NUMERICAL SOLUTIONS OF MAGNETIC FIELD RESPONSE FROM TWO DIMENSIONAL CONTINUOUSLY CONDUCTIVE GROUND

By

Yaowared Khonkhem

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Department of Mathematics
Graduate School, Silpakorn University
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ผลเฉลยเชิงตัวเลขของการตอบสนองทางสนามแม่เหล็ก
จากสภาพนำไฟฟ้าสองมิติอย่างต่อเนื่องใต้พื้นดิน

โดย
นางสาวเยาวเรศ ขนเข็ม

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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ภาควิชาคณิตศาสตร์
บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
ปีการศึกษา 2558

The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Numerical Solutions of Magnetic Field Response from Two Dimensional Continuously Conductive Ground” submitted by Miss Yaowared Khonkhem as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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(Associate Professor Panjai Tantatsanawong, Ph.D.)

Dean of Graduate School

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The Thesis Examination Committee

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(Warin Sripanya, Ph.D.)

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(Associate Professor Suabsagun Yooyuanyong, Ph.D.)

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YAOWARED KHONKHEM : NUMERICAL SOLUTIONS OF MAGNETIC FIELD RESPONSE FROM TWO DIMENSIONAL CONTINUOUSLY CONDUCTIVE GROUND. THESIS ADVISOR : ASSOC. PROF. SUABSAGUN YOOYUANYONG, Ph.D. 112 pp.

In this thesis, mathematical models of Finite Difference Method and Finite Element Method for the magnetic field response from a two dimensional continuously conductive ground are presented. The magnetic field at various locations are plotted by assuming the Earth structure having a two dimensional exponential conductivity profile. There is a source providing a Direct Current (DC) voltage and receiver on the ground surface. Finite Difference and Finite Element techniques are applied to solve the partial differential equation. MATLAB programing is used to perform both values and graphs of magnetic field at various locations. The results show the intensity of magnetic field for cross−section of the ground structure very well. The behaviour of magnetic field clearly performs very good relation to the conductive ground. The research results are very useful in geophysical exploration since normally we can measure magnetic field on the ground surface. The magnetic field intensity can be able to inform the location of the ore body under the ground via the technique of inverse problem.

Department of Mathematics Graduate School, Silpakorn University

Student's signature .... Academic Year 2015

Thesis Advisor's signature ....

56305207 : สาขาวิชาคณิตศาสตร์
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เยาวเรศ ขนเข็ม : ผลเฉลยเชิงตัวเลขของการตอบสนองทางสนามแม่เหล็กจากสภาพนำไฟฟ้าสองมิติอย่างต่อเนื่องใต้พื้นดิน. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : รศ.ดร. สืบสกุล อยู่ยืนยง. 112 หน้า.

วิทยานิพนธ์นี้ได้นำเสนอการศึกษาแบบจำลองทางคณิตศาสตร์โดยใช้วิธีสมาชิกจำกัดและวิธีผลต่างอันตะสำหรับการตอบสนองทางสนามแม่เหล็กจากพื้นดินที่มีสภาพนำไฟฟ้าแบบต่อเนื่องสองมิติ โดยการสมมติว่าโครงสร้างพื้นโลกมีสภาพนำไฟฟ้าเป็นแบบสองมิติอย่างต่อเนื่อง สนามแม่เหล็กที่ตอบสนองมาจากโครงสร้างโลกที่มีสภาพนำไฟฟ้าแบบเอกโปเนนเชียล 2 มิติ ซึ่งมีแหล่งกำเนิดที่เกิดจากไฟฟ้ากระแสตรงและเครื่องรับสัญญาณที่ถูกติดตั้งไว้บริเวณพื้นผิวดิน เทคนิคผลต่างอันตะและสมาชิกจำกัดถูกนำมาใช้เพื่อหาผลเฉลยเชิงตัวเลขของสมการเชิงอนุพันธ์ย่อย โปรแกรม MATLAB ถูกใช้สำหรับการคำนวณค่าสนามแม่เหล็กและเขียนกราฟหลายๆ ตำแหน่ง ผลลัพธ์แสดงความเข้มสนามแม่เหล็กสำหรับหน้าตัดพื้นดินอย่างดีมากกับสภาพนำไฟฟ้าของพื้นดิน ผลของงานวิจัยมีประโยชน์มากในวงการสำรวจทางธรณีฟิสิกส์ อันเนื่องจากปกติเราสามารถวัดสนามแม่เหล็กได้บนพื้นผิวดิน ความเข้มของสนามแม่เหล็กแสดงถึงตำแหน่งของแร่ธาตุใต้พื้นดินได้ โดยอาศัยเทคนิคของปัญหาย้อนกลับ

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Chapter 1

Introduction

At present, natural resources are utilized extensively such as minerals, petroleum and groundwater. Geophysical survey is very important to geological structure survey or exploring the natural resources. The purpose of the geophysical survey are knowing the geological features and physical properties (i.e. density, magnetization, elasticity and electric conductivity.) at the area that is interesting. These data can be used for planning to well drilling in the area to explore them even more effective. There are many ways and methods to explore the geological structure. For instance, gravitational, magnetic, seismic, electrical, electromagnetic and magnetometic resistivity.

The magnetometic resistivity method is a method of using the low-frequency magnetic field. The working principle is that when we leave the DC voltage to the electrode, it will cause a magnetic field radiate through the medium. The magnetic field can be measured by using magnetic filed receiver. The information of magnetic fields can be interpreted to show the basic information of the various physical properties of the target.

