SOME PROPERTIES OF 3-I-VERTEX-CRITICAL GRAPHS

By
Miss Sriphan Ruangthampisan

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Graduate School, Silpakorn University
Academic Year 2015

SOME PROPERTIES OF 3-I-VERTEX-CRITICAL GRAPHS

By

Miss Sriphan Ruangthampisan

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science Program in Mathematics

Department of Mathematics

Graduate School, Silpakorn University

Academic Year 2015

สมบัติของกราฟ 3-i-vertex-critical

โดย
นางสาวศรีพรรณ เรืองธรรมพิศาล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์
ภาควิชาคณิตศาสตร์
บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
ปีการศึกษา 2558

The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Some properties of 3-i-vertex-critical graphs” submitted by Miss Sriphan Ruangthampisan as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

....
(Associate Professor Panjai Tantatsanawong,Ph.D.)
Dean of Graduate School
........../..................../..........

The Thesis Advisor

Associate Professor Nawarat Ananchuen, Ph.D.

The Thesis Examination Committee

.... Chairman
(Chalermpong Worawannotai, Ph.D.)
............/......................../..............

.... Member
(Associate Professor Watcharaphong Ananchuen, Ph.D.)
............/......................../..............

.... Member
(Associate Professor Nawarat Ananchuen, Ph.D.)
............/......................../..............

54305205 : MAJOR : MATHEMATICS
KEY WORD : INDEPENDENT DOMINATION / CRITICAL / PERFECT MATCHING
SRIPHAN RUANGTHAMPISAN : SOME PROPERTIES OF 3-I-VERTEX-CRITICAL GRAPHS. THESIS ADVISOR : ASSOC. PROF.NAWARAT ANANCHUEN, Ph.D.. 31 pp.

Let $i(G)$ denote the independent domination number of a graph G. A graph G is said to be n-i-vertex-critical if $i(G) = n$ and $i(G-v) < i(G)$ for all $v \in V(G)$.

A matching M in G is called a perfect matching if all vertices of G are incident with some edge of M.

In this thesis, we provide characterizations of connected 3-i-vertex-critical graphs with a cutset S for $1 \leq |S| \leq 2$. In addition, we present properties of 3-i-vertex-critical graphs G with a minimum cutset S where $\Delta(G[S]) \leq 1$ in terms of $\omega(G-S)$. Moreover, we show that $\omega(G - S) \leq |S| - 1$ with some condition on $|S|$. Finally, we provide a sufficient condition for 3-i-vertex-critical graphs of even order to have a perfect matching.

Program of Mathematics Graduate School, Silpakorn University
Student'ssignature .... Academic Year 2015
Thesis Advisor's signature ....

54305205 : สาขาวิชาคณิตศาสตร์
คำสำคัญ : จำนวนควบคุมที่เป็นอิสระ / วิกฤติ / การจับคู่สมบูรณ์

ศรีพรรณ เรืองธรรมพิศาล : สมบัติของกราฟ 3-i-vertex-critical. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : รศ.ดร.นวรัตน์ อนันต์ชื่น. 31 หน้า.

กำหนดให้ i(G) แทนขนาดของเซตควบคุมอิสระที่เล็กที่สุดของกราฟ G เราจะเรียกกราฟ G ว่า n-i-vertex-critical เมื่อ $i(G) = n$ และ $i(G - v) < i(G)$ สำหรับแต่ละจุด $v \in V(G)$

การจับคู่ M ใน G เรียกว่า การจับคู่สมบูรณ์ ถ้าทุกจุดในกราฟ G ตกกระทบกับบางเส้นใน M

ในวิทยานิพนธ์นี้เราได้ให้ลักษณะเฉพาะเจาะจงของกราฟเชื่อมโยง 3-i-vertex-critical ที่มี S เป็นเซตตัด โดยที่ $1 \leq |S| \leq 2$ และให้สมบัติของกราฟ G ที่เป็นกราฟ 3-i-vertex-critical ที่มี S เป็นเซตตัดขนาดเล็กสุด โดยที่ $\Delta G([S]) \leq 1$ ในเทอมของ $\omega(G - S)$ ยิ่งไปกว่านั้นเราแสดงว่า $\omega(G - S) \leq |S| - 1$ สำหรับเงื่อนไขบน S บางประการและสุดท้ายเราให้เงื่อนไขที่เพียงพอสำหรับกราฟ 3-i-vertex-critical อันดับคู่ที่มีการจับคู่สมบูรณ์

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
ลายมือชื่อนักศึกษา.. ปีการศึกษา 2558
ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

ACKNOWLEDGEMENTS

I am deeply indebted to many people providing me with advice and support to complete this thesis. I am grateful to Assoc. Prof. Dr. Nawarat Ananchuen, my advisor who devoted her time to guide me and gave me valuable suggestions and encouragement to overcome challenges.

I would like to acknowledge great help and academic advice to make my work most accurate and reliable from Dr. Chalermpong Worawannotai and Assoc. Prof. Dr. Watcharaphong Ananchuen, Chairman and Member of the thesis committee.

I would also like to thank Miss Supalak Sumalroj for constructive suggestions. Her willingness to give her time so generously has been much appreciated.

I would like to extend my gratitude to all instructors including all staff at Department of Mathematics. My grateful thanks also express to my friends for interesting ideas and encouragement. Most importantly, I am also grateful for sincere support given by my beloved family throughout my study.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

In this chapter, we introduce some definitions and notations used in this thesis. Most of them follows Clark and Holton[3] and Chartrand and Oellermann[4].

A graph $G = (V(G), E(G))$ consists of two finite sets : $V(G)$, the **vertex set** of the graph which is a nonempty set of elements called **vertices** and $E(G)$, the **edge set** of the graph which is a possibly empty set of elements called **edges** such that each edge e in $E(G)$ is assigned an unordered pair of vertices (u, v), called the **end vertices** of e. An edge that joins itself is a **loop**. If two (or more) edges of G have the same end vertices then these edges are called **parallel**. A graph is called **simple** if it has no loops and no parallel edges. Let G denote a simple graph with a vertex set $V(G)$ and an edge set $E(G)$. If $e = uv$ is an edge of a graph G, then we say that u and v are **adjacent**, and we say that e and u (and e and v) are **incident** with each other. The **complement** $\overline{G}$ of G is defined to be the simple graph with the same vertex set as G and where two vertices u and v are adjacent precisely when they are not adjacent in G. The **open neighborhood** $N_G(v)$ of a vertex v consists of the set of vertices adjacent to v and the **closed neighborhood** of v denoted by $N_G[v]$ is $N_G(v) \cup \{v\}$. Further, $N_H(v)$ denotes either $N_G(v) \cap V(H)$ if H is a subgraph of G or $N_G(v) \cap H$ if H is a subset of $V(G)$. For simplicity, $\overline{N}_G(v)$ denotes **non-open neighborhood** of v in G such that if $x \in \overline{N}_G(v)$ for $x \in V(G) - \{v\}$, then $xv \notin E(G)$. Let v be a vertex of the graph G, the **degree** $d(v)$ of v is the number of edges of G incident with v. In other words, it is the number of times which v is an end vertex of an edge. For a graph G, we let $\Delta(G)$=max$\{d(v) : v \text{ is a vertex of } G\}$. Thus, $\Delta(G)$ is the **maximum degree** of G.

Let H be a graph with vertex set $V(H)$ and edge set $E(H)$ and, similarly, let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The **induced subgraph** of G with vertex set $S \subseteq V(G)$, denoted by $G[S]$, is the graph with the vertex set S and the edge set of $G[S]$ consists of all the edges of G with both end vertices in S. Two simple graphs G_1 and G_2 are **isomorphic** if there is a one-to-one function ϕ from $V(G_1)$ onto $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. If G_1 and G_2 are isomorphic, then we write $G_1 \cong G_2$. The function ϕ is called an **isomorphism**. If G is a graph of order n and every two distinct vertices are adjacent, we say that G is a **complete graph** and is denoted by K_n. If the vertex set $V(G)$ can be partitioned into two nonempty subsets X and Y ($X \cup Y = V(G)$ and $X \cap Y = \emptyset$) in such a way that each edge of G has one end in X and one end in Y then G is called **bipartite**. The partition $V(G) = X \cup Y$ is called a **bipartition** of G. A **complete bipartite graph** is a simple bipartite graph G, with bipartition $V(G) = X \cup Y$, in which every vertex in X is joined to every vertex in Y. If X has m vertices and Y has n vertices, such a graph is denoted by $K_{m,n}$.

Let G_1 and G_2 be two graphs with no vertex in common. We define the **join** of G_1 and G_2, denoted by $G_1 \vee G_2$, to be the graph with vertex set and edge set given as follows : $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup J$ where $J = \{x_1x_2 | x_1 \in V(G_1)$ and $x_2 \in V(G_2)\}$. Thus J consists of edges which join every vertex of G_1 to every vertex of G_2.

A **walk** in a graph G is an alternating sequence of vertices and edges, begining and ending with vertices. A walk in which no vertex is repeated, is called a **path**. Let u and v be vertices in a graph G. We say that u is connected to v if G contains a $u-v$ path. We say that G is a **connected graph** if u is connected to v for every pair u, v of vertices of G. For a pair u, v of vertices of G, the **distance** $d_G(u, v)$ between u and v of G is the length of a shortest $u-v$ path in G if such a path exists. A **diameter** of G is given by $max\{d_G(u, v) : u, v \in V(G)\}$.

Given any vertex u of a graph G, let $C(u)$ denote the set of all vertices in G that are connected to u. Then the subgraph of G induced by $C(u)$ is called a **connected component** containing u. We denote the number of components and the number of odd components of G by $\omega(G)$ and $\omega_0(G)$, respectively. For $S \subseteq V(G)$, S is called a **cutset** if $\omega(G-S) > \omega(G)$. If $S = \{v\}$ is a cutset, then v is also called a **cut-vertex**. The **toughness** of a graph G, denoted by $tough(G)$, is defined as $min\{\frac{|S|}{\omega(G-S)} | S \subseteq V(G)\}$.

A set of edges in a graph G is called a **matching** if no two edges have a vertex in common. A matching M in G is called a **perfect matching** if all vertices of G are incident with some edge of M.

A set $S \subseteq V(G)$ is **independent** if no two vertices in S are adjacent. For $S \subseteq V(G)$, S is a **dominating set** for G if every vertex of G either belongs to S or is adjacent to a vertex of S. An **independent dominating set** in a graph is a set that is both dominating and independent. The **independent domination number** of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. We will write $S \succ_i G$ if S is an independent dominating set for G. For any $v \in V(G)$, an independent dominating set for $G - \{v\}$ is denoted by I_v. For simplicity, if $u \in V(G)$ and $T \subseteq N_G[u]$, we shall write $u \succ_i T$. A graph G is called **n-i-vertex-critical graph** if $i(G) = n$ but $i(G - v) < n$ for all $v \in V(G)$. We also say that G is i-vertex-critical if G is n-i-vertex-critical for some n. The concept of n-i-vertex-critical graphs was introduced by Ao [1] in 1994. Her results concerning this concept are reviewed in Chapter 2.

The next two results are used in establishing our results in this thesis. They are :

Theorem 1.1. *[2](Pigoenhole's Principle)*

If $n+1$ objects are put into n boxes, then at least one box contains two or more of the objects.

Theorem 1.2. *[5](Tutte's Theorem)*

A graph G has a perfect matching if and only if $\omega_o(G-S) \leq |S|$, for all $S \subseteq V(G)$.

The next three chapters in this thesis provide some previous results and our new results. More precisely, the previous results are contained in Chapter 2. Chapter 3 and Chapter 4 contain new results where Chapter 3 provide characterizations of connected 3-i-vertex-critical graphs with a minimum cutset S for $1 \leq |S| \leq 2$. Properties of 3-i-vertex-critical graphs with a minimum cutset in terms of the number of components and result concerning having a perfect matching are in Chapter 4.

Chapter 2

Literature Review

In this chapter, we provide some previous studies concerning our study. As we mention in Chapter 1 that the concept of n-i-vertex-critical graphs was introduced by Ao [1]. In her study, she established some properties of n-i-vertex-critical graphs. She characterized n-i-vertex-critical graphs for $n = 1$ and $n = 2$. It is shown that 1-i-vertex-critical graphs are K_1 and 2-i-vertex-critical graphs are complete graphs K_{2n} without a perfect matching for some positive integer n. The following five results established by Ao[1] are fundamental results used in establishing on results.

Lemma 2.1. *[1] A graph G is n-i-vertex-critical if and only if for every* $v \in V(G)$, $i(G - v) = n - 1$.

Lemma 2.2. *[1] If G is i-vertex-critical, then every vertex* $v \in V(G)$ *belongs to some minimum independent dominating set.*

Lemma 2.3. *[1] If there exist distinct vertices* $u, v \in V(G)$ *such that* $N_G[v] \subseteq N_G[u]$*, then G is not i-vertex-critical.*

Lemma 2.3 can be restated as : If G is i-vertex-critical, then for each $v \in V(G)$, there is no $v \neq v' \in V(G)$ such that $N_G[v] \subseteq N_G[v']$.

