beta^3 + 1) > 0$.

Case 1.2 $b = F_{n-1}$. Then by Lemma 2.5,

$$B = \sqrt{5}\beta^{n-1} + \beta^{n-1} - [n \equiv 1 \pmod 2] + 1.$$

If n is even, then $B < 1$ and $B \ge 1 + \beta^5 + \sqrt{5}\beta^5 = 1 - 2\beta^4 > 0$. If n is odd, then $B > 0$ and $B \le \sqrt{5}\beta^4 + \beta^4 = -2\beta^3 < 1$.

Case 1.3 $b = F_k$ and $3 \le k \le n-3$. This case occurs only when $n \ge 6$. By Lemma 2.5,

$$B = \sqrt{5}\beta^{n-1} + \beta^k - [k \equiv 0 \pmod 2] + 1.$$

We first consider the case that k is even. Then $B = \sqrt{5}\beta^{n-1} + \beta^k$. If n is odd, then $B > 0$ and $B \le \sqrt{5}\beta^4 + \beta^4 = -2\beta^3 < 1$. If n is even, then $B < \beta^k \le \beta^4 < 1$, $k \le n-4$, and $B \ge \sqrt{5}\beta^{n-1} + \beta^{n-4} = \beta^{n-4}(\sqrt{5}\beta^3 + 1) > 0$. Next, suppose k is odd. Then $B = \sqrt{5}\beta^{n-1} + \beta^k + 1$. If n is even, then $B < 1$ and $B \ge \sqrt{5}\beta^5 + \beta^3 + 1 = 1 - 3\beta^4 > 0$. If n is odd, then $k \le n-4$, $B > \sqrt{5}\beta^{n-1} > 0$ and $B \le \sqrt{5}\beta^{n-1} + \beta^{n-4} + 1 < 1$.

Case 2 $F_k < b < F_{k+1}$ for some $k \in \{4, 5, \ldots, n\}$. We apply Lemma 2.5 without further reference. By Zeckendorf's theorem, we can write

$$b = F_{a_1} + F_{a_2} + \cdots + F_{a_\ell},$$

where $\ell \ge 2$, $k = a_1 > a_2 > \cdots > a_\ell \ge 2$ and $a_{i-1} - a_i \ge 2$ for every $i = 2, 3, \ldots, \ell$. Then by Lemma 2.4, we obtain

$$\{b\alpha\} = \{\{F_{a_1}\alpha\} + \{F_{a_2}\alpha\} + \cdots + \{F_{a_\ell}\alpha\}\},$$

which is equal to

$$\{(1 - \beta^{b_1} + 1 - \beta^{b_2} + \cdots + 1 - \beta^{b_r}) + (-\beta^{c_1} - \beta^{c_2} - \cdots - \beta^{c_s})\},$$

where $\{b_1, b_2, \dots, b_r\} \cup \{c_1, c_2, \dots, c_s\} = \{a_1, a_2, \dots, a_\ell\}$, $b_1 > b_2 > \cdots > b_r$ are even numbers, and $c_1 > c_2 > \cdots > c_s$ are odd numbers. Notice that one of the sets $\{b_1, b_2, \dots, b_r\}$ and $\{c_1, c_2, \dots, c_s\}$ may be empty. In this case, such a set disappears from the subsequence calculation. Also, for convenience, we let $A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} + \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s}$. Then. by Lemma 2.1, $\{b\alpha\} = \{-A\}$. To show that $0 < B < 1$, it is enough to prove

$$\sqrt{5}\beta^{n-1} < \{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}.$$

Case 2.1 $\{b_1, b_2, \dots, b_r\}$ is empty. Then

$$A = \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^3 + \beta^5 + \cdots = \frac{\beta^3}{1-\beta^2} = -\beta^2.$$