During the past several decades, many researchers such as Chen and Oldenburg [7] derived the magnetic field directly by solving a boundary value problem of a horizontally stratified layered Earth. Kim and Lee [5] derived a new resistivity kernel function for calculating apparent resistivity of a multilayered Earth with layers having exponentially varying conductivities. Siew and Yooyuanyong [11] also used this assumption to create the mathematical model of electromagnetic response of a conductive thin disc beneath an exponentially varying conductive overburden. Chumchob [8] employed this conductivity variation to formulate the mathematical model of electromagnetic sounding for a conductive circular cylinder ore body embedded in an inhomogeneous conducting half-space. Sripanya [14] derived solutions of the steady state magnetic field due to a DC current source in a layered Earth with some layer having exponentially or binomially or linearly varying conductivity. Tunnurak et al [12] derived the mathematical model of finite element method for the magnetic field of an exponential conductivity ground profile. The results are computed to find the value of the magnetic field at various locations of the ground.

All the relevant research works mentioned above are one dimentional conductivity profile. The two dimentional conductivity profile is more realistic than one dimensional profile. It challenges me to explore our problem with the use of numerical techniques such Finite Difference and Finite Element methods.

1.1 Outline of the Thesis

This thesis deals with a mathematical model for the magnetic field from a two dimensional continuously conductive ground. The magnetic field solutions are computed and plotted to show the intensity of the field at many location of the ground. The following outline is a list of that require further investigation:

Chapter 2, we present a geometric model of the problem and the governing equation is the Maxwell's equations.

Chapter 3, we present the numerical computations for finding approximate solutions. Finite Difference Method is introduced to calculate the magnetic field intensity at various locations. MATLAB program is used to compute and plot magnetic field intensity via the source-receiver spacing and the depth.

Chapter 4, we present the numerical computations for finding approximate solutions. Finite Element Method is introduced to calculate the magnetic field intensity at various locations. MATLAB program is used to compute and plot magnetic field intensity via the source-receiver spacing and the depth.

Finally, in Chapter 5, we sumarize the results of our work and also suggest the future works.

Chapter 2

Formulation of Mathematical Problem

Geometric model of the problem, which we considered, is shown in Figure 2.1.

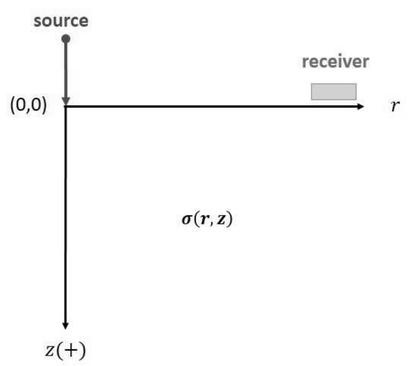


Figure 2.1: Geometric model of the problem.

Cylindrical coordinate system (r, ϕ, z) is introduced with the z-axis positive downward, where z represents the depth, $\hat{e}_r, \hat{e}_\phi$ and $\hat{e}_z$ are basis vectors in r, ϕ and z direction, respectively. We note that the plane $z = 0$ is the ground surface where the receiver is located at r from the source.

For the half space $z > 0$, it is a region of the ground where the two dimensional conductivity is given by

$$\sigma(r, z) = \sigma_0 e^{(az+br)},$$

here σ_0 is a positive constant and a, b are constants. This may represent the ground near seashore.

2.1 Governing equations

Consider the Maxwell's equations [14]

$$\nabla \times \vec{E} = \vec{0}, \tag{2.1}$$

$$\nabla \times \vec{H} = \sigma \vec{E}, \tag{2.2}$$

where $\vec{E}$ is the electric field intensity (Volt/meter), $\vec{H}$ is the magnetic field intensity (Ampere/meter), σ is conductivity (Siemen/meter). The gradient operator, in cylindrical coordinates, is defined by

$$\nabla = \frac{\partial \hat{e}_r}{\partial r} + \frac{1}{r}\frac{\partial \hat{e}_\phi}{\partial \phi} + \frac{\partial \hat{e}_z}{\partial z}.$$

Substituting equation (2.2) into (2.1), we obtain

$$\nabla \times \frac{1}{\sigma}(\nabla \times \vec{H}) = 0. \tag{2.3}$$

Note that

$$\frac{1}{\sigma}(\nabla \times \vec{H}) = \frac{1}{\sigma r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}.$$

Then

$$\frac{1}{\sigma}(\nabla \times \vec{H}) = \frac{1}{\sigma}[\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}]\hat{e}_r + \frac{1}{\sigma}[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}]\hat{e}_\phi + \frac{1}{\sigma}[\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi}]\hat{e}_z$$

and

$$\begin{aligned}\nabla \times \frac{1}{\sigma}(\nabla \times \vec{H}) = &[\frac{1}{r}\frac{\partial}{\partial \phi}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial (H_r)}{\partial \phi}) - \frac{\partial}{\partial z}\frac{1}{\sigma}(\frac{\partial (H_r)}{\partial z} - \frac{\partial (H_z)}{\partial r})]\hat{e}_r \\ &+ [\frac{\partial}{\partial z}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial (H_z)}{\partial \phi} - \frac{\partial (H_\phi)}{\partial z}) - \frac{\partial}{\partial r}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial (H_r)}{\partial \phi})]\hat{e}_\phi \\ &+ [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{1}{\sigma}(\frac{\partial (H_r)}{\partial z} - \frac{\partial (H_z)}{\partial r})) - \frac{1}{r}\frac{\partial}{\partial \phi}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial (H_z)}{\partial \phi} - \frac{\partial (H_\phi)}{\partial z})]\hat{e}_z.\end{aligned}$$