Corollary 2.4. *[1] If G has a vertex v with* $d_G(v) \geq 1$ *such that* $G[N_G[v]]$ *is complete, then G is not n-i-vertex-critical.*

Corollary 2.5. *[1] If G is connected and n-i-vertex-critical, then the minimum degree of G is greater than or equal to 2.*

In 2013, Wang[6] provided the upper bound on the diameter of n-i-vertex-critical graphs.

Theorem 2.6. *[6] If G is a connected n-i-vertex-critical graph, then* $diam(G) \leq 2(n - 1)$.

In this thesis, we provide characterizations of connected 3-i-vertex-critical graphs with a cutset S for $1 \leq |S| \leq 2$ and we study toughness result in 3-i-vertex-critical graphs. These results are in Chapter 3 and Chapter 4.

Our latest search shows that there are no other results concerning n-i-vertex-critical graphs besides results stated in Lemma 2.1 - Theorem 2.6. Hence, our results are new.

Chapter 3

Characterizations of connected 3-i-vertex-critical graphs with a minimum cutset of small order

In this chapter, we provide characterizations of connected 3-i-vertex-critical graphs with a cutset S for $1 \leq |S| \leq 2$. We begin our chapter with classes of connected 3-i-vertex-critical graphs.

3.1 Classes of connected 3-i-vertex-critical graphs

In this section, we present five classes of connected 3-i-vertex-critical graphs.

Class $\mathscr{H}$

For positive integers m and n and for $G \in \mathscr{H}$, let G be a graph of order $2m + 2n + 3$ where $V(G) = X \cup Y \cup \{u, v, w\}$ and $|X| = 2m$ and $|Y| = 2n$. Form complete graphs on X and Y with a perfect matching deleted. Join v to every vertex of X ; join w to every vertex of Y and finally join u to every vertex of $X \cup Y$. Observe that for $G \in \mathscr{H}$, G is a connected 3-i-vertex-critical graph containing u as a cut-vertex. Further, $\omega(G-u) = 2$. Figure 1 illustrates our construction.

Class $\mathscr{R}$

For positive integers m and n, let G be a graph of order $2m+2n+5$ where $V(G) = X \cup Y \cup \{u, v, w, x, y\}$ and $|X| = 2m$ and $|Y| = 2n$. Form complete graphs on X and Y with a perfect matching deleted. Join w to every vertex of $X \cup Y$; join u to every vertex of $X \cup \{x, y\}$ and finally join v to every vertex of $Y \cup \{x, y\}$. Let $G' \in \mathscr{R}$ where $V(G') = V(G)$ and $E(G') = E(G) \cup E'$ where $E' \subseteq \{e = x^*y^* | x^* \in X \text{and}\, y^* \in Y\}$. Note that if $E' = \emptyset$, then $G' = G$. It is not difficult to show that $G' \in \mathscr{R}$ is a connected 3-i-vertex-critical graph where $\{u, v\}$

is a minimum cutset. Observe that $\omega(G' - \{u, v\}) = 3$ and $G' - \{u, v\}$ contains exactly two singleton components. Figure 2 illustrates our construction.

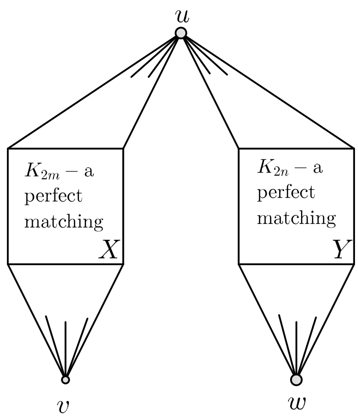


Figure 1: The structure of a graph in $\mathscr{H}$

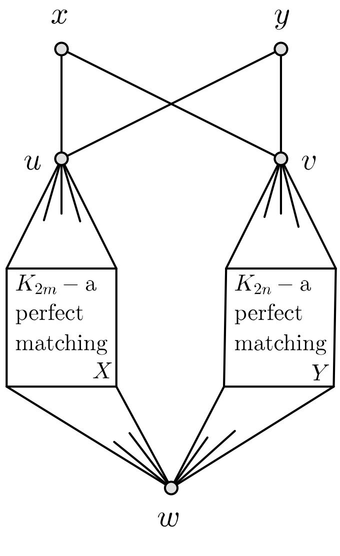
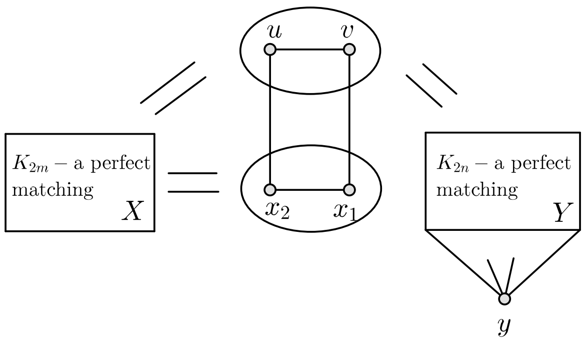


Figure 2: The structure of a graph in $\mathscr{R}$ where $E' = \emptyset$

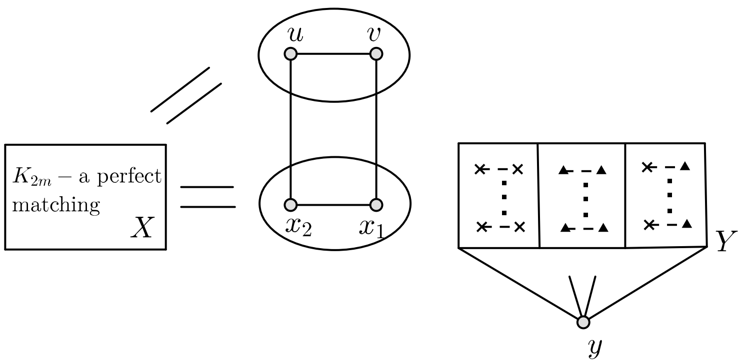
Note that in our diagrams, in the rest of this section, double line denotes the join, the vertices that are adjacent to both u and v are represented by the triangle vertices while the vertices that are adjacent to u but not v and v but not u are represented by the cross and diamond vertices, respectively.

Class $\mathscr{M}$

For a positive integer n and non-negative integer m and for $G \in \mathscr{M}$, let G be a graph of order $2m + 2n + 5$ where $V(G) = X \cup Y \cup \{u, v, y, x_1, x_2\}$ and $|X| = 2m$ and $|Y| = 2n$. Let $\emptyset \neq Y_1 \subseteq Y$. Now join u to every vertex of $\{v, x_2\} \cup X \cup Y$; join v to every vertex of $\{x_1\} \cup X \cup Y_1$; join y to every vertex of Y and then add the edge x_1x_2. Further, if $X \neq \emptyset$, join each vertex of X to every vertex of $\{x_1, x_2\}$ and then form a complete graph on X with a perfect matching deleted. Now form a complete graph on Y with a perfect matching $F = F_1 \cup F_2 \cup F_3$ deleted where $F_1 = \{y_1y_2 \in E(G) | y_1, y_2 \in N_Y(u) - N_Y(v)\}$, $F_2 = \{y_1y_2 \in E(G) | y_1, y_2 \in N_Y(u) \cap N_Y(v)\}$, $F_3 = \{y_1y_2 \in E(G) | y_1 \in N_Y(u) - N_Y(v)$ and $y_2 \in N_Y(u) \cap N_Y(v)\}$ and F_i might be empty for $1 \leq i \leq 3$. Note that if $Y_1 = Y$, then $F = F_2$ and if $Y_1 \neq Y$, then $F_1 \cup F_3 \neq \emptyset$. Observe that $G \in \mathscr{M}$ is a connected 3-i-vertex-critical with a cutset $\{u, v\}$ where $\omega(G - \{u, v\}) = 2$. Figure 3 illustrates our construction.



$G \in \mathscr{M}$ where $Y_1 = Y$



$G \in \mathscr{M}$ where $Y_1 \neq Y$

Figure 3: The structure of a graph in $\mathscr{M}$

Class $\mathscr{N}$

For non-negative integers m and $n_i \geq 1$ where $1 \leq i \leq 6$, let H be a graph of order $2m + \sum_{i=1}^{6} 2n_i + 5$ where $V(H) = X \cup \bigcup_{i=1}^{6} Y_i \cup \{u, v, x, y, z\}$, $|X| = 2m$ and $|Y_i| = 2n_i$ for $1 \leq i \leq 6$. Let $H[Y_i] = K_{2n_i} -$ a perfect matching. Further, for $4 \leq i \leq 6$, let $Y_i = Y_i^{'} \cup Y_i^{''}$ where $H[Y_i^{'}] = H[Y_i^{''}] = K_{n_i}$. Join u to every vertex of $X \cup Y_1 \cup Y_3 \cup Y_4 \cup Y_5' \cup Y_6' \cup \{y\}$; join v to every vertex of $X \cup Y_2 \cup Y_3 \cup Y_4'' \cup Y_5 \cup Y_6'' \cup \{x\}$; join x, y to every vertex of X ; join z to every vertex of $Y = \bigcup_{i=1}^{6} Y_i$ and then add the edge xy.

Further, if $X \neq \emptyset$, then form a complete graph on X with a perfect matching deleted. The class $\mathscr{N}$ consists of G_i, G_i' for $1 \leq i \leq 32$, where G_i and G_i' are constructed from induced subgraph of H as follows

$G_1 = H[\{u, v, x, y, z\} \cup Y_6]$
$G_2 = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_6]$
$G_3 = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2]$
$G_4 = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_6]$
$G_5 = H[\{u, v, x, y, z\} \cup Y_3 \cup Y_6]$
$G_6 = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_3 \cup Y_6]$
$G_7 = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_3]$
$G_8 = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_6]$
$G_9 = H[\{u, v, x, y, z\} \cup Y_2 \cup Y_4]$
$G_{10} = H[\{u, v, x, y, z\} \cup Y_4 \cup Y_6]$
$G_{11} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_4]$
$G_{12} = H[\{u, v, x, y, z\} \cup Y_2 \cup Y_4 \cup Y_6]$
$G_{13} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_4 \cup Y_6]$
$G_{14} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_4 \cup Y_6]$
$G_{15} = H[\{u, v, x, y, z\} \cup Y_2 \cup Y_3 \cup Y_4]$
$G_{16} = H[\{u, v, x, y, z\} \cup Y_3 \cup Y_4 \cup Y_6]$
$G_{17} = H[\{u, v, x, y, z\} \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_6]$
$G_{18} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4]$
$G_{19} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_3 \cup Y_4 \cup Y_6]$
$G_{20} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_6]$
$G_{21} = H[\{u, v, x, y, z\} \cup Y_4 \cup Y_5]$
$G_{22} = H[\{u, v, x, y, z\} \cup Y_4 \cup Y_5 \cup Y_6]$
$G_{23} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_4 \cup Y_5]$
$G_{24} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_4 \cup Y_5]$
$G_{25} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_4 \cup Y_5 \cup Y_6]$
$G_{26} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_4 \cup Y_5 \cup Y_6]$
$G_{27} = H[\{u, v, x, y, z\} \cup Y_3 \cup Y_4 \cup Y_5]$
$G_{28} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_3 \cup Y_4 \cup Y_5]$
$G_{29} = H[\{u, v, x, y, z\} \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6]$
$G_{30} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5]$
$G_{31} = H[\{u, v, x, y, z\} \cup Y_1 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6]$

$G_{32} = H[\{u,v,x,y,z\} \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6]$
$G'_1 = H[\{u,v,x,y,z\} \cup X \cup Y_6]$
$G'_2 = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_6]$
$G'_3 = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2]$
$G'_4 = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_6]$
$G'_5 = H[\{u,v,x,y,z\} \cup X \cup Y_3 \cup Y_6]$
$G'_6 = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_3 \cup Y_6]$
$G'_7 = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_3]$
$G'_8 = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_6]$
$G'_9 = H[\{u,v,x,y,z\} \cup X \cup Y_2 \cup Y_4]$
$G'_{10} = H[\{u,v,x,y,z\} \cup X \cup Y_4 \cup Y_6]$
$G'_{11} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_4]$
$G'_{12} = H[\{u,v,x,y,z\} \cup X \cup Y_2 \cup Y_4 \cup Y_6]$
$G'_{13} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_4 \cup Y_6]$
$G'_{14} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_4 \cup Y_6]$
$G'_{15} = H[\{u,v,x,y,z\} \cup X \cup Y_2 \cup Y_3 \cup Y_4]$
$G'_{16} = H[\{u,v,x,y,z\} \cup X \cup Y_3 \cup Y_4 \cup Y_6]$
$G'_{17} = H[\{u,v,x,y,z\} \cup X \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_6]$
$G'_{18} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4]$
$G'_{19} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_3 \cup Y_4 \cup Y_6]$
$G'_{20} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_6]$
$G'_{21} = H[\{u,v,x,y,z\} \cup X \cup Y_4 \cup Y_5]$
$G'_{22} = H[\{u,v,x,y,z\} \cup X \cup Y_4 \cup Y_5 \cup Y_6]$
$G'_{23} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_4 \cup Y_5]$
$G'_{24} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_4 \cup Y_5]$
$G'_{25} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_4 \cup Y_5 \cup Y_6]$
$G'_{26} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_4 \cup Y_5 \cup Y_6]$
$G'_{27} = H[\{u,v,x,y,z\} \cup X \cup Y_3 \cup Y_4 \cup Y_5]$
$G'_{28} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_3 \cup Y_4 \cup Y_5]$
$G'_{29} = H[\{u,v,x,y,z\} \cup X \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6]$
$G'_{30} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5]$
$G'_{31} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6]$
$G'_{32} = H[\{u,v,x,y,z\} \cup X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6]$

Observe that G_i and G'_i belonging to $\mathscr{N}$ are connected 3-i-vertex-critical having $\{u,v\}$ as a minimum cutset and $\omega(G - \{u,v\}) = 2$. Figure 4 shows the graphs G_4, G'_4, G_{22} and G'_{22}.