Therefore $0 < -A < \beta^2 < 1$ and so $\{b\alpha\} = \{-A\} = -A$. If n is even, then obviously $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = -A < \beta^2 < 1 + \sqrt{5}\beta^3 < 1 + \sqrt{5}\beta^{n-1}$. So assume that n is odd. Then $\{b\alpha\} = -A < \beta^2 < 1 + \sqrt{5}\beta^{n-1}$, and $\{b\alpha\} = -A = |\beta|^{c_1} + |\beta|^{c_2} + \cdots + |\beta|^{c_s}$. If $\ell \geq 3$, then $s \geq 3$, and so

$$\{b\alpha\} \geq |\beta|^{c_1} + |\beta|^{c_2} + |\beta|^{c_3} \geq |\beta|^n + |\beta|^{n-2} + |\beta|^{n-4} > |\beta|^n + |\beta|^{n-2} = \sqrt{5}\beta^{n-1}.$$

Suppose $\ell = 2$. Then $s = 2$ and $\{b\alpha\} = |\beta|^{c_1} + |\beta|^{c_2}$. If $c_1 \neq n$, then

$$|\beta|^{c_1} + |\beta|^{c_2} \geq |\beta|^{n-2} + |\beta|^{n-4} > |\beta|^{n-2} + |\beta|^n = -(\beta^n + \beta^{n-2}) = \sqrt{5}\beta^{n-1}.$$

Since $L_{n-1} = F_n + F_{n-2}$ and $b \neq L_{n-1}$, we see that $\{c_1, c_2\} \neq \{n, n-2\}$. Therefore, if $c_1 = n$, then $c_2 \neq n-2$, and so $c_2 \leq n-4$

$$|\beta|^{c_1} + |\beta|^{c_2} \geq |\beta|^n + |\beta|^{n-4} > |\beta|^n + |\beta|^{n-2} = \sqrt{5}\beta^{n-1}.$$

In any case, $\{b\alpha\} > \sqrt{5}\beta^{n-1}$, as required.
Case 2.2 $\{c_1, c_2, \dots, c_s\}$ is empty. Then

$$A = \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^2 + \beta^4 + \cdots = \frac{\beta^2}{1-\beta^2} = -\beta.$$

Therefore $-1 < \beta < -A < 0$ and $\{b\alpha\} = \{-A\} = 1 - A$. Suppose n is even. Then $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = 1 - A = 1 - \beta^{b_1} - \beta^{b_2} - \cdots - \beta^{b_r}$. As in the proof of Case 2.1, if $\ell \geq 3$, then $r \geq 3$ and

$$\{b\alpha\} \leq 1 - \beta^{b_1} - \beta^{b_2} - \beta^{b_3} \leq 1 - \beta^n - \beta^{n-2} - \beta^{n-4} < 1 - \beta^n - \beta^{n-2} = 1 + \sqrt{5}\beta^{n-1}.$$

If $\ell = 2$ and $b_1 \neq n$, then

$$\{b\alpha\} = 1 - \beta^{b_1} - \beta^{b_2} \leq 1 - \beta^{n-2} - \beta^{n-4} < 1 - \beta^{n-2} - \beta^{n} = 1 + \sqrt{5}\beta^{n-1}.$$

If $\ell = 2$ and $b_1 = n$, then $b_2 \leq n-4$ and

$$\{b\alpha\} = 1 - \beta^{b_1} - \beta^{b_2} \leq 1 - \beta^{n} - \beta^{n-4} < 1 - \beta^{n} - \beta^{n-2} = 1 + \sqrt{5}\beta^{n-1}.$$

If n is odd, then $\{b\alpha\} < 1 < 1 + \sqrt{5}\beta^{n-1}$ and $\{b\alpha\} = 1 - A > 1 + \beta = \beta^2 \geq \beta^{n-3} \geq \sqrt{5}\beta^{n-1}$.

Case 2.3 $\{b_1, b_2, \ldots, b_r\}$ and $\{c_1, c_2, \ldots, c_s\}$ are not empty. Then there is some cancellation in the sum defining A. As in Cases 2.1 and 2.2, we have $A < \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < -\beta$ and $A > \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > -\beta^2$.