So, equation (2.3) becomes

$$\begin{aligned}0 = &[\frac{1}{r}\frac{\partial}{\partial \phi}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial (H_r)}{\partial \phi}) - \frac{\partial}{\partial z}\frac{1}{\sigma}(\frac{\partial (H_r)}{\partial z} - \frac{\partial (H_z)}{\partial r})]\hat{e}_r \\ &+ [\frac{\partial}{\partial z}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial (H_z)}{\partial \phi} - \frac{\partial (H_\phi)}{\partial z}) - \frac{\partial}{\partial r}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial (H_r)}{\partial \phi})]\hat{e}_\phi \\ &+ [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{1}{\sigma}(\frac{\partial (H_r)}{\partial z} - \frac{\partial (H_z)}{\partial r})) - \frac{1}{r}\frac{\partial}{\partial \phi}\frac{1}{\sigma}(\frac{1}{r}\frac{\partial (H_z)}{\partial \phi} - \frac{\partial (H_\phi)}{\partial z})]\hat{e}_z, \qquad (2.4)\end{aligned}$$

where H_r, H_ϕ and H_z are the components of $\vec{H}$ in $\hat{e}_r, \hat{e}_\phi$ and $\hat{e}_z$ directions, respectively. Since the problem is axis symmetric and $\vec{H}$ has only the azimuthal component in cylindrical coordinates, for simplicity, we use H to represent the azimuthal component in following derivations. Simplifying equation (2.4) yields

$$-\frac{\partial}{\partial z}(\frac{1}{\sigma}\frac{\partial (H)}{\partial z}) - \frac{\partial}{\partial r}(\frac{1}{\sigma r}\frac{\partial (rH)}{\partial r}) = 0. \qquad (2.5)$$

Substituting $\sigma(r, z) = \sigma_0 e^{(az+br)}$ to equation (2.5) , we obtain

$$\frac{\partial^2 H}{\partial z^2} - a\frac{\partial H}{\partial z} + \frac{\partial^2 H}{\partial r^2} + (\frac{1}{r} - b)\frac{\partial H}{\partial r} - (\frac{b}{r} + \frac{1}{r^2})H = 0. \qquad (2.6)$$

Chapter 3

Finite Difference Method for Magnetic Field Response

In this chapter, the numerical methods for finding the approximate solution is presented by using Finite Difference Method (FDM).

The FDM proceeds by replacing those derivatives in the DEs by finite difference approximations, which are algebraic in form. They relate the value of dependent variable at a point in the solution region to the values at some neighboring points. Finite Difference Method basically involves the following step [6,9]:
(1) Discretize the domain region Ω into a mesh of discrete points called nodes.
(2) Approximate all derivatives using the finite difference approximations. In this step the DE is approximated by a large system of algebraic equations.
(3) Solve the linear or nonlinear system of algebraic equations.

We now rewrite equation (2.6) as

$$\frac{\partial^2 H(r,z)}{\partial z^2} - a\frac{\partial H(r,z)}{\partial z} + \frac{\partial^2 H(r,z)}{\partial r^2} + (\frac{1}{r} - b)\frac{\partial H(r,z)}{\partial r} - (\frac{b}{r} + \frac{1}{r^2})H(r,z) = 0,\ for (r,z) \in \Omega, \tag{3.1}$$

where $r \in [10, 190]$ and $z \in [0, 180]$.

$$\begin{aligned} H(r,z) &= 0 && on \quad \partial\Omega_1, \\ H(r,z) &= -4.4 \times 10^{-5} z + 0.0008 && on \quad \partial\Omega_2, \\ H(r,z) &= H(\bar{n}-1, 0), \bar{n} = 1, 2, \ldots, 10 && on \quad \partial\Omega_3, \\ H(r,z) &= -8.83 \times 10^{-5} z + 0.0159 && on \quad \partial\Omega_4. \end{aligned} \tag{3.2}$$

The boundary conditions of problem (3.1) and the notation of the magnetic field intensity in boundary domain modeled, using square elements, are shown in Figure 3.1.

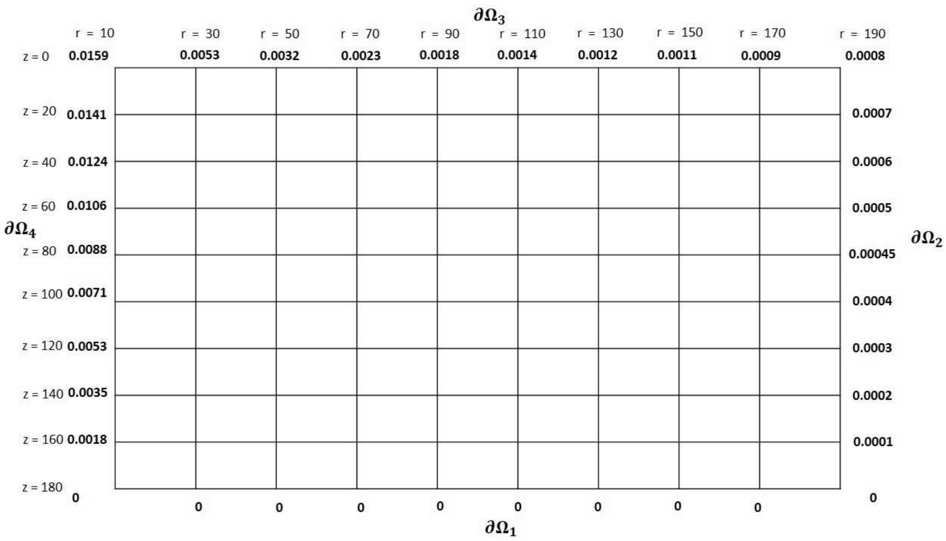


Figure 3.1: Boundary conditions of the Earth structure.

The values of magnetic field on $\partial\Omega_1, \partial\Omega_2, \partial\Omega_3$ and $\partial\Omega_4$ are obtained from [12]. See Figure 3.2 for the case $\hat{m}_r = \hat{m}_z = \hat{m}$ and $\hat{m} = 8$ points, where $\hat{m}_r$ and $\hat{m}_z$ are interior points in r and z direction, respectively. $\hat{m}$ is interior points. Mesh width or the distance between grid points in this chapter is 20 m.