Class $\mathscr{O}$

For positive integers m,n and k, let G be a graph of order $2m + 2n + 2k + 3$ where $V(G) = X \cup Y \cup Z \cup \{u,v,z\}$ where $|X| = 2m$, $|Y| = 2n$ and $|Z| = 2k$. Form complete graphs on X,Y and Z with a perfect matching deleted. Join u to every vertex of $X \cup Y$; join v to every vertex of $X \cup Z$; and finally join z to every vertex of $Y \cup Z$. Let $G' \in \mathscr{O}$ where $V(G') = V(G)$ and $E(G') = E(G) \cup E'$

where $E' \subseteq \{e = yz | y \in Y \text{and} \, z \in Z\}$. Note that if $E' = \emptyset$, then $G' = G$. It is easy to see that $G' \in \mathscr{O}$ is a connected 3-i-vertex-critical graph having $\{u, v\}$ as a minimum cutset and $\omega(G' - \{u, v\}) = 2$. Figure 5 illustrates our construction

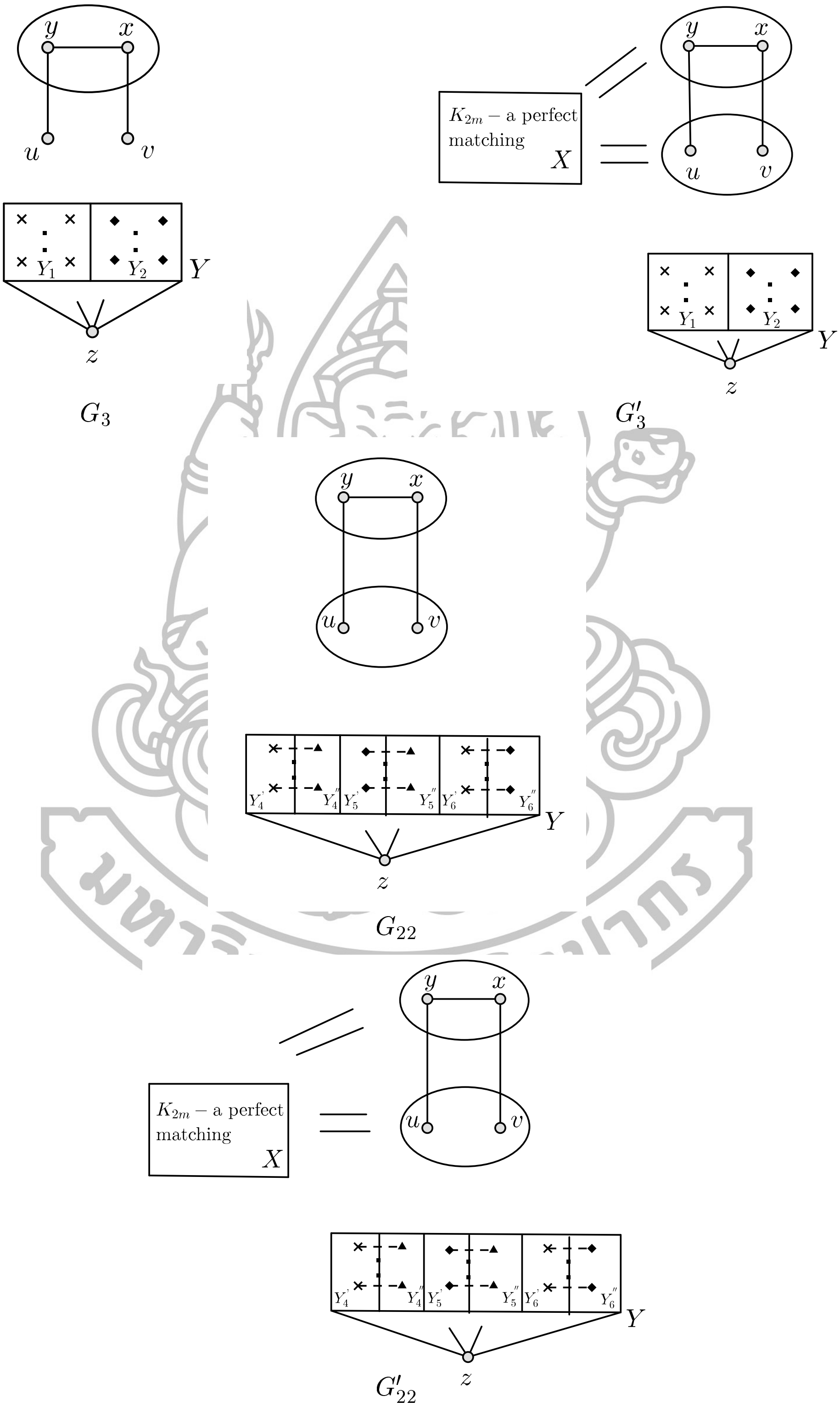


Figure 4: Some graphs in the class $\mathscr{N}$

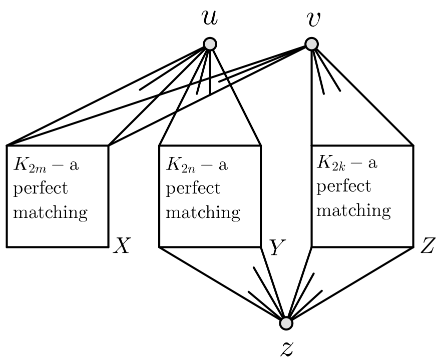


Figure 5: The structure of a graph in $\mathscr{O}$ where $E' = \emptyset$

3.2 Characterizations of connected 3-i-vertex-critical graphs with a minimum cutset S where $1 \leq |S| \leq 2$.

In this section we provide characterizations of connected 3-i-vertex-critical graphs with a cutset S for $|S| = 1$ and $|S| = 2$. We begin with an easy useful result.

Lemma 3.2.1. *For a positive integer $n \geq 2$, let G be an n-i-vertex-critical graph. Then, for each $v \in V(G)$, $I_v \cap N_G[v] = \emptyset$.*

Proof. Suppose to the contrary that $I_v \cap N_G[v] \neq \emptyset$. Then there is a vertex $x \in I_v \cap N_G[v]$. Clearly, $x \neq v$. Since $x \in I_v$ and $xv \in E(G)$, it follows that $I_v \succ_i G$, a contradiction since $|I_v| = n - 1$ but $i(G) = n$. This proves our lemma. □

Theorem 3.2.2. *Suppose G is a connected 3-i-vertex-critical graph with a cut-vertex u. Then $\omega(G - u) = 2$ and G belongs to $\mathscr{H}$ defined in Section 3.1*

Proof. **Claim 1** : $\omega(G - u) = 2$.

Suppose to the contrary that $\omega(G-u) \geq 3$. Consider $G-u$. Since $|I_u| = 2$ and $\omega(G-u) \geq 3$, it follows that I_u does not dominate some component of $G-u$, a contradiction. Hence, $\omega(G - u) = 2$ as required. This proves our claim.

Now $G - u$ contains exactly two components, say C_1 and C_2. It is easy to see that $I_u \cap V(C_1) \neq \emptyset$ and $I_u \cap V(C_2) \neq \emptyset$. Put $I_u \cap V(C_1) = \{v\}$ and $I_u \cap V(C_2) = \{w\}$. By Lemma 3.2.1, $vu \notin E(G)$ and $wu \notin E(G)$. Further, $v \succ_i V(C_1)$ and $w \succ_i V(C_2)$. Since G is connected, $N_{C_1}(u) \neq \emptyset$ and $N_{C_2}(u) \neq \emptyset$.

Claim 2 : For each $x \in N_{C_1}(u)$, there exists a unique vertex $y \in N_{C_1}(u)$ such that $y \in I_x$ and $y \succ_i V(C_1) - \{x\}$ and $yx \notin E(G)$.

Let $x \in N_{C_1}(u)$. Consider $G - x$. By Lemma 3.2.1, $\{v, u\} \cap I_x = \emptyset$ since $vx, ux \in E(G)$. Then $I_x \cap V(C_1) \neq \emptyset$ and $I_x \cap V(C_2) \neq \emptyset$. Put $\{y\} = I_x \cap V(C_1)$. Then $y \succ_i V(C_1) - \{x\}$. Observe that $N_{C_1}[v] = V(C_1)$ and $V(C_1) - \{x\} \subseteq N_{C_1}[y]$. If $y \notin N_{C_1}(u)$, then $N_{C_1}[y] = V(C_1) - \{x\}$ and thus $N_{C_1}[y] \subseteq N_{C_1}[v]$, contradicting Lemma 2.3. Thus $yu \in E(G)$. If there is $y' \in N_{C_1}(u) - \{y\}$ such that $I_x \cap V(C_1) = \{y'\}$, then $N_G[y'] = (V(C_1) - \{x\}) \cup \{u\} = N_G[y]$, again contradicting Lemma 2.3. This proves our claim.

Claim 3 : $N_{C_1}(u) = V(C_1) - \{v\}$.

If there is a vertex $x \in V(C_1) - \{v\}$ where $x \notin N_{C_1}(u)$, then $N_G[x] \subseteq N_G[v]$. But this contradicts Lemma 2.3. Hence, Claim 3 is proved.

The following claim follows immediately from Claims 2 and 3.

Claim 4 : $G[V(C_1) - \{v\}] \cong K_{2m}$- a perfect matching for some positive integer m.

By similar arguments as in the proof of Claims 2,3 and 4, we have following claims.

Claim 5 : For each $x \in N_{C_2}(u)$, there exists a unique vertex $y \in N_{C_2}(u)$ such that $y \in I_x$ and $y \succ_i V(C_2) - \{x\}$ and $yx \notin E(G)$.

Claim 6 : $N_{C_2}(u) = V(C_2) - \{w\}$.

Claim 7 : $G[V(C_2) - \{w\}] \cong K_{2n}$ - a perfect matching for some positive integer n.

By Claims 3,4,6 and 7, G belongs to $\mathscr{H}$ as required. This completes the proof of our theorem. $\square$

We now turn our attention to a minimum cutset S where $|S| = 2$.

Theorem 3.2.3. *Suppose G is a connected 3-i-vertex-critical graph and S is a minimum cutset in G with $|S| = 2$. Then*
(1) $\omega(G - S) \leq 3$.
(2) If $\omega(G - S) = 3$, then there are exactly 2 singleton components in $G - S$ and G belongs to $\mathscr{R}$, defined in Section 3.1.

Proof. Let $S = \{u, v\}$ and let $C_1, ..., C_t$ be components of $G - S$.

Claim 1 : Suppose $t = \omega(G - S) \geq 3$. If $a \in V(C_i)$ for some $1 \leq i \leq t$ where $|V(C_i)| \geq 2$, then $a \notin N_G(u) \cap N_G(v)$.

Suppose to the contrary that $a \in N_G(u) \cap N_G(v)$. Then $I_a \cap \{u, v\} = \emptyset$ by Lemma 3.2.1. Thus, $I_a \subseteq \bigcup_{i=1}^{t} V(C_i)$. Since $|I_a| = 2$, $t \geq 3$ and $|V(C_i) - \{a\}| \geq 1$, it follows that there is a component of $G - S$ which is not dominated by I_a, a contradiction. This proves our claim.

We are ready to prove (1).

(1) Suppose to the contrary that $t = \omega(G-S) \geq 4$. If $uv \in E(G)$, then $v \notin I_u$ and thus $|I_u| \geq 3$, a contradiction. Thus $uv \notin E(G)$. Note that $u \in I_v$ and $v \in I_u$ since $\omega(G-S) \geq 4$. Consider $G-u$. Then v must dominate at least $t-1$ components. We may suppose without loss of generality that $v \succ_i \bigcup_{i=2}^{t} V(C_i)$. We next consider $G-v$. Since $v \succ_i \bigcup_{i=2}^{t} V(C_i)$ $I_v \cap \bigcup_{i=2}^{t} V(C_i) = \emptyset$ by Lemma 3.2.1. It follows that $I_v \cap V(C_1) \neq \emptyset$. Then u must dominate $\bigcup_{i=2}^{t} V(C_i)$. By Claim 1, $|V(C_i)| = 1$ for $2 \leq i \leq t$. Let $\{z\} = V(C_2)$. Then $I_z \cap \{u,v\} = \emptyset$ and thus $I_z \subseteq \bigcup_{i=1}^{t} V(C_i) - \{z\}$. But this is not possible since $|I_z| = 2$ and $t = \omega(G-S) \geq 4$. This proves (1).

(2) We now suppose that $t = \omega(G-S) = 3$. If $|V(C_1)| = |V(C_2)| = |V(C_3)| = 1$, then $i(G) \leq 2$ since S is a minimum cutset, a contradiction. Without loss of generality, we may assume that $|V(C_1)| \geq 2$. Choose $z \in N_{C_1}(u)$. By Claim 1, $zv \notin E(G)$. Consider $G-z$. Clearly, $v \in I_z$ since $\omega(G-S) = 3$ and $|V(C_1) - \{z\}| \geq 1$. Put $\{z'\} = I_z - \{v\}$. We first suppose that $z' \notin V(C_1)$. Without loss of generality, assume that $z' \in V(C_2)$. Then $v \succ_i (V(C_1) - \{z\}) \cup V(C_3)$. By Claim 1, $N_{C_1}(u) = \{z\}$. Now consider $G-v$. By Lemma 3.2.1, $I_v \cap ((V(C_1) - \{z\}) \cup V(C_3)) = \emptyset$. Since $\omega(G-S) = 3$, $u \in I_v$ otherwise no vertex of I_v dominates $V(C_3)$. Then the only vertex of $I_v - \{u\}$ dominates $V(C_1) - \{z\}$ since $N_{C_1}(u) = \{z\}$. Consequently, $I_v - \{u\} = \{z\}$. But this contradicts the fact that I_v is independent since $z \in N_{C_1}(u)$. Hence, $z' \in V(C_1)$. Thus $v \succ_i V(C_2) \cup V(C_3)$. Since S is a minimum cutset, $N_{C_i}(u) \neq \emptyset$ for $1 \leq i \leq 3$. It then follows, by Claim 1, that $|V(C_2)| = |V(C_3)| = 1$.

Put $\{x\} = V(C_2)$ and $\{y\} = V(C_3)$. Since S is a minumum cutset, $N_G(x) = N_G(y) = \{u,v\}$. Since $\omega(G-S) = 3$, $u \in I_v$ and thus $uv \notin E(G)$. Consider $G-x$. Then $I_x \cap \{u,v\} = \emptyset$. Since $N_G(y) = \{u,v\}$, $y \in I_x$. Put $\{w\} = I_x - \{y\}$. Clearly, $w \in V(C_1)$ since $\{y\} = V(C_3)$. Further, $w \succ_i V(C_1)$. If $uw \in E(G)$, then, by Claim 1, $vw \notin E(G)$ and thus $\{w,v\}$ is an independent dominating set for G, a contradiction. Thus $uw \notin E(G)$. Similarly, $vw \notin E(G)$. If there is $w' \in V(C_1)$ such that $w'u \notin E(G)$ and $w'v \notin E(G)$, then $N_G[w'] \subseteq N_G[w]$, contradicting Lemma 2.3. Hence, $\{w\} = V(C_1) - (N_{C_1}(u) \cup N_{C_1}(v))$ or $N_{C_1}(u) \cup N_{C_1}(v) = V(C_1) - \{w\}$. It follows by Claim 1 that $N_{C_1}(u) \cap N_{C_1}(v) = \emptyset$.

Claim 2 : For each $a \in N_{C_1}(u)$, there exists a unique vertex $b \in N_{C_1}(u)$ such that $b \in I_a$ and $b \succ_i N_{C_1}(u) - \{a\}$.

Let $a \in N_{C_1}(u)$. Then $au \in E(G)$ and $aw \in E(G)$. By Claim 1, $av \notin E(G)$. Consider $G-a$. It is easy to see that $v \in I_a$. Put $\{b\} = I_a - \{v\}$. Clearly, $b \in V(C_1) - \{a\}$. Then $bv \notin E(G)$ and thus $b \in N_{C_1}(u)$. Note that $b \succ_i N_{C_1}(u) - \{a\}$ since $N_{C_1}(u) \cap N_{C_1}(v) = \emptyset$. If there is $b' \in N_{C_1}(u) - \{b\}$ such that $I_a = \{v, b'\}$, then $N_G[b'] \subseteq N_G[b]$, contradicting Lemma 2.3. This proves our claim.

By similar arguments, we have the following claim.

Claim 3 : For each $a \in N_{C_1}(v)$, there exists a unique vertex $b \in N_{C_1}(v)$ such that $b \in I_a$ and $b \succ_i N_{C_1}(v) - \{a\}$.

It follows by Claims 2 and 3 that $G[N_{C_1}(u)] \cong K_{2m}$ - a perfect matching and $G[N_{C_1}(v)] \cong K_{2n}$ - a perfect matching for some positive integers m and n. Therefore, G belongs to $\mathscr{R}$. This completes the proof of our theorem. $\square$

Lemma 3.2.4. *Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| = 2$ and $\omega(G - S) = 2$. Suppose $S = \{u, v\}$ and $G[S] = K_2$. Let C_1 and C_2 be the components of $G - S$. Then*

(1) There exist $x_1, x_2 \in V(C_1)$ and $y \in V(C_2)$ such that $N_G[x_1] = V(C_1) \cup \{v\}$, $N_G[x_2] = V(C_1) \cup \{u\}$ and $N_G[y] = V(C_2)$. Further, $V(C_1) - \{x_1, x_2\} = N_{C_1}(u) \cap N_{C_1}(v)$ and $V(C_2) - \{y\} = N_{C_2}(u) \cup N_{C_2}(v)$. Consequently, $N_{C_1}(u) = V(C_1) - \{x_1\}$ and $N_{C_1}(v) = V(C_1) - \{x_2\}$.

(2) If $|V(C_1) - \{x_1, x_2\}| \geq 1$, then $G[V(C_1) - \{x_1, x_2\}] \cong K_{2n}$- a perfect matching for some positive integer n.

(3) $u \succ_i V(C_2) - \{y\}$ or $v \succ_i V(C_2) - \{y\}$.

(4) $G[V(C_2) - \{y\}] \cong K_{2m}$- a perfect matching for some positive integer m.

Proof. (1) Consider $G - u$. Clearly, by Lemma 3.2.1, $v \notin I_u$ and then $I_u \cap V(C_1) \neq \emptyset$ and $I_u \cap V(C_2) \neq \emptyset$. Put $I_u \cap V(C_1) = \{x_1\}$ and $I_u \cap V(C_2) = \{y\}$. Then $x_1 \succ_i V(C_1)$, $y \succ_i V(C_2)$ and $\{x_1, y\} \subseteq \overline{N}_G(u)$. Since I_u must dominate v, without loss of generality, we may assume that $x_1 v \in E(G)$. Now consider $G - v$. Clearly, $I_v \cap \{u, x_1\} = \emptyset$ by Lemma 3.2.1. Further, $I_v \cap (V(C_1) - \{x_1\}) \neq \emptyset$ and $I_v \cap V(C_2) \neq \emptyset$. Put $\{x_2\} = I_v \cap (V(C_1) - \{x_1\})$ and $\{y_1\} = I_v \cap V(C_2)$. So $x_2 \succ_i V(C_1)$ and $y_1 \succ_i V(C_2)$. Clearly, $x_2 v, y_1 v \notin E(G)$. If $x_2 u \notin E(G)$, then $N_G[x_2] \subseteq N_G[x_1]$, contradicting Lemma 2.3. Thus $x_2 u \in E(G)$. Hence, $N_G[x_1] = V(C_1) \cup \{v\}$ and $N_G[x_2] = V(C_1) \cup \{u\}$.

We now show that $V(C_1) - \{x_1, x_2\} = N_{C_1}(u) \cap N_{C_1}(v)$. Clearly, $N_{C_1}(u) \cap N_{C_1}(v) \subseteq V(C_1) - \{x_1, x_2\}$. Let $z \in V(C_1) - \{x_1, x_2\}$. If $z \notin N_G(u) \cup N_G(v)$, $N_G[z] \subseteq N_G[x_1]$, contradicting Lemma 2.3. Hence, $z \in N_G(u) \cup N_G(v)$. Suppose $z \in N_G(u)$ but $z \notin N_G(v)$. Then $N_G[z] \subseteq N_G[x_2]$, again a contradiction. Hence, $z \in N_G(u) \cap N_G(v)$. By similar arguments, if $z \in N_G(v)$, then $z \in N_G(u)$ and thus $N_G(u) \cup N_G(v) = N_G(u) \cap N_G(v)$. Hence, $V(C_1) - \{x_1, x_2\} = N_{C_1}(u) \cap N_{C_1}(v)$.

Recall that $\{y\} = I_u \cap V(C_2)$. Clearly, $yv \notin E(G)$ otherwise $\{y, x_2\} \succ_i G$. We next show that $y_1 = y$. Suppose this is not the case. Then $y_1 u \in E(G)$ otherwise $N_G[y_1] \subseteq N_G[y]$. It then follows that $\{x_1, y_1\} \succ_i G$, a contradiction. Hence, $y_1 = y$ as required. Since $\{y\} = I_u \cap V(C_2)$ and $\{y_1\} = I_v \cap V(C_2)$, it follows that $y \in V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$. Thus $N_G[y] = V(C_2)$. By Lemma 2.3, it is easy to see that $V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v)) = \{y\}$. This proves (1).

We now let x_1, x_2 and y are vertices in (1).

(2) By (1), $x_1 \succ_i V(C_1)$ and $x_2 \succ_i V(C_1)$. Suppose $V(C_1) - \{x_1, x_2\} \neq \emptyset$. Let $z_1 \in V(C_1) - \{x_1, x_2\}$. Then $z_1 \in N_{C_1}(u) \cap N_{C_1}(v)$ by (1). Consider $G - z_1$. By Lemma 3.2.1, $\{x_1, x_2, u, v\} \cap I_{z_1} = \emptyset$. Thus $I_{z_1} \cap V(C_1) \neq \emptyset$ and $I_{z_1} \cap V(C_2) \neq \emptyset$. Let $\{z_1'\} = I_{z_1} \cap V(C_1)$. Then $z_1' \in V(C_1) - \{x_1, x_2, z_1\}$. Thus $z_1' \succ_i V(C_1) - \{z_1\}$ and $\{z_1'u, z_1'v\} \subseteq E(G)$. Consider $G - z_1'$. By Lemma 3.2.1, $I_{z_1'} \cap ((V(C_1) - \{z_1\}) \cup \{u, v\}) = \emptyset$ then $\{z_1\} = I_{z_1'} \cap V(C_1)$. If $V(C_1) - \{x_1, x_2, z_1, z_1'\} \neq \emptyset$, then, continuing in this fashion, $G[V(C_1) - \{x_1, x_2\}] \cong K_{2n}$ - a perfect mathching for some positive integer $n \geq 1$. This proves (2).

(3) Since S is a minimum cutset and $\{y\} = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$, it follows that $|N_{C_2}(u) \cup N_{C_2}(v)| \geq 2$ and thus $|V(C_2)| \geq 3$. Consider $G - y$. By Lemma 3.2.1, $I_y \cap V(C_2) = \emptyset$. Since $|V(C_2)| \geq 3$, $I_y \cap S \neq \emptyset$. However, $|I_y \cap S| = 1$ since $uv \in E(G)$. Therefore, $u \succ_i V(C_2) - \{y\}$ or $v \succ_i V(C_2) - \{y\}$. This proves (3).

(4) By (3) suppose, without loss of generality, that $u \succ_i V(C_2) - \{y\}$. Choose $w_1 \in V(C_2) - \{y\}$. Clearly, $w_1 y \in E(G)$ and $w_1 u \in E(G)$. Consider $G - w_1$. If $v \in I_{w_1}$, then the only vertex of $I_{w_1} - \{v\}$ must dominate $\{x_2, y\}$. But this is not possible since $x_2 \in V(C_1)$ and $y \in V(C_2)$. Hence, $v \notin I_{w_1}$. It follows that $I_{w_1} \cap V(C_1) \neq \emptyset$ and $I_{w_1} \cap V(C_2) \neq \emptyset$. Suppose $\{w_1'\} = I_{w_1} \cap V(C_2)$. Then $w_1' \succ_i V(C_2) - \{w_1\}$. It is easy to see that $\{w_1\} = I_{w_1'} \cap V(C_2)$. Then $w_1 \succ_i V(C_2) - \{w_1'\}$. If $V(C_2) - \{y, w_1, w_1'\} \neq \emptyset$, then, continuing in this fashion, $G[V(C_2) - \{y\}] \cong K_{2m}$ - a perfect matching for some positive integer $m \geq 1$. This proves (4) and completes the proof of our lemma. □

Theorem 3.2.5. *Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| = 2$ and $\omega(G - S) = 2$. Suppose $G[S] = K_2$ and C_1, C_2 are components of $G - S$. Then G belongs to $\mathscr{M}$ defined in Section 3.1.*

Proof. Let $S = \{u, v\}$ where $uv \in E(G)$. By Lemmas 3.2.4(1) and 3.2.4(2), there exist $x_1, x_2 \in V(C_1)$ and $y \in V(C_2)$ such that $N_G[x_1] = V(C_1) \cup \{v\}$, $N_G[x_2] = V(C_1) \cup \{u\}$ and $N_G[y] = V(C_2)$. Further, if $V(C_1) - \{x_1, x_2\} = N_{C_1}(u) \cap N_{C_1}(v) \neq \emptyset$, then $G[V(C_1) - \{x_1, x_2\}] \cong K_{2n}$ - a perfect matching for some positive integer n. Again, by Lemmas 3.2.4(1) and 3.2.4(4), $G[V(C_2) - \{y\}] = G[N_{C_2}(u) \cup N_{C_2}(v)] \cong K_{2m}$ - a perfect matching for some positive integer m. Let F be such a perfect matching in $\overline{G}[V(C_2) - \{y\}]$. We may now assume that $u \succ_i V(C_2) - \{y\}$ by Lemma 3.2.4(3). Since S is a minimum cutset, $\emptyset \neq N_{C_2}(v) \subseteq V(C_2) - \{y\}$. Put $F_1 = \{zz' \in F | z, z' \in N_{C_2}(u) - N_{C_2}(v)\}$, $F_2 = \{zz' \in F | z, z' \in N_{C_2}(u) \cap N_{C_2}(v)\}$ and $F_3 = \{zz' \in F | z \in N_{C_2}(u) - N_{C_2}(v), z' \in N_{C_2}(u) \cap N_{C_2}(v)\}$. Clearly, $F_1 \cup F_2 \cup F_3 = F$. If $N_{C_2}(v) = V(C_2) - \{y\}$, then $F = F_2$ and if $N_{C_2}(v) \neq V(C_2) - \{y\}$, then $F_1 \cup F_3 \neq \emptyset$. In either case, G belongs to $\mathscr{M}$. This completes the proof of our theorem. □

Lemma 3.2.6. *Let G be a connected 3-i-vertex-critical graph with a minimum* *cutset S where $|S| = 2$. Suppose $S = \{u, v\}$ is an independent set and C_1 and C_2 are components of $G - S$. If $v \notin I_u$, then*

(1) *There exist $x \neq y \in V(C_1)$ and $z \in V(C_2)$ such that $x \succ_i V(C_1)$, $y \succ_i V(C_1)$ and $z \succ_i V(C_2)$. Further, $N_{C_1}(u) = V(C_1) - \{x\}$, $N_{C_1}(v) = V(C_1) - \{y\}$, and $\{z\} = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$.*

(2) *$N_{C_2}(v) - N_{C_2}(u) \neq \emptyset$ and $N_{C_2}(u) - N_{C_2}(v) \neq \emptyset$.*

(3) *If $|V(C_1) - \{x, y\}| \geq 1$, then $V(C_1) - \{x, y\}$ is isomorphic to a K_{2m} - a perfect matching for some positive integer m.*

(4) *$V(C_2) - \{z\}$ is isomorphic to a K_{2n} - a perfect matching for some positive integer n.*

Proof. Since S is independent, $uv \notin E(G)$. Consider $G - u$. Since $v \notin I_u$, it follows that $|I_u \cap V(C_1)| = 1$ and $|I_u \cap V(C_2)| = 1$. Put $\{x\} = I_u \cap V(C_1)$ and $\{z\} = I_u \cap V(C_2)$.

(1) Since $\{x\} = I_u \cap V(C_1)$ and $\{z\} = I_u \cap V(C_2)$, it follows that $xu, zu \notin E(G)$ and $x \succ_i V(C_1)$, $z \succ_i V(C_2)$. Note that $|V(C_1)| \geq 2$ otherwise v becomes a cut-vertex. Since $I_u = \{x, z\}$ and $I_u \succ_i G - u = V(C_1) \cup V(C_2) \cup \{v\}$, we may assume that $xv \in E(G)$. Consider $G - x$. Since $x \succ_i V(C_1)$ and $xv \in E(G)$, it follows that $u \in I_x$ and $u \succ_i V(C_1) - \{x\}$. So $N_{C_1}(u) = V(C_1) - \{x\}$. We next show that $vz \notin E(G)$. Suppose to the contrary that $vz \in E(G)$. Consider $G - z$. Then, $u \in I_z$ by Lemma 3.2.1 since $z \succ_i V(C_2)$ and $vz \in E(G)$. Thus $u \succ_i V(C_2) - \{z\}$. It follows that $u \succ_i (V(C_1) - \{x\}) \cup (V(C_2) - \{z\})$ and $\{xv, zv\} \subseteq E(G)$. Hence, $\{u, v\} \succ_i G$, a contradiction. Therefore, $vz \notin E(G)$ and thus $z \in V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$. Observe that if there is a vertex $z^* \in V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v) \cup \{z\})$, then $N_G[z^*] \subseteq N_G[z]$ since $z \succ_i V(C_2)$, contradicting Lemma 2.3. Hence, $\{z\} = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$.

We now consider $G - v$. If $u \in I_v$, then the only vertex of $I_v - \{u\}$ must dominate x and z. But this is not possible since x and z belongs to different components of $G - \{u, v\}$. Thus $u \notin I_v$ and it follows that $I_v \cap V(C_1) \neq \emptyset$ and $I_v \cap V(C_2) \neq \emptyset$. Let $\{y\} = I_v \cap V(C_1)$ and $\{y^*\} = I_v \cap V(C_2)$. Clearly, $yv, y^*v \notin E(G)$ and $y \succ_i V(C_1)$ and $y^* \succ_i V(C_2)$. If $y^* \neq z$, then $N_{C_2}[z] \subseteq N_{C_2}[y^*]$. So $y^* = z$. Since $y^*u = zu \notin E(G)$, $yu \in E(G)$. Hence, $y \neq x$ since $ux \notin E(G)$. Consider $G - y$. Since $y \succ_i V(C_1)$ and $yu \in E(G)$, it follows that $v \in I_y$ and $v \succ_i V(C_1) - \{y\}$. Hence, $N_{C_1}(v) = V(C_1) - \{y\}$. This proves (1).

In what follows we now assume that x, y and z are vertices of $V(G) - \{u, v\}$ satisfying (1)

(2) It is easy to see that $v \in I_y$ and $I_y - \{v\} \subseteq V(C_2)$. Put $\{y^*\} = I_y - \{v\}$. Clearly, $y^*v \notin E(G)$. Since $uv \notin E(G)$, $uy^* \in E(G)$. This proves that $N_{C_2}(u) - N_{C_2}(v) \neq$

$\emptyset$. By similar arguments, $N_{C_2}(v) - N_{C_2}(u) \neq \emptyset$. This proves (2)

(3) Let $x_1 \in V(C_1) - \{x, y\}$. Then $x_1 \in N_{C_1}(u) \cap N_{C_1}(v)$ by (1). Consider $G - x_1$. Clearly, $\{u, v, x, y\} \cap I_{x_1} = \emptyset$. Thus $I_{x_1} \cap (V(C_1) - \{x, y\}) \neq \emptyset$ and $I_{x_1} \cap V(C_2) \neq \emptyset$. Put $\{x_1^*\} = I_{x_1} \cap V(C_1)$. Then $x_1^* \in V(C_1) - \{x_1, x, y\}$. So $x_1^* \succ_i V(C_1) - \{x_1\}$ and thus $N_G[x_1^*] = (V(C_1) \cup \{u, v\}) - \{x_1\}$. It is easy to see that $\{x_1\} = I_{x_1^*} \cap V(C_1)$. Then $x_1 \succ_i V(C_1) - \{x_1^*\}$. If $V(C_1) - \{x, y, x_1, x_1^*\} \neq \emptyset$, then, continuing in this fashion, $G[V(C_1) - \{x, y\}] \cong K_{2m}$ - a perfect matching for some positive integer $m \geq 1$. This proves (3).

Recall that $I_{x_1} \cap V(C_2) \neq \emptyset$. Let $\{y_1\} = I_{x_1} \cap V(C_2)$ where $y_1 \succ_i V(C_2)$. If $y_1 \neq z$, then $N_{C_2}[z] \subseteq N_{C_2}[y_1]$. Hence, $I_{x_1} = \{x_1^*, z\}$. Morever, if $V(C_1) - \{x, y, x_1, x_1^*\} \neq \emptyset$, for each $x_i \in V(C_1) - \{x, y, x_1, x_1^*\}$, $I_{x_i} = \{x_i^*, z\}$ where $x_i^* \in V(C_1) - \{x, y, x_1, x_1^*, x_i\}$. By similar argument, $I_{x_i^*} = \{x_i, z\}$.

(4) By (2), $N_{C_2}(v) - N_{C_2}(u) \neq \emptyset$ and $N_{C_2}(u) - N_{C_2}(v) \neq \emptyset$. Let $a \in N_{C_2}(v) - N_{C_2}(u)$. Consider $G - a$. If $u \in I_a$, then the only vertex of $I_a - \{u\}$ dominates x and z. But this is not possible since x and z belong to different components. Hence, $u \notin I_a$. It follows that $I_a \cap V(C_1) \neq \emptyset$ and $I_a \cap V(C_2) \neq \emptyset$. Note that, by (3), it is easy to see that either $I_a \cap V(C_1) = \{x\}$ or $I_a \cap V(C_1) = \{y\}$. Let $I_a \cap V(C_2) = \{a^*\}$. Clearly, $a^* \neq z$, $aa^* \notin E(G)$ and $a^* \succ_i V(C_2) - \{a\}$. Observe that $a^* \in (N_{C_2}(u) - N_{C_2}(v)) \cup (N_{C_2}(v) - N_{C_2}(u)) \cup (N_{C_2}(u) \cap N_{C_2}(v))$. If $V(C_2) - \{z, a, a^*\} \neq \emptyset$, then, continuing in this fashion, $G[V(C_2) - \{z\}] \cong K_{2n}$ - a perfect matching for some positive integer $n \geq 1$. This proves (4) and completes the proof of our lemma.

□

Theorem 3.2.7. *Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| = 2$. Suppose $S = \{u, v\}$ is an independent set and C_1,C_2 are components of $G - S$. If $v \notin I_u$, then G belongs to $\mathcal{N}$ defined in Section 3.1.*

Proof. By Lemma 3.2.6(1), there exist $x, y \in V(C_1)$ and $z \in V(C_2)$ such that $x \succ_i V(C_1)$, $y \succ_i V(C_1)$ and $z \succ_i V(C_2)$. Moreover, $N_{C_1}(u) = V(C_1) - \{x\}$, $N_{C_1}(v) = V(C_1) - \{y\}$ and $\{z\} = V(C_2) - (N_{C_2}(u) \cup N_{C_2}(v))$. By Lemma 3.2.6(3), if $V(C_1) - \{x, y\} \neq \emptyset$, then $V(C_1) - \{x, y\}$ is isomorphic to a complete graph without a perfect matching. By Lemma 3.2.6(2), $N_{C_2}(v) - N_{C_2}(u) \neq \emptyset$ and $N_{C_2}(u) - N_{C_2}(v) \neq \emptyset$. By Lemma 3.2.6(4), $V(C_2) - \{z\}$ is isomorphic to a complete graph without a perfect matching. Let F be such a perfect matching in $V(C_2) - \{z\}$. Put

$$Y_1 = \{x \in N_{C_2}(u) - N_{C_2}(v) | \text{ there is } y \in N_{C_2}(u) - N_{C_2}(v) \text{ such that } xy \in F\}$$

$$Y_2 = \{x \in N_{C_2}(v) - N_{C_2}(u) | \text{ there is } y \in N_{C_2}(v) - N_{C_2}(u) \text{ such that } xy \in F\}$$

$$Y_3 = \{x \in N_{C_2}(u) \cap N_{C_2}(v) | \text{ there is } y \in N_{C_2}(u) \cap N_{C_2}(v) \text{ such that } xy \in F\}$$

$Y_4 = Y_4' \cup Y_4''$ where

$Y_4' = \{x \in N_{C_2}(u) - N_{C_2}(v) |$ there is $y \in N_{C_2}(u) \cap N_{C_2}(v)$ such that $xy \in F\}$

$Y_4'' = \{x \in N_{C_2}(u) \cap N_{C_2}(v) |$ there is $y \in N_{C_2}(u) - N_{C_2}(v)$ such that $xy \in F\}$

$Y_5 = Y_5' \cup Y_5''$ where

$Y_5' = \{x \in N_{C_2}(v) - N_{C_2}(u) |$ there is $y \in N_{C_2}(u) \cap N_{C_2}(v)$ such that $xy \in F\}$

$Y_5'' = \{x \in N_{C_2}(u) \cap N_{C_2}(v) |$ there is $y \in N_{C_2}(v) - N_{C_2}(u)$ such that $xy \in F\}$

$Y_6 = Y_6' \cup Y_6''$ where

$Y_6' = \{x \in N_{C_2}(u) - N_{C_2}(v) |$ there is $y \in N_{C_2}(v) - N_{C_2}(u)$ such that $xy \in F\}$

$Y_6'' = \{x \in N_{C_2}(v) - N_{C_2}(u) |$ there is $y \in N_{C_2}(u) - N_{C_2}(v)$ such that $xy \in F\}$.

Note that $V(C_2) - \{z\} = \bigcup_{i=1}^{6} Y_i$ and $Y_i \cap Y_j = \emptyset$, $1 \leq i \neq j \leq 6$. We distinguish two cases.

Case 1 : $N_{C_2}(u) \cap N_{C_2}(v) = \emptyset$.

Then $V(C_2) - \{z\} = Y_1 \cup Y_2 \cup Y_6$. We first suppose that $Y_6 = \emptyset$. By Lemma 3.2.6(2), $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$. Thus $G \cong G_3$ if $V(C_1) - \{x, y\} = \emptyset$ or $G \cong G_3'$ if $V(C_1) - \{x, y\} \neq \emptyset$. We now suppose that $Y_6 \neq \emptyset$. Then

Then

$$G \in \begin{cases} \{G_1, G_2, G_4\}, & \text{if } V(C_1) - \{x, y\} = \emptyset \\ \{G_1', G_2', G_4'\}, & \text{if } V(C_1) - \{x, y\} \neq \emptyset. \end{cases}$$

Case 2 : $N_{C_2}(u) \cap N_{C_2}(v) \neq \emptyset$. Thus $Y_3 \cup Y_4 \cup Y_5 \neq \emptyset$.

Subcase 2.1 : $Y_3 \neq \emptyset$ but $Y_4 = Y_5 = \emptyset$.

Then either $Y_6 \neq \emptyset$ or $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$. Thus

$$G \in \begin{cases} \{G_5, G_6, ..., G_8\}, & \text{if } V(C_1) - \{x, y\} = \emptyset \\ \{G_5', G_6', ..., G_8'\}, & \text{if } V(C_1) - \{x, y\} \neq \emptyset. \end{cases}$$

Subcase 2.2 : $Y_4 \neq \emptyset$ but $Y_3 = Y_5 = \emptyset$.

Then $Y_2 \cup Y_6 \neq \emptyset$ and thus

$$G \in \begin{cases} \{G_9, G_{10}, ..., G_{14}\}, & \text{if } V(C_1) - \{x, y\} = \emptyset \\ \{G'_9, G'_{10}, ..., G'_{14}\}, & \text{if } V(C_1) - \{x, y\} \neq \emptyset. \end{cases}$$

Subcase 2.3 : $Y_3 \neq \emptyset$, $Y_4 \neq \emptyset$ but $Y_5 = \emptyset$.

Then $Y_2 \cup Y_6 \neq \emptyset$. Thus

$$G \in \begin{cases} \{G_{15}, G_{16}, ..., G_{20}\}, & \text{if } V(C_1) - \{x, y\} = \emptyset \\ \{G'_{15}, G'_{16}, ..., G'_{20}\}, & \text{if } V(C_1) - \{x, y\} \neq \emptyset. \end{cases}$$

Subcase 2.4 : $Y_4 \neq \emptyset$, $Y_5 \neq \emptyset$ but $Y_3 = \emptyset$.

Then

$$G \in \begin{cases} \{G_{21}, G_{22}, ..., G_{26}\}, & \text{if } V(C_1) - \{x, y\} = \emptyset \\ \{G'_{21}, G'_{22}, ..., G'_{26}\}, & \text{if } V(C_1) - \{x, y\} \neq \emptyset. \end{cases}$$

Subcase 2.5 : $Y_3 \neq \emptyset$, $Y_4 \neq \emptyset$ and $Y_5 \neq \emptyset$.

Then

$$G \in \begin{cases} \{G_{27}, G_{28}, ..., G_{32}\}, & \text{if } V(C_1) - \{x, y\} = \emptyset \\ \{G'_{27}, G'_{28}, ..., G'_{32}\}, & \text{if } V(C_1) - \{x, y\} \neq \emptyset. \end{cases}$$

Therefore, G belongs to $\mathscr{N}$. This completes the proof of our theorem. □

Lemma 3.2.8. *Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| = 2$. Suppose $S = \{u, v\}$ is an independent set and C_1 and C_2 are components of $G - S$. If $v \in I_u$ and $|V(C_i)| \geq 2$ for $1 \leq i \leq 2$, then*

(1) $V(C_1) \subseteq N_G(u) \cap N_G(v)$.

(2) For each $a \in V(C_1)$, there exists unique $b \in V(C_1)$ such that $b \in I_a$ and $b \succ_i V(C_1) - \{a\}$.

(3) $V(C_1) \cong K_{2m}$ - a perfect matching for some positive integer m.

(4) There exists $z \in V(C_2)$ *such that* $\{z\} = \overline{N}_{C_2}(u) \cap \overline{N}_{C_2}(v)$ *and* $z \succ_i V(C_2)$ *and* $V(C_2) = (N_{C_2}(u) - N_{C_2}(v)) \cup (N_{C_2}(v) - N_{C_2}(u)) \cup \{z\}$.

Put $A = N_{C_2}(u) - N_{C_2}(v)$, $B = N_{C_2}(v) - N_{C_2}(u)$.

(5) For each $a \in A$*, there exists unique* $b \in A - \{a\}$ *such that* $b \in I_a$ *and* $b \succ_i (A - \{a\}) \cup \{z\}$*. Consequently,* $G[A] \cong K_{2n}$ *- a perfect matching for some positive integer* n.

(6) For each $a \in B$*, there exists unique* $b \in B - \{a\}$ *such that* $b \in I_a$ *and* $b \succ_i (B - \{a\}) \cup \{z\}$*. Consequently,* $G[B] \cong K_{2k}$ *- a perfect matching for some positive integer* k.

Proof. (1) Since $v \in I_u$, v must dominate at least 1 component of $G - S$. Without loss of generality, we may assume that $v \succ_i V(C_1)$. Consider $G - v$. Clearly, $I_v \cap V(C_1) = \emptyset$ and then $u \in I_v$. So $u \succ_i V(C_1)$. Therefore, $V(C_1) \subseteq N_G(u) \cap N_G(v)$.

(2) Let $a \in V(C_1)$. Since $V(C_1) \subseteq N_G(u) \cap N_G(v)$, $au \in E(G)$ and $av \in E(G)$. Consider $G - a$. By Lemma 3.2.1, $I_a \cap \{u, v\} = \emptyset$. Since $|V(C_1)| > 1$, $I_a \cap V(C_1) \neq \emptyset$. Let $b \in I_a \cap V(C_1)$. Then $b \succ_i V(C_1) - \{a\}$. If there is $b' \in V(C_1) - \{a, b\}$ such that $b' \in I_a \cap V(C_1)$, then $N_G[b'] = (V(C_1) - \{a\}) \cup \{u, v\} = N_G[b]$, contradicting Lemma 2.3. This proves (2).

(3) follows by (2).

(4) Let $x \in V(C_1)$. By (2), there is $y \in V(C_1)$ such that $y \in I_x$ and $y \succ_i V(C_1) - \{x\}$. Put $I_x - \{y\} = \{z\}$. Then, by Lemma 3.2.1 and (1), $z \in V(C_2)$ and $z \succ_i V(C_2)$. Consider $G - z$. Since $z \succ_i V(C_2)$, $I_z \cap \{u, v\} \neq \emptyset$. Without loss of generality, we may assume that $u \in I_z$. Clearly, $uz \notin E(G)$. If $zv \in E(G)$, then $\{z, u\} \succ_i G$, a contradiction. So $zv \notin E(G)$. It follows that $z \in \overline{N}_{C_2}(u) \cap \overline{N}_{C_2}(v)$. If there is $z' \in (\overline{N}_{C_2}(u) \cap \overline{N}_{C_2}(v)) - \{z\}$, $N_{C_2}[z'] \subseteq N_{C_2}[z]$, a contradiction. Hence, $\overline{N}_{C_2}(u) \cap \overline{N}_{C_2}(v) = \{z\}$. We next show that $N_{C_2}(u) \cap N_{C_2}(v) = \emptyset$. Suppose to the contrary that $N_{C_2}(u) \cap N_{C_2}(v) \neq \emptyset$. Let $a \in N_{C_2}(u) \cap N_{C_2}(v)$. Then $I_a \cap \{u, v\} = \emptyset$. It follows that $I_a \cap V(C_1) \neq \emptyset$ and $I_a \cap V(C_2) \neq \emptyset$. Thus the only vertex of $I_a \cap V(C_1)$ must dominate $V(C_1)$, contradicting (2). Hence, $N_{C_2}(u) \cap N_{C_2}(v) = \emptyset$. Since S is minimum cutset and $N_{C_2}(u) \cap N_{C_2}(v) = \emptyset$, it follows that $N_{C_2}(u) - N_{C_2}(v) \neq \emptyset$ and $N_{C_2}(v) - N_{C_2}(u) \neq \emptyset$. Therefore, $V(C_2) = (N_{C_2}(u) - N_{C_2}(v)) \cup (N_{C_2}(v) - N_{C_2}(u)) \cup \{z\}$.

(5) Let $a \in A$. Clearly, $au \in E(G)$ and $av \notin E(G)$. Consider $G - a$. By (3), if $v \notin I_a$, $|I_a \cap V(C_1)| \geq 2$ and thus no vertex of I_a dominates $V(C_2) - \{a\}$ since $|I_a| = 2$, a contradiction. Hence, $v \in I_a$. Because $vz \notin E(G)$, $I_a \cap V(C_2) \neq \emptyset$. In fact, $I_a \cap V(C_2) \subseteq N_{C_2}(u)$ since $vu \notin E(G)$. Put $\{b\} = I_a - \{v\}$. Since I_a is independent, $b \notin N_{C_2}(v)$. Thus $b \in A - \{a\}$. Clearly, $bz \in E(G)$ and $b \succ_i A - \{a\}$. We

next show that there exists unique $b \in A - \{a\}$ such that $b \in I_a$. Suppose to the contrary that there exists $b' \in A - \{a, b\}$ such that $b' \in I_a$ and $b' \succ_i (A - \{a\}) \cup \{z\}$. Consider $G - b'$. By similar arguments as above, $v \in I_{b'}$ and $I_{b'} - \{v\} \subseteq A$. It then follows that $I_{b'} - \{v\} = \{a\}$. But then no vertex of $I_{b'}$, dominates b, a contradiction. Hence, (5) is proved.

By similar arguments as in the proof of (5), (6) follows. This completes the proof of our lemma.

□

Theorem 3.2.9. *Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| = 2$. Suppose $S = \{u, v\}$ is an independent set and C_1, C_2 are components of $G - S$. If $v \in I_u$ and $|V(C_i)| \geq 2$ for $i \in \{1, 2\}$, then G belongs to $\mathscr{O}$ defined in Section 3.1*

Proof. By Lemma 3.2.8(1), $V(C_1) \subseteq N_G(u) \cap N_G(v)$. Moreover, $V(C_1) \cong K_{2m}$ - a perfect matching for some positive integer m by Lemma 3.2.8(3). Note that $m \geq 2$ otherwise $\omega(G - S) = 3$. By Lemma 3.2.8(4), there exists $z \in V(C_2)$ such that $\{z\} = \overline{N}_{C_2}(u) \cap \overline{N}_{C_2}(v)$, $z \succ_i V(C_2)$ and $V(C_2) = (N_{C_2}(u) - N_{C_2}(v)) \cup (N_{C_2}(v) - N_{C_2}(u)) \cup \{z\}$. Further, by Lemma 3.2.8(5) and 3.2.8(6), $G[N_{C_2}(u) - N_{C_2}(v)] \cong K_{2n}$- a perfect matching for some positive integer n and $G[N_{C_2}(v) - N_{C_2}(u)] \cong K_{2k}$ - a perfect matching for some positive integer k Therefore, G belongs to $\mathscr{O}$. This completes the proof of our theorem. □

We conclude this chapter by pointing out that if we have hypothesis as in Theorem 3.2.9 but one of the components in $G - S$ is singleton, then we still do not know the structure of such graphs.

Chapter 4

Matching property and toughness results in 3-i-vertex-critical graphs

In this chapter, we present properties of 3-i-vertex-critical graphs G with a minimum cutset S where $\Delta(G[S]) \leq 1$ in terms of $\omega(G - S)$. In fact, we show that $\omega(G - S) \leq |S| - 1$ with some condition on $|S|$. We also provide a sufficient condition for G to have a perfect matching.

4.1 Results on toughness

Theorem 4.1.1. *Let G be a connected 3-i-vertex-critical with a minimum cutset S where $|S| = 3$. Then $\omega(G - S) \leq 3$*

Proof. Suppose to the contrary that $\omega(G - S) = t \geq 4$. Since $\omega(G - S) \geq 4$, $I_x \cap S \neq \emptyset$ for each $x \in V(G)$. Let $S = \{x_1, x_2, x_3\}$.

Claim 1 : $|E(S)| \leq 1$

Suppose to the contrary that $|E(S)| \geq 2$. Without loss of generality, we may assume that $x_1x_2 \in E(G)$ and $x_2x_3 \in E(G)$. Consider $G - x_2$. Since $I_x \cap S \neq \emptyset$ for all $x \in V(G)$, $I_{x_2} \cap \{x_1, x_3\} \neq \emptyset$. But this contradicts Lemma 3.2.1. Hence, $|E(S)| \leq 1$. This settles our claim.

Claim 2 : For each $x \in \bigcup_{i=1}^{t} V(C_i)$, there exists a vertex $x' \in S$ such that $xx' \notin E(G)$.

Suppose to the contrary that every vertex in S is adjacent to x. It then follows that $I_x \cap S = \emptyset$. But this contradicts the fact that $I_x \cap S \neq \emptyset$. This settles our claim.

Claim 3 : For $1 \leq i \leq t$, $|V(C_i)| \geq 2$

Claim 3 follows by Claim 2 and the fact that S is a minimum cutset.

Let $y_1 \in N_{C_1}(x_1)$. Consider $G - y_1$. Without loss of generality, we may assume that $x_2 \in I_{y_1}$. So $x_2y_1 \notin E(G)$. Put $\{y_2\} = I_{y_1} - \{x_2\}$.

Case 1 : $y_2 \in V(C_1)$

Then $y_1y_2 \notin E(G)$ and $y_2x_2 \notin E(G)$. So $x_2 \succ_i \bigcup_{i=2}^{t} V(C_i)$. Since S is a minimum cutset, $N_{C_j}(x_i) \neq \emptyset$ for $1 \leq i \leq 3$ and $1 \leq j \leq t$. Choose $y_3 \in N_{C_2}(x_1)$ and $y_4 \in N_{C_3}(x_1)$. Then $\{y_3x_1, y_4x_1, y_3x_2, y_4x_2\} \subseteq E(G)$. By Claim 2, $y_3x_3 \notin E(G)$ and $y_4x_3 \notin E(G)$. Consider $G - y_3$. By Lemma 3.2.1, $x_3 \in I_{y_3}$. Since $x_3y_4 \notin E(G)$, $I_{y_3} \cap V(C_3) \neq \emptyset$. It follows that $x_3 \succ_i \bigcup_{i=1}^{t} V(C_i) - (V(C_3) \cup \{y_3\})$. Then all vertices in C_4 is adjacent to x_2 and x_3. So no vertex in C_4 is adjacent to x_1 by Claim 2. It follows that $\{x_2, x_3\}$ is a cutset, contradicting the fact that S is a minimum cutset. Hence, this case cannot occur.

Case 2 : $y_2 \in \bigcup_{i=2}^{t} V(C_i)$

Without loss of generality, we may assume that $y_2 \in V(C_2)$. Then $y_2x_2 \notin E(G)$. So $x_2 \succ_i (V(C_1) - \{y_1\}) \cup \bigcup_{i=3}^{t} V(C_i)$. Since S is minimum cutset, $N_{C_j}(x_i) \neq \emptyset$ for $1 \leq i \leq 3$ and $1 \leq j \leq t$. Choose $y_3 \in N_{C_3}(x_1)$ and $y_4 \in N_{C_4}(x_1)$. Then $\{y_3x_1, y_4x_1, y_3x_2, y_4x_2\} \subseteq E(G)$. By Claim 2, $y_3x_3 \notin E(G)$ and $y_4x_3 \notin E(G)$. Consider $G - y_3$. It is easy to see that $x_3 \in I_{y_3}$. Since $x_3 \in I_{y_3}$ and $x_3y_4 \notin E(G)$, it follows that $I_{y_3} \cap V(C_4) \neq \emptyset$. Let $\{y_5\} = I_{y_3} - \{x_3\}$. Then $y_5 \in V(C_4)$. Thus $x_3 \succ_i \bigcup_{i=1}^{t} V(C_i) - (V(C_4) \cup \{y_3\})$. Then $y_1x_3 \in E(G)$ and $y_2x_3 \in E(G)$. Consider $G - y_4$. By Lemma 3.2.1, $x_3 \in I_{y_4}$ since $y_4x_1, y_4x_2 \in E(G)$. Since $x_3y_3 \notin E(G)$, $I_{y_4} \cap V(C_3) \neq \emptyset$. Because $x_3 \succ_i V(C_3) - \{y_3\}$, $I_{y_4} \cap V(C_3) = \{y_3\}$. It then follows that $x_3 \succ_i V(C_4) - \{y_4\}$. It follows that $y_5 = y_4$. We now have $x_3 \succ_i \bigcup_{i=1}^{t} V(C_i) - \{y_3, y_4\}$. Consider $G - x_3$. Observe that $\{x_1, y_3\}$, $\{x_1, y_4\}$, $\{x_2, y_3\}$ and $\{x_2, y_4\}$ are not independent. Thus $I_{x_3} \notin \{\{x_1, y_3\}, \{x_1, y_4\}, \{x_2, y_3\}, \{x_2, y_4\}\}$. Since $x_3 \succ_i \bigcup_{i=1}^{t} V(C_i) - \{y_3, y_4\}$, $I_{x_3} = \{x_1, x_2\}$. Recall that $x_2 \succ_i (V(C_1) - \{y_1\}) \cup \bigcup_{i=3}^{t} V(C_i)$. Then $V(C_3) - \{y_3\} \subseteq N_G(x_2) \cap N_G(x_3)$ and $V(C_4) - \{y_4\} \subseteq N_G(x_2) \cap N_G(x_3)$. By Claim 2, $N_{C_3}(x_1) = \{y_3\}$ and $N_{C_4}(x_1) = \{y_4\}$. Choose $z \in V(C_1) - \{y_1\}$. Observe that $zx_2, zx_3 \in E(G)$. Then $I_z \cap S = \{x_1\}$ and thus the only vertex of $I_z - \{x_1\}$ dominates $(V(C_3) \cup V(C_4)) - \{y_3, y_4\}$. But this is not possible. Hence, this case cannot occur.

Case 3 : $y_2 \in S$

Then $I_{y_1} = \{x_2, x_3\}$. Clearly, $y_1x_3 \notin E(G)$ and $x_2x_3 \notin E(G)$. Without loss of generality, we may assume that $x_1x_2 \in E(G)$. Since S is a minimum cutset, $N_{C_j}(x_i) \neq \emptyset$, for $1 \leq i \leq 3$ and $1 \leq j \leq t$. Let $y_3 \in N_{C_2}(x_1)$. By Claim 1, we have $x_1x_3 \notin E(G)$. Consider $G - x_1$. It is easy to see that $x_3 \in I_{x_1}$. Since $y_1 \in V(C_1)$ and $y_1x_3 \notin E(G)$, $I_{x_1} \cap V(C_1) \neq \emptyset$. Then $x_3 \succ_i \bigcup_{i=2}^{t} V(C_i)$. So $y_3x_3 \in E(G)$ because $y_3 \in V(C_2)$. By Claim 2, we have $y_3x_2 \notin E(G)$. Consider $G - y_3$. Clearly, $x_2 \in I_{y_3}$. Since $y_1 \in V(C_1)$ and $y_1x_2 \notin E(G)$, it follows that $I_{y_3} \cap V(C_1) \neq \emptyset$.

Thus $x_2 \succ_i \bigcup_{i=2}^{t} V(C_i) - \{y_3\}$. Therefore, each vertex of $V(C_3)$ is adjacent to x_2 and x_3. By Claim 2, no vertex of $V(C_3)$ is adjacent to x_1. It follows that $\{x_2, x_3\}$ is a cutset, contradicting the fact that S is a minimum cutset. Hence, this case cannot occur.

Hence, $\omega(G - S) \leq 3$. This completes the proof of our theorem. □

It is easy to see that $K_{3,3}$ satisfies the hypothesis in Theorem 4.1.1. Hence, the bound on the number of components in Theorem 4.1.1 is best possible.

Theorem 4.1.2. *Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| \geq 4$ and $\Delta(G[S]) = 0$. Then $\omega(G - S) \leq |S| - 1$.*

Proof. Suppose to the contrary that $\omega(G - S) = t \geq k = |S|$. Since $|S| \geq 4$, $\omega(G - S) \geq 4$. It follows that $I_x \cap S \neq \emptyset$ for each $x \in V(G)$. Let $C_1, C_2, ..., C_t$ be components of $G - S$.

Claim 1 : For each $x \in V(G)$, $|I_x \cap S| = 1$.

Since $\omega(G - S) \geq 4$, it is not difficult to see that $I_x \cap S \neq \emptyset$ for all $x \in V(G)$. If $I_x \subseteq S$ for some $x \in V(G)$ then there exists at least one vertex in S is not dominated by I_x, since S is independent and $|S| \geq 4$. Hence, $I_x \nsubseteq S$ and thus $|I_x \cap S| = 1$ as required. This settles our claim.

The next two claims follow by Claim 1, Lemma 3.2.1 and the fact that S is a minimum cutset.

Claim 2 : For each $x \in \bigcup_{i=1}^{t} V(C_i)$, there exists a vertex $x' \in S$ such that $xx' \notin E(G)$.

Claim 3 : For $1 \leq i \leq t$, $|V(C_i)| \geq 2$.

Claim 4 : If $x \in V(C_i)$ where $1 \leq i \leq t$, then $I_x - S \subseteq V(C_i) - \{x\}$.

Consider $G - x$. By Claim 1, $|I_x \cap S| = 1$. Put $\{x_i\} = I_x \cap S$. Let $\{x_i^*\} = I_x - \{x_i\}$. Suppose to the contrary that $x_i^* \notin V(C_i)$. Then $x_i^* \in V(C_j)$ where $j \neq i$. Then x_i^* is adjacent to every vertex of $S - \{x_i\}$ since S is independent and $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - (V(C_j) \cup \{x\})$. Consider $G - x_i^*$. Since x_i^* is adjacent to every vertex of $S - \{x_i\}$, $I_{x_i^*} \cap S = \{x_i\}$ by Claim 1 and Lemma 3.2.1. Since $x_i x \notin E(G)$ and $x \in V(C_i)$, it follows that $I_{x_i^*} \cap V(C_i) \neq \emptyset$. Then $I_{x_i^*} - \{x_i\} = \{x\}$ because $x_i \succ_i V(C_i) - \{x\}$. So x is adjacent to every vertex of $S - \{x_i\}$ and $x_i \succ_i V(C_j) - \{x_i^*\}$. Now x_i is adjacent to every vertex of $\bigcup_{l=1}^{t} V(C_l) - \{x, x_i^*\}$. Consider $G - x_i$. Since $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - \{x, x_i^*\}$, either $x \in I_{x_i}$ or $x_i^* \in I_{x_i}$. By Claim 1, $I_{x_i} \cap (S - \{x_i\}) \neq \emptyset$. But this contradicts the fact that I_{x_i} is independent since $S - \{x_i\} \subseteq N_G(x) \cap N_G(x_i^*)$. Hence, $x_i^* \in V(C_i)$ as required. This settles our claim.

Claim 5 : For $1 \le i \ne j \le t$, if $\{x_i\} = I_{y_i} \cap S$ and $\{x_j\} = I_{y_j} \cap S$ where $y_i \in V(C_i)$ and $y_j \in V(C_j)$, then $x_i \ne x_j$.

Put $\{z_i\} = I_{y_i} - \{x_i\}$. By Claim 4, $z_i \in V(C_i)$. Then $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - V(C_i)$. Thus $x_i y_j \in E(G)$. By Lemma 3.2.1, $x_i \ne x_j$. This settles our claim.

For $1 \le i \le t$, choose $y_i \in V(C_i)$. It follows by Claims 1 and 5 that $t = k$ since $|S| = k$. Put $S = \{x_1, x_2, ..., x_k\}$. We may assume without loss of generality that $I_{y_i} \cap S = \{x_i\}$. Put $\{z_i\} = I_{y_i} - \{x_i\}$. By Claim 4, $z_i \in V(C_i)$ and thus $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - V(C_i)$. Since S is independent, $z_i \succ_i S - \{x_i\}$.

We now consider $G - z_i$. By Lemma 3.2.1 and Claim 1, $I_{z_i} \cap S = \{x_i\}$. Observe that each vertex of $V(C_i) - \{y_i, z_i\}$ is adjacent to either x_i or z_i since $I_{y_i} = \{x_i, z_i\}$. It then follows by Claim 4 that $I_{z_i} = \{x_i, y_i\}$. Because I_{y_i} and I_{z_i} are independent, $\{x_i, y_i, z_i\}$ is independent. Since S is a minimum cutset, there exists $w \in V(C_i) - \{y_i, z_i\}$ such that $wx_i \in E(G)$. Consequently, w is adjacent to every vertex of S since $x_i \succ_i \bigcup_{l=1}^{k} V(C_l) - V(C_i)$ for $1 \le i \le k$. But this contradicts Claim 2 and completes the proof of our theorem. □

Theorem 4.1.3. *Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| \ge 6$ and $\Delta(G[S]) = 1$. then $\omega(G - S) \le |S| - 1$.*

Proof. Suppose to the contrary that $\omega(G - S) = t \ge |S| = k$. Since $|S| \ge 6$, $\omega(G - S) \ge 6$. It follows that $I_x \cap S \ne \emptyset$ for each $x \in V(G)$. Let $C_1, C_2, ..., C_t$ be components of $G - S$.

By similar arguments as in the proof of Theorem 4.1.2, we have following claims.

Claim 1 : For each $x \in V(G)$ and $|S| \ge 6$, $|I_x \cap S| = 1$.

Claim 2 : For each $x \in \bigcup_{i=1}^{t} V(C_i)$, there exists a vertex $x' \in S$ such that $xx' \notin E(G)$.

Claim 3 : For $1 \le i \le t$, $|V(C_i)| \ge 2$.

Claim 4 : If $y_i, y_j \in \bigcup_{l=1}^{t} V(C_l)$ such that y_i and y_j are in different components, then $I_{y_i} \cap S \ne I_{y_j} \cap S$.

Let $y_i \in V(C_i)$ and $y_j \in V(C_j)$ where $i \ne j$. Suppose to the contrary that $I_{y_i} \cap S = I_{y_j} \cap S$. Put $\{x\} = I_{y_i} \cap S = I_{y_j} \cap S$. By Lemma 3.2.1, $xy_i, xy_j \notin E(G)$. Then $I_{y_i} - \{x\} \subseteq V(C_j)$ and $I_{y_j} - \{x\} \subseteq V(C_i)$. It follows that $x \succ_i \bigcup_{l=1}^{t} V(C_l) - \{y_i, y_j\}$. We now consider $G - x$. Then $I_x \subseteq \{y_i, y_j\} \cup (S - \{x\})$. Since $|I_x \cap S| = 1$, by Claim 1, either $y_i \in I_x$ or $y_j \in I_x$. Put $\{z\} = I_x - \{y_i, y_j\}$. Then $z \in S - \{x\}$ and $zx \notin E(G)$. We first suppose that $I_x = \{z, y_i\}$. Since $I_{y_j} = \{x, y_i\}$, and $zx \notin E(G)$, it follows that $zy_i \in E(G)$. But this contradicts the fact that I_x is independent. Hence, $I_x \ne \{z, y_i\}$ and thus $I_x = \{z, y_j\}$. By similar arguments as above and the fact that $I_{y_i} = \{x, y_j\}$, $zy_j \in E(G)$, again a contradiction. This settles our claim.

Claim 5 : If $y_i \in V(C_i)$ for some $1 \leq i \leq t$, then $I_{y_i} - S \subseteq V(C_i) - \{y_i\}$.

Consider $G - y_i$. By Claim 1, $|I_{y_i} \cap S| = 1$. Put $\{x_i\} = I_{y_i} \cap S$. Suppose to the contrary that $I_{y_i} - S \nsubseteq V(C_i) - \{y_i\}$. Let $I_{y_i} - S = \{y_j\}$ where $y_j \in V(C_j)$ and $j \neq i$. So $x_iy_i, x_iy_j \notin E(G)$ and $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - (V(C_j) \cup \{y_i\})$. Since $\Delta(G[S]) = 1$ and $I_{y_i} = \{x_i, y_j\}$, $|N_{S-\{x_i\}}(y_j)| \geq k - 2$. Let $N_{S-\{x_i\}}(y_j) = S'$. Consider $G - y_j$. By Claim 4, $I_{y_j} \cap S \neq \{x_i\}$. Suppose that $I_{y_j} \cap S = \{x_j\}$. Since $I_{y_i} = \{x_i, y_j\} \succ_i G - y_i$ and $y_jx_j \notin E(G)$, it follows that $x_ix_j \in E(G)$. Since $\Delta(G[S]) = 1$, x_i is not adjacent to any vertex of $S - \{x_i, x_j\}$ and x_j is not adjacent to any vertex of $S - \{x_i, x_j\}$. Hence, $|S'| = k - 2$ and $S' = S - \{x_i, x_j\}$. Put $\{z\} = I_{y_j} - \{x_j\}$.Then $zx_j \notin E(G)$. We distinguish four cases.

Case 1 : $z = y_i$.

Then $x_j \succ_i \bigcup_{l=1}^{t} V(C_l) - (V(C_i) \cup \{y_j\})$. Since S is minimum cutset, $N_{C_{l'}}(x_l) \neq \emptyset$ for $1 \leq l \leq k$ and $1 \leq l' \leq t$. Let $y_{j'} \in N_{C_j}(x_j)$. Consider $G - y_{j'}$. By Claim 4 and the fact that $x_i \in I_{y_i}$, it follows that $x_i \notin I_{y_{j'}}$. Clearly, by Lemma 3.2.1, $x_j \notin I_{y_{j'}}$ because $y_{j'}x_j \in E(G)$. Then $I_{y_{j'}} \cap S \subseteq S'$. Let $I_{y_{j'}} \cap S = \{x_{j'}\}$. Observe that $|S - \{x_i, x_j, x_{j'}\}| = k - 3$ and $|\omega(G - S)| - |\{C_i, C_j\}| = t - 2$. For $1 \leq \Lambda \leq t$ where $\Lambda \notin \{i, j\}$, let $y_\Lambda \in V(C_\Lambda)$. Clearly, $|\{y_\Lambda | 1 \leq \Lambda \leq t$ and $\Lambda \notin \{i, j\}\}| = t - 2$. By Claim 1, $|I_{y_\Lambda} \cap S| = 1$. Further, by Claim 4, $I_{y_\Lambda} \cap S \subseteq S - \{x_i, x_j, x_{j'}\}$. Since $t \geq k$, $t - 2 > k - 3$. By Pigoenhole principle (Theorem 1.1), there exist $y_{\Lambda'} \in V(C_{\Lambda'})$ and $y_{\Lambda''} \in V(C_{\Lambda''})$ where $1 \leq \Lambda' \neq \Lambda'' \leq t$, $\{\Lambda', \Lambda''\} \cap \{i, j\} = \emptyset$, such that $I_{y_{\Lambda'}} \cap S = I_{y_{\Lambda''}} \cap S$. But this contradicts Claim 4. This proves Case 1.

Case 2 : $z \in V(C_i) - \{y_i\}$.

Since $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - (V(C_j) \cup \{y_i\})$, $zx_i \in E(G)$. Further, $x_j \succ_i \bigcup_{l=1}^{t} V(C_l) - (V(C_i) \cup \{y_j\})$ and $z \succ_i S'$. Thus, $N_S(z) = S - \{x_j\}$. Consider $G - z$. By Lemma 3.2.1, $\{x_j\} = I_z \cap S$. But this contradicts Claim 4 since $x_j \in I_{y_j}$ and $y_j \in V(C_j)$. This settles Case 2.

Case 3 : $z \in V(C_j) - \{y_j\}$.

Then, $zy_j \notin E(G)$. So $x_j \succ_i (\bigcup_{l=1}^{t} V(C_l)) - V(C_j)$ and $z \succ_i S'$. Since S is minimum cutset, $N_{C_{l'}}(x_l) \neq \emptyset$ for $1 \leq l \leq k$ and $1 \leq l' \leq t$. Let $y_{j'} \in N_{C_j}(x_j)$. Consider $G - y_{j'}$. By Claim 4, $I_{y_{j'}} \cap S \subseteq S'$. Then applying similar arguments as in the proof of Case 1, we have a contradiction. This proves Case 3.

Case 4 : $z \in (\bigcup_{l=1}^{t} V(C_l)) - (V(C_i) \cup V(C_j))$.

Let $z \in V(C_n)$. Recall that $I_{y_i} = \{x_i, y_j\}$. Since $z \in V(C_n)$, $x_i z \in E(G)$ because $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - (V(C_j) \cup \{y_i\})$. Further, since $x_i x_j \in E(G)$ and $\Delta(G[S]) = 1$, it follows that $z \succ_i S'$. Thus, $z \succ_i S - \{x_j\}$. It then follows that $I_z \cap S = \{x_j\}$ by Claim 1. But this contradicts Claim 4 since $\{x_j\} = I_{y_j} \cap S$ and $y_j \in V(C_j)$. This proves Case 4 and settles our claim.

It then follows from Claims 1 and 4 that $t = k$. For $1 \leq i \leq k$, choose $y_i \in V(C_i)$. Put $\{x_i\} = I_{y_i} \cap S$ for $1 \leq i \leq k$. Then $x_i \succ_i \bigcup_{l=1}^{t} V(C_l) - V(C_i)$ by Claim 5. Since S is a minimum cutset, there exists $w \in N_{C_i}(x_i)$. But then $w \succ_i S$. But this contradicts Claim 2 and completes the proof of our theorem. □

We now post the following conjecture.

Conjecture Let G be a connected 3-i-vertex-critical graph with a minimum cutset S where $|S| \geq 4$. Then $\omega(G - S) \leq |S| - 1$.

We conclude this section by pointing out that if G is a connected 3-i-vertex-critical graphs, then $tough(G) \leq \frac{1}{2}$ by our results in Chapter 3 and in this section.

4.2 Results on matching

We now present a property of a 3-i-vertex-critical graph with a perfect matching.

Theorem 4.2.1. *If G is a connected $K_{1,7}$-free 3-i-vertex-critical graph of even order, then G has a perfect matching.*

Proof. Suppose to the contrary that G has no perfect matching. Then by Tutte's Theorem (Theorem 1.2) and the fact that $|V(G)|$ is even, there is a subset $S \subseteq V(G)$ such that $\omega_o(G - S) \geq |S| + 2$. Among of those sets, choose S_o such that $\omega_o(G - S_o) \geq |S_o| + 2$ and S_o is the minimum cutset. It follows by Theorems 3.2.2, 3.2.3 and 4.1.1 that $|S_o| \geq 4$. So $\omega_o(G - S_o) \geq 6$. Since S_o is minimum cutset, for each $x \in S_o$, $N_{C_i}(x) \neq \emptyset$. It follows that $\omega(G - S_o) \leq 6$ because G is $K_{1,7}$-free. Thus $|S_o| = 4$ and $\omega_o(G - S_o) = 6 = \omega(G - S_o)$. Since $\omega(G - S_o) = 6$ and $|I_x| = 2$ for all $x \in V(G)$, we have the following claim.

Claim 1 : $I_x \cap S_o \neq \emptyset$ for all $x \in V(G)$.

If there is a vertex $x \in S_o$ where $d_{S_o}(x) = 3$, then $I_x \cap S_o = \emptyset$ by Lemma 3.2.1 which contradicts Claim 1. Thus $\Delta(G[S_o]) \leq 2$. If $\Delta(G[S_o]) = 0$, then $\omega(G - S_o) \leq 3$ by Theorem 4.1.2 which contradicts the fact that $\omega(G - S_o) = 6$. Hence, $1 \leq \Delta(G[S_o]) \leq 2$. We now put $S = \{x_1, x_2, x_3, x_4\}$. Without loss of generality, we may assume that $x_1 x_2 \in E(G)$. Consider $G - x_1$. It is easy to see that $I_{x_1} \cap \{x_3, x_4\} \neq \emptyset$. We distinguish two cases.

Case 1 : $|I_{x_1} \cap S_o| = 1$

Without loss of generality, we may assume that $I_{x_1} \cap S_o = \{x_4\}$. Then $I_{x_1} - \{x_4\} \subseteq \bigcup_{i=1}^{t} V(C_i)$. Without loss of generality, we may assume that $I_{x_1} - \{x_4\} \subseteq V(C_1)$. It follows that $x_4 \succ_i \bigcup_{i=2}^{6} V(C_i)$. For $2 \leq i \leq 6$, let $y_i \in N_{C_i}(x_3)$. Then $y_i x_3, y_i x_4 \in E(G)$. By Claim 1 and Lemma 3.2.1, either $I_{y_i} \cap S_o = \{x_1\}$ or $I_{y_i} \cap S_o = \{x_2\}$ since I_{y_i} is independent. By Pigoenhole Principle (Theorem 1.1), either x_1 or x_2 belongs to at least three independent dominating sets, say $I_{y_{i'}}, I_{y_{i''}}$ and $I_{y_{i'''}}$ where $\{i', i'', i'''\} \subseteq \{2, 3, ..., 6\}$. Let $x^* \in \{x_1, x_2\}$ where $x^* \in I_{y_{i'}} \cap I_{y_{i''}} \cap I_{y_{i'''}}$. Then $x^* y_{i'}, x^* y_{i''}, x^* y_{i'''} \notin E(G)$. Thus the only vertex of $I_{y_{i'}} - \{x^*\}$ which belongs to $\bigcup_{i=1}^{6} V(C_i) - \{y_{i'}\}$ dominates $\{y_{i''}, y_{i'''}\}$. But this is not possible. Hence, Case 1 cannot occur.

Case 2 : $|I_{x_1} \cap S_o| = 2$

Then $I_{x_1} = \{x_3, x_4\}$. Without loss of generality, we may assume that $x_2 x_3 \in E(G)$. Consider $G - x_2$. By Claim 1 and Lemma 3.2.1, $I_{x_2} \cap S_o = \{x_4\}$ and $I_{x_2} - \{x_4\} \subseteq \bigcup_{i=1}^{6} V(C_i)$. Suppose that $I_{x_2} - \{x_4\} \subseteq V(C_1)$. Thus $x_4 \succ_i \bigcup_{i=2}^{6} V(C_i)$. Choose $y_i \in N_{C_i}(x_3)$ for $2 \leq i \leq 6$. Then $y_i x_3, y_i x_4 \in E(G)$. By Claim 1 and Lemma 3.2.1, either $I_{y_i} \cap S_o = \{x_1\}$ or $I_{y_i} \cap S_o = \{x_2\}$. By similar arguments as in the proof of Case 1, Case 2 cannot occor.

This completes the proof of our theorem.

$\square$

Bibliography

[1] S. Ao, *Independent domination critical graphs*, M.Sc.Dissertation, University of Victoria (1994).

[2] P.A. Brualdi, *Introductory Combinatorics*, Pearson Prentice Hall, United S-tates of America (2004).

[3] J. Clark and D.A. Holton, *A First Look at Graph Theory*, World Scientific Publishing Co. Pte. Ltd., Singapore (1991).

[4] G. Chartrand and O. R. Oellermann, *Applied and Algorithmic Graph Theory*, McGraw-Hill, Inc., Singapore (1993).

[5] L. Lovasz and M.D. Plummer, *Matching Theory*, Elsevier Science Publishers B.V., Netherland (1986).

[6] T. Wang, *On the diameter for various types of domination vertex critical graphs*, arXiv : 1304.0274v2[math.CO] 23 Apr 2013.

Biography

Name	Miss Sriphan Ruangthampisan
Address	38 Moo 6 Tambol Thatalat, Amphur Sampharn,
	Nakhonprathom, 73110
Institution Attended	
2010	Bachelor of Science in Mathematics,
	Silpakorn University
2015	Master of Science in Mathematics,
	Silpakorn University