Case 2.3.1 A is positive. Then $-1 < \beta < -A < 0$, and so $\{b\alpha\} = \{-A\} = 1 - A$. If n is odd, then $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$ and

$$\{b\alpha\} = 1 - A > 1 + \beta > \sqrt{5}\beta^4 \geq \sqrt{5}\beta^{n-1}.$$

Assume that n is even. Then $\{b\alpha\} > 0 > \sqrt{5}\beta^{n-1}$. It remains to show that $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$. Let $u = \min\{b_1, b_2, \ldots, b_r\}$ and $v = \min\{c_1, c_2, \ldots, c_s\}$. Since $a_{i-1} - a_i \geq 2$ for all $i = 2, 3, \ldots, \ell$ and $a_1 = k \leq n$, we obtain that $u \leq n$ and $|v - u| \geq 3$. Then

$$\beta^u \leq \beta^{b_1} + \beta^{b_2} + \cdots + \beta^{b_r} < \beta^u + \beta^{u+2} + \beta^{u+4} + \cdots = \frac{\beta^u}{1-\beta^2} = -\beta^{u-1}, \tag{3.2}$$

$$\beta^v \geq \beta^{c_1} + \beta^{c_2} + \cdots + \beta^{c_s} > \beta^v + \beta^{v+2} + \beta^{v+4} + \cdots = \frac{\beta^v}{1-\beta^2} = -\beta^{v-1}. \tag{3.3}$$

By (3.2) and (3.3), we obtain $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v-u| \geq 3$, we see that either $v - u \geq 3$ or $v - u \leq -3$. Suppose for a contradiction that $v - u \leq -3$. Since $v \leq u - 3$ and both v and $u - 3$ are odd, we have $\beta^v \leq \beta^{u-3}$. Thus $A < \beta^v - \beta^{u-1} \leq \beta^{u-3} - \beta^{u-1} = \beta^{u-3}(1-\beta^2) = -\beta^{u-2} < 0$, which contradicts the assumption that A is positive. Hence $v - u \geq 3$. Since $v - 1 \geq u + 2$ and both $v - 1$ and $u + 2$ are even, $\beta^{v-1} \leq \beta^{u+2}$. So $A > \beta^u - \beta^{u+2} = \beta^u(1-\beta^2) = -\beta^{u+1}$. We have $u \leq v - 3 \leq n - 3$. Thus $u + 1 \leq n - 2$. Since $n - 2$ is even and $u + 1$ is odd, we have $u + 1 \leq n - 3$. Then $\{b\alpha\} = 1 - A < 1 + \beta^{u+1} \leq 1 + \beta^{n-3}$. Since $\sqrt{5}\beta^2 < 1$ and $n - 3$ is odd, $\sqrt{5}\beta^{n-1} > \beta^{n-3}$. Therefore $\{b\alpha\} < 1 + \beta^{n-3} < 1 + \sqrt{5}\beta^{n-1}$, as required.

Case 2.3.2 A is negative. Then $0 < -A < \beta^2 < 1$. Then $\{b\alpha\} = \{-A\} = -A$. We first show that $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1}$. If n is odd, then $\{b\alpha\} < 1 < 1+$