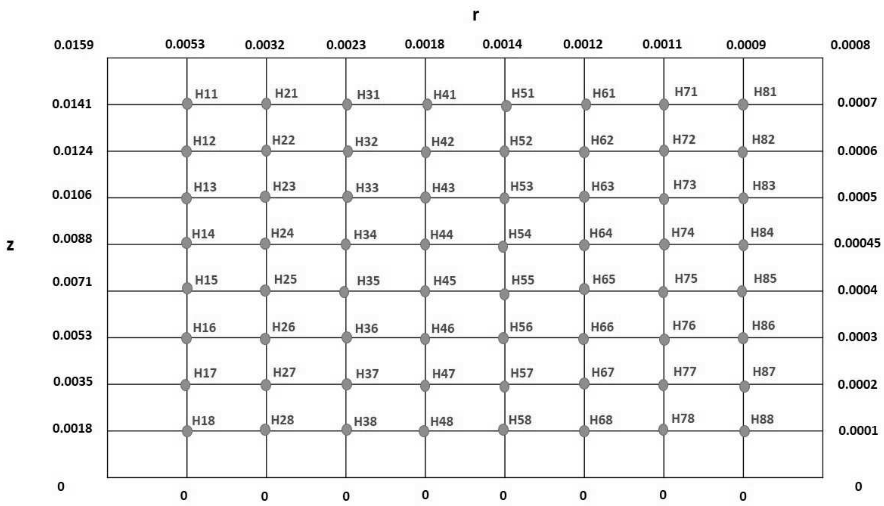


Figure 3.2: Discretizing the domain Ω using a uniform grid.

We define $H_{i,j} = H(r_i, z_j)$ to denote the associated grid function. To discretize the PDE in (3.1), we replace the $r-$ and $z-$derivatives with the central finite difference approximations as follows,

$$\frac{[H_{i,j-1} - 2H_{i,j} + H_{i,j+1}]}{h_z^2} - a\frac{[H_{i,j+1} - H_{i,j-1}]}{2h_z} + \frac{[H_{i-1,j} - 2H_{i,j} + H_{i+1,j}]}{h_r^2}$$
$$+ (\frac{1}{r_i} - b)\frac{[H_{i+1,j} - H_{i-1,j}]}{2h_r} - (\frac{b}{r_i} + \frac{1}{r_i^2})H_{i,j} = 0. \quad (3.3)$$

This leads us to define the numerical approximation $H_{i,j}$ as the solution of the linear system of $\hat{m}^2 = 64$ equations.

The linear system can be written in matrix form as

$$A\bar{H} = F, \quad (3.4)$$

where $\bar{H}$ represents the unknown vector magnetic field such that

$\bar{H} = [H_{11}, H_{12}, \ldots, H_{18}, H_{21}, H_{22}, \ldots, H_{28}, \ldots\ldots, H_{81}, H_{82}, \ldots, H_{88}]^T_{64\times1}$ and

F is a constant column vector.

The solution vector $\bar{H}$ consists of all interior points (the unknowns). Matrix A in equation (3.4) has the form

$$A = \begin{bmatrix} B_1 & D_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 & B_2 & D_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_3 & B_3 & D_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3 & B_3 & D_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_5 & B_5 & D_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_6 & B_6 & D_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_7 & B_7 & D_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_8 & B_8 \end{bmatrix}_{64\times 64},$$

where

$$B_i = \begin{bmatrix} m_i & p & 0 & 0 & 0 & 0 & 0 & 0 \\ l & m_i & p & 0 & 0 & 0 & 0 & 0 \\ 0 & l & m_i & p & 0 & 0 & 0 & 0 \\ 0 & 0 & l & m_i & p & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m_i & p & 0 & 0 \\ 0 & 0 & 0 & 0 & l & m_i & p & 0 \\ 0 & 0 & 0 & 0 & 0 & l & m_i & p \\ 0 & 0 & 0 & 0 & 0 & 0 & l & m_i \end{bmatrix}_{8\times 8},$$

$$l = \frac{1}{h_z^2} + \frac{a}{2h_z},$$

$$m_i = -\left(\frac{2}{h_z^2} + \frac{2}{h_r^2} + \frac{1}{r_i^2} + \frac{b}{r_i}\right),$$

$$p = \frac{1}{h_z^2} - \frac{a}{2h_z} \quad \text{for } i = 1, 2, \ldots, 8,$$

$$C_i = \begin{bmatrix} n_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_i \end{bmatrix}_{8\times 8},$$

$$n_i = \frac{1}{h_r^2} - \frac{1}{2r_ih_r} + \frac{b}{2h_r}, \text{ for } i = 2, 3, \ldots, 8,$$

$$D_i = \begin{bmatrix} t_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_i \end{bmatrix}_{8\times 8},$$

and
$$t_i = \frac{1}{h_r^2} + \frac{1}{2r_ih_r} - \frac{b}{2h_r}, \text{ for } i = 1, 2, \ldots, 7.$$

The boundary conditions are absorbed into F as

$$F = \begin{bmatrix} f_1 \\ -l(0.0032) \\ f_2 \\ -l(0.0023) \\ f_2 \\ -l(0.0018) \\ f_2 \\ -l(0.0014) \\ f_2 \\ -l(0.0012) \\ f_2 \\ -l(0.0011) \\ f_2 \\ f_3 \end{bmatrix}_{64\times 1},$$

where f_2 is zeros matrix size 7×1,

$$f_1 = \begin{bmatrix} -n_1(0.0141) - l(0.0053) \\ -n_1(0.0124) \\ -n_1(0.0106) \\ -n_1(0.0088) \\ -n_1(0.0071) \\ -n_1(0.0053) \\ -n_1(0.0035) \\ -n_1(0.0018) \end{bmatrix}_{8\times 1},$$

for $$n_1 = \frac{1}{h_r^2} - \frac{1}{2r_1 h_r} + \frac{b}{2h_r},$$

$$l = \frac{1}{h_z^2} + \frac{a}{2h_z},$$

and

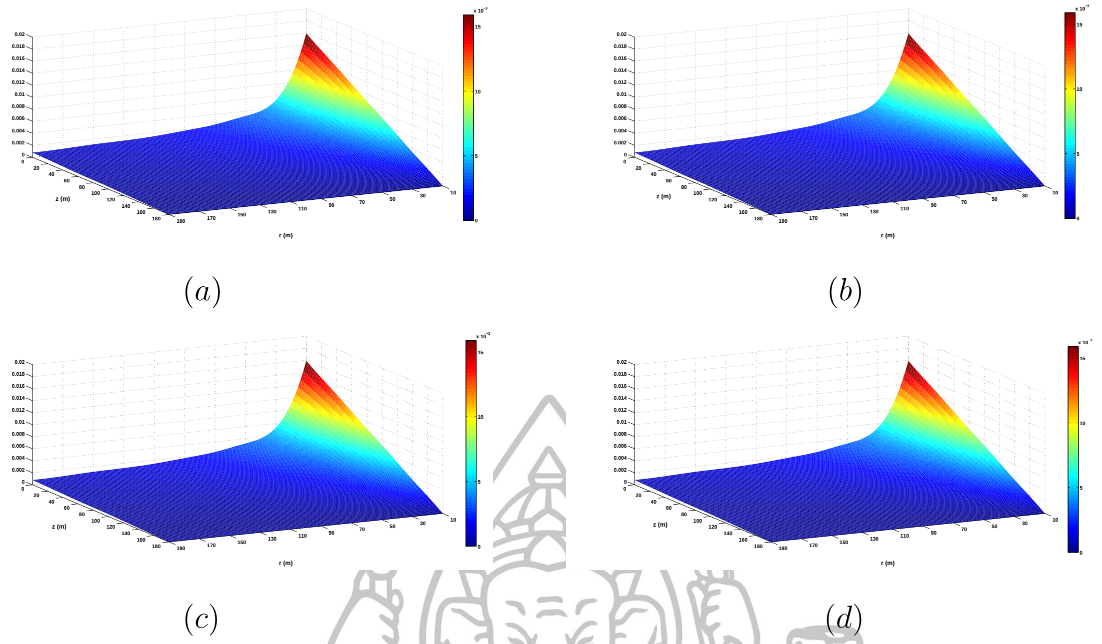
$$f_3 = \begin{bmatrix} -l(0.0009) - t_8(0.0007) \\ -t_8(0.0006) \\ -t_8(0.0005) \\ -t_8(0.00045) \\ -t_8(0.0004) \\ -t_8(0.0003) \\ -t_8(0.0002) \\ -t_8(0.0001) \end{bmatrix}_{8\times 1},$$

for
$$t_8 = \frac{1}{h_r^2} + \frac{1}{2r_8 h_r} - \frac{b}{2h_r}.$$

3.1 Numerical Experiments

Numerical results of the magnetic field intensity from equation (3.1) are obtained by using the Finite Difference Method. There is a source providing a DC voltage of direct current $I = 1A$ and receiver on the ground surface which picks up the signal from $r = 10$ to $r = 190\ m$. The depth z start from the ground surface $z = 0$ to $z = 180\ m$. The grid size $h = 20\ m$. a and b are given constants . The magnetic field intensity is computed by using MATLAB programing.

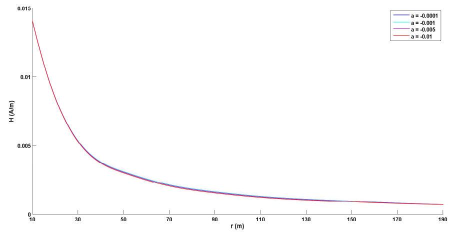
Consider for the case of an exponentially decreasing conductivity $\sigma(r, z) = \sigma_0 e^{(az+br)}$ when $a < 0$ and $b = 0$, the graphs of the relationship between magnetic field intensity and spacing of source-receiver at various depths are plotted as shown in Figure 3.3 and 3.4.



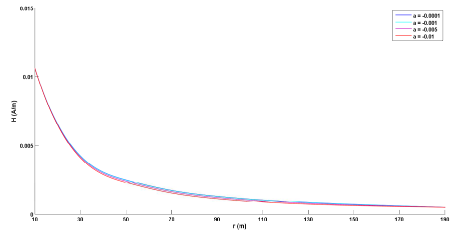
(a) (b) (c) (d)

Figure 3.3: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0$, a is varied and z is fixed (a) $a = -0.0001 \ m^{-1}$ (b) $a = -0.001 \ m^{-1}$ (c) $a = -0.005 \ m^{-1}$ and (d) $a = -0.01 \ m^{-1}$.

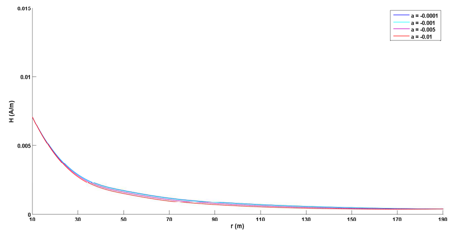
From Figure 3.3 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and $-0.01 \ m^{-1}$, respectively, we can see that the values of magnetic field decrease exponentially as r and z increase. The values of magnetic field decrease as a decreases.



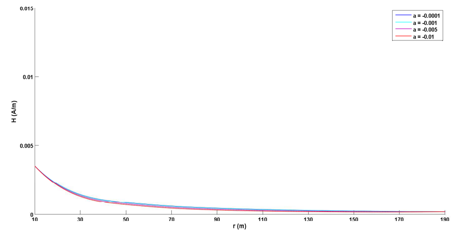
(a)



(b)



(c)

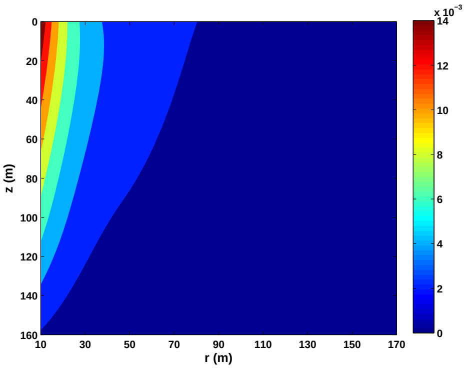


(d)

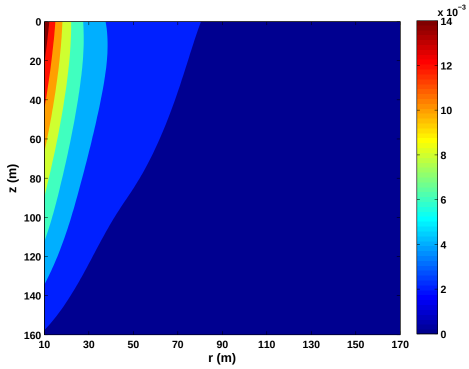
Figure 3.4: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0$, a varies from -0.0001, -0.001, -0.005 and -0.01 m^{-1} and z is fixed. (a) $z = 20$ m (b) $z = 60$ m (c) $z = 100$ m and (d) $z = 140$ m.

From Figure 3.4 (a) to (d) represents the values of magnetic field which are plotted against r whereas a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic field decrease to zero and have values near zero where z increases as a varies.

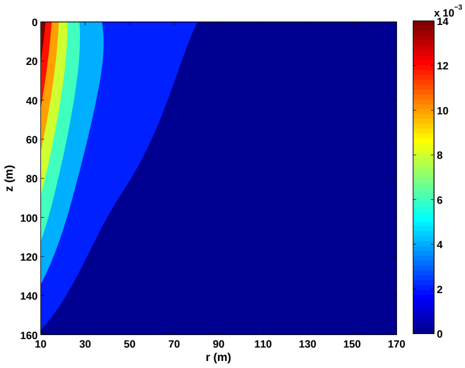
Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 3.5.



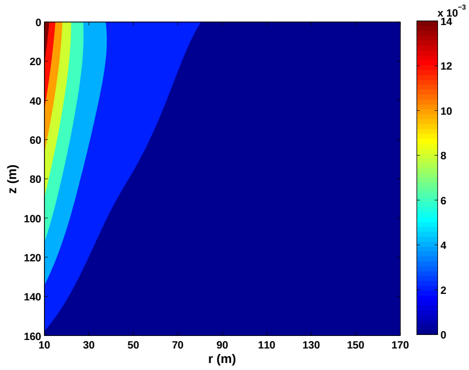
(a)



(b)



(c)

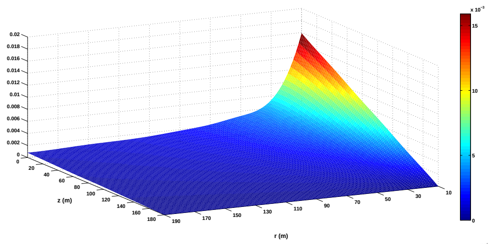


(d)

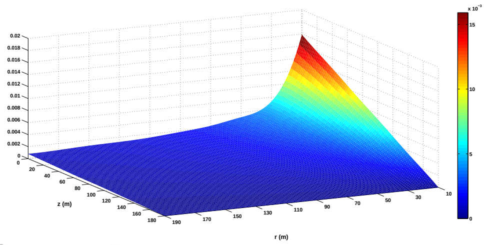
Figure 3.5: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0$, (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

From Figure 3.5 (a) to (d), where $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1}, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area where the values of magnetic field is low.

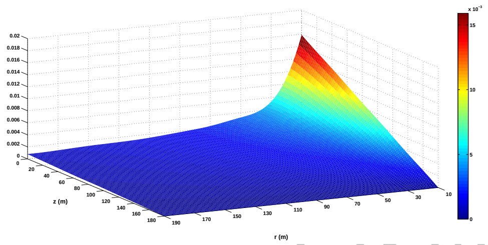
Consider for the case of an exponentially increasing conductivity $\sigma(r, z) = \sigma_0 e^{(az+br)}$ when $a > 0$ and $b = 0$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 3.6 and 3.7.



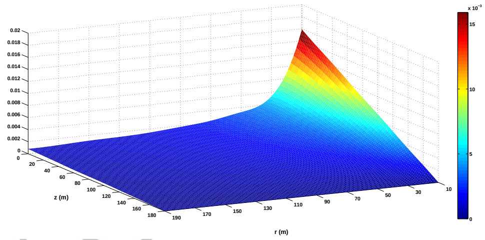
(a)



(b)



(c)



(d)

Figure 3.6: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0$, a varies and z is fixed (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

From Figure 3.6 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1}, respectively, we can see that the values of magnetic fields decrease exponentially as r and z increase. The values of magnetic fields increase whereas a increases. The results agree to Tunnurak et al. [12].

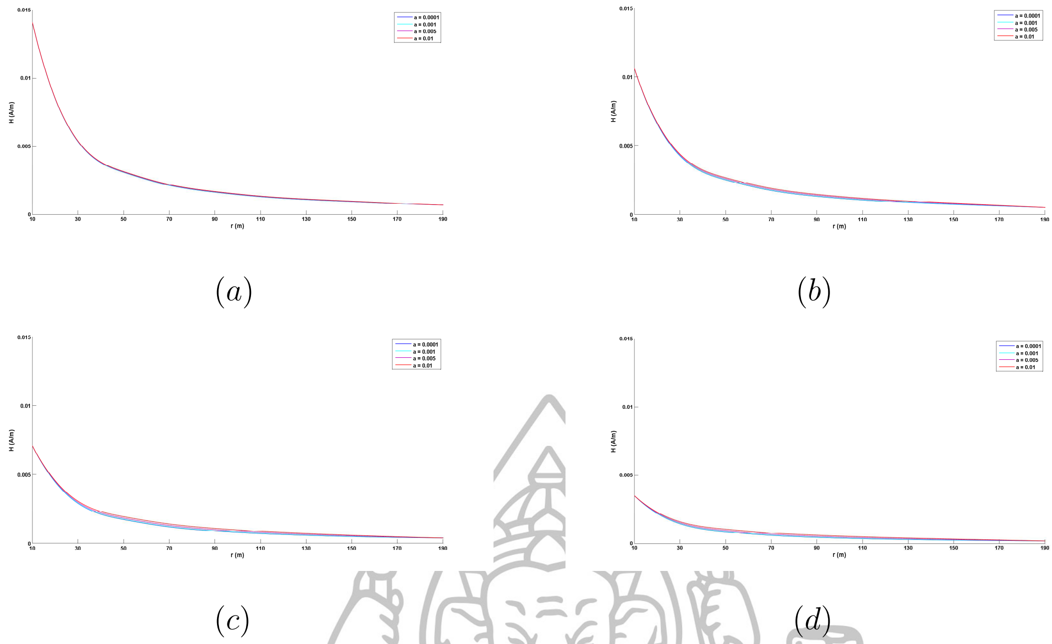


Figure 3.7: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0$, a varies from 0.0001, 0.001, 0.005 and 0.01 m^{-1} and z is fixed. (a) $z = 20\ m$ (b) $z = 60\ m$ (c) $z = 100\ m$ and (d) $z = 140\ m$.

From Figure 3.7 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and $140\ m$, respectively. We can see that the values of magnetic fields where $a = 0.0001, 0.001, 0.005$ and $0.01\ m^{-1}$ decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic fields decrease to zero and has value near zero where z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 3.8.

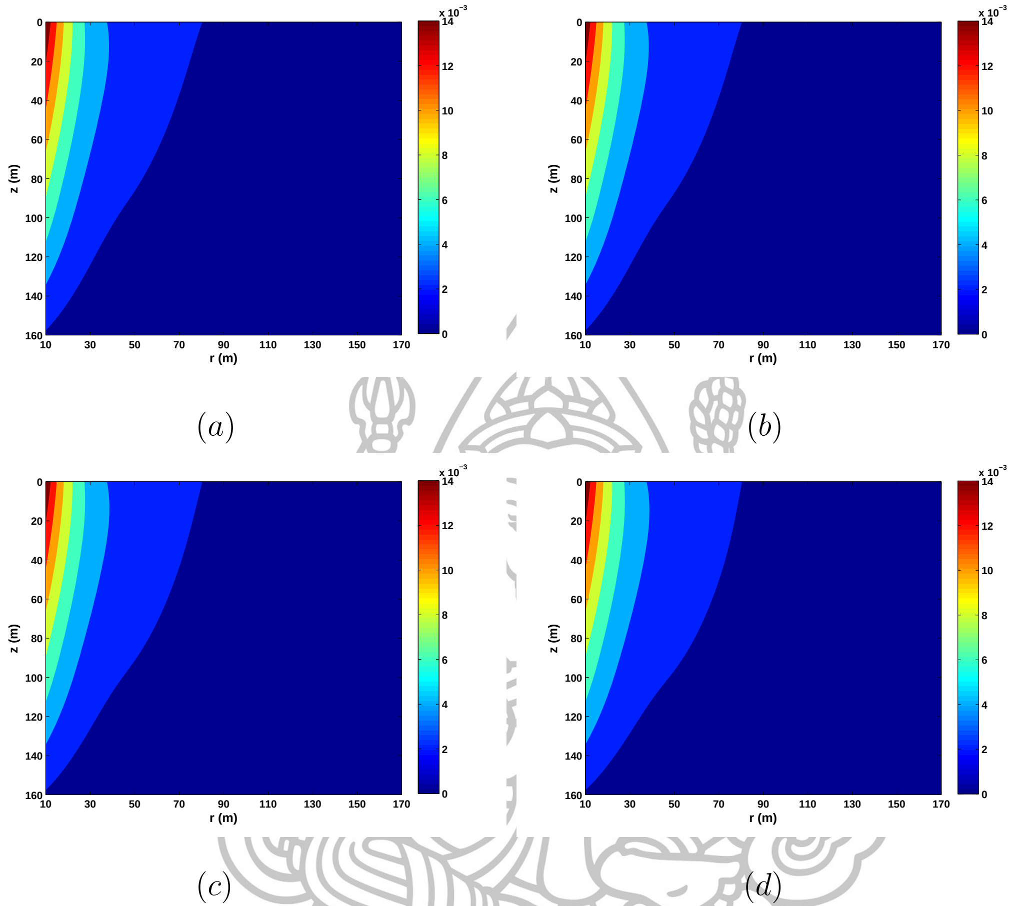


Figure 3.8: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0$, (a) $a = 0.0001 \ \ m^{-1}$ (b) $a = 0.001 \ \ m^{-1}$ (c) $a = 0.005 \ m^{-1}$ and (d) $a = 0.01 \ m^{-1}$.

From Figure 3.8 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and $0.01 \ m^{-1}$, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area where the values of magnetic field is low. The results agree to Tunnurak et al. [12].

Consider for the case of an exponentially decreasing conductivity $\sigma(r,z) = \sigma_0 e^{(az+br)}$ when $a < 0$ and $b = -0.001$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 3.9 and 3.10.

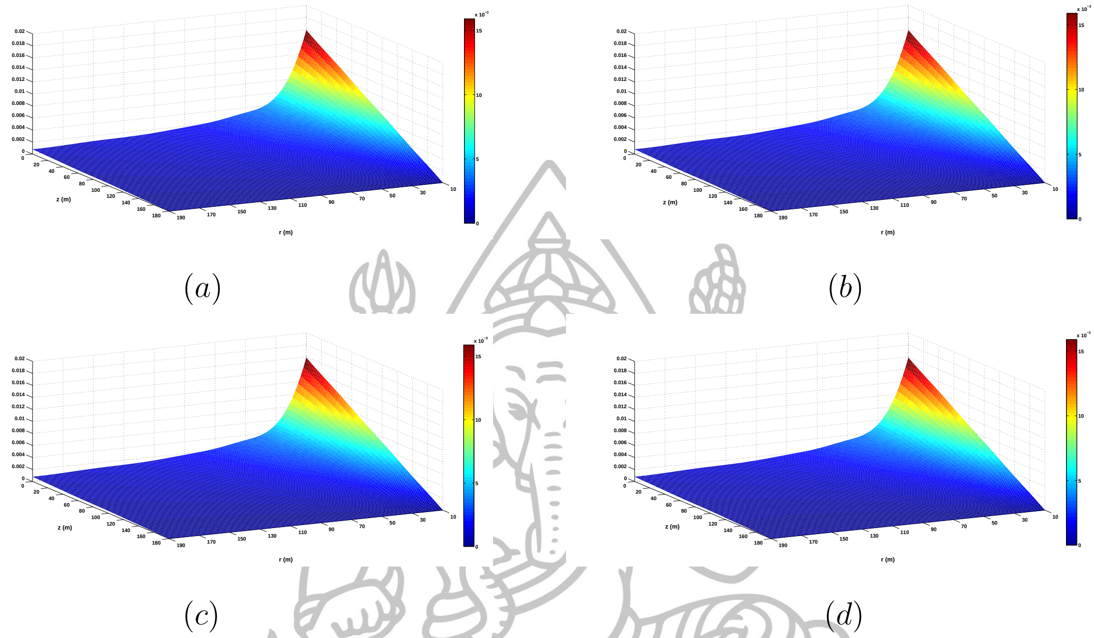
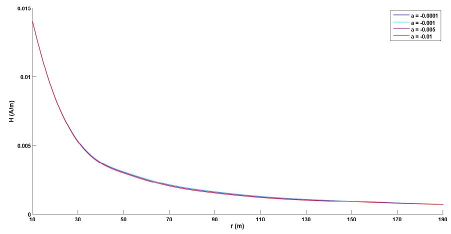
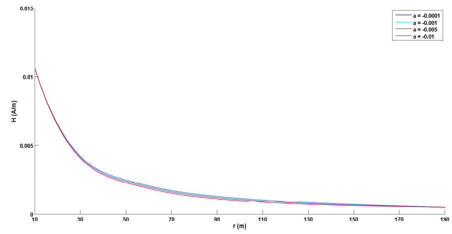


Figure 3.9: Graphs of the magnetic field intensity via distance of receiver from source where $b = -0.001$, a varies and z is fixed (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

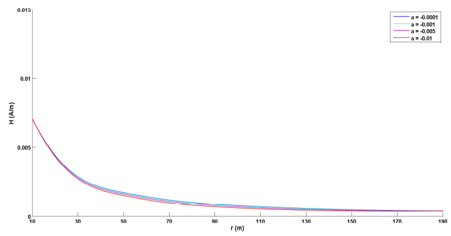
From Figure 3.9 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1}, respectively, we can see that the values of magnetic field decrease exponentially as r and z increase. The values of magnetic field decrease where a decreases.



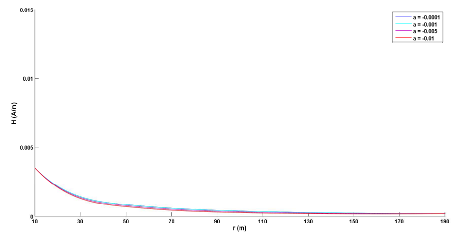
(*a*)



(*b*)



(*c*)

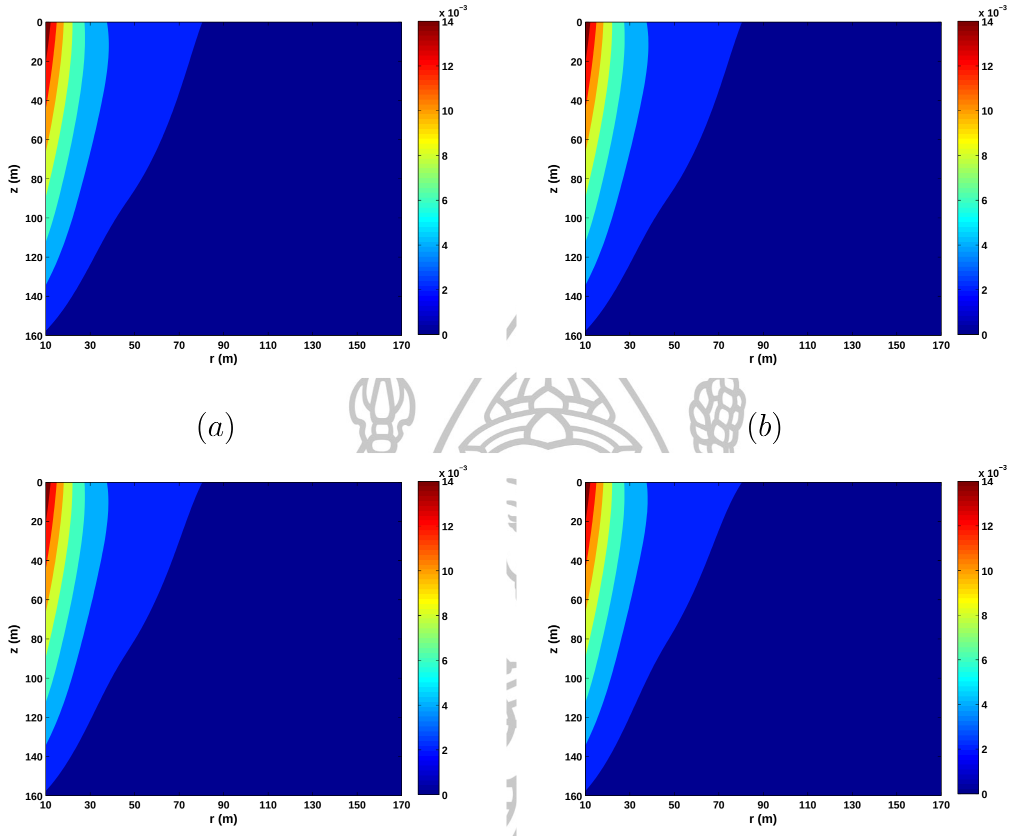


(*d*)

Figure 3.10: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = -0.001$, a varies from -0.0001, -0.001, -0.005 and -0.01 m^{-1} as z is fixed. (a) $z = 20$ m (b) $z = 60$ m (c) $z = 100$ m and (d) $z