$\sqrt{5}\beta^{n-1}$. If n is even, then $\{b\alpha\} = -A < \beta^2 < 1+\sqrt{5}\beta^3 < 1+\sqrt{5}\beta^{n-1}$. Next, we show that $\{b\alpha\} > \sqrt{5}\beta^{n-1}$. If n is even, then $\sqrt{5}\beta^{n-1} < 0 < \{b\alpha\}$. So assume that n is odd. Let $u = \min\{b_1, b_2, \ldots, b_r\}$ and $v = \min\{c_1, c_2, \ldots, c_s\}$. As in Case 2.3.1, we have $u \le n$, $|v-u| \ge 3$, the equalities (3.2) and (3.3) hold, and $\beta^u - \beta^{v-1} < A < \beta^v - \beta^{u-1}$. Since $|v-u| \ge 3$, we see that either $v-u \ge 3$ or $v-u \le -3$. If $v-u \ge 3$, then $\beta^{v-1} \le \beta^{u+2}$ and $A > \beta^u - \beta^{v-1} \ge \beta^u - \beta^{u+2} = -\beta^{u+1} > 0$, which contradicts the assumption that $A < 0$. Thus $v-u \le -3$, and so $A < \beta^{u-3} - \beta^{u-1} = -\beta^{u-2}$. Since $u \le n$, u is even and n is odd, we have $u-2 \le n-3$. Then $-A > \beta^{u-2} \ge \beta^{n-3} > \sqrt{5}\beta^{n-1}$. Therefore $\{b\alpha\} = -A > \sqrt{5}\beta^{n-1}$ as desired. This completes the proof. □

Theorem 3.1 leads to a short proof of [4, Theorem 3.3].

Corollary 3.2. [4, Theorem 3.3] *Let $n \ge 5$, $1 \le b \le F_{n+1}$, and $b \ne F_n$. Then $0 < \{b\alpha\} + \beta^n < 1$.*

Proof. If $b = F_{n-2}$ or $b = L_{n-1}$, we can apply Lemma 2.5 to obtain the desired result. So suppose that $b \ne F_{n-2}$ and $b \ne L_{n-1}$. We first consider the case n is odd. Then it is obvious that $\{b\alpha\} + \beta^n < 1$. For the other inequality, we apply Theorem 3.1 to obtain $\{b\alpha\} > \sqrt{5}\beta^{n-1} > -\beta^n$. Similarly, if n is even, then it is immediate that $\{b\alpha\} + \beta^n > 0$ and by using Theorem 3.1, we obtain $\{b\alpha\} < 1 + \sqrt{5}\beta^{n-1} < 1 - \beta^n$. This completes the proof. □

It is possible to extend the range of b in Theorem 3.1 and Corollary 3.2 but the results are not nice and we do not need them in our application. Therefore, we only give some special cases in an example and leave the general case to the interested readers.

Example 3.3. *Let $n \ge 5$, $k \ge n+2$, $b = F_k$, and $B = \{b\alpha\} + \beta^n$. Then the following statements hold.*

(i) *If k and n are odd, then $-1 < B < 0$.*

(ii) *If $k \not\equiv n \pmod 2$, then $0 < B < 1$.*

(iii) *If k and n are even, then $1 < B < 2$.*

Proof. By Lemma 2.5, $B = \beta^n - \beta^k + [k \equiv 0 \pmod 2]$.
Case 1 k is odd. Then $B = \beta^n - \beta^k$. If n is odd, then $-1 < \beta^n < \beta^{n+2} \le \beta^k < 0$, and so $-1 < B < 0$. If n is even, then $k \ge n+3$, and $0 < B \le \beta^n - \beta^{n+3} = -2\beta^{n+1} < 1$.
Case 2 k is even. Then $B = \beta^n - \beta^k + 1$. If n is odd, then $B < 1$, $k \ge n+3$, and $B \ge 1 + \beta^n - \beta^{n+3} = 1 - 2\beta^{n+1} > 0$. If n is even, then $0 < \beta^k \le \beta^{n+2} < \beta^n < 1$, and so $1 < B < 2$. This completes the proof. □

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VITA

NAME	Sutasinee Kawsumarng
DATE OF BIRTH	30 December 1995
PLACE OF BIRTH	Nakhonpathom
INSTITUTIONS ATTENDED	Bachelor of Science in Mathematics, Silpakorn University in 2017.
HOME ADDRESS	48 Swine 4, Bangchang Sub-district, Sampran District, Nakhonpathom, 73110.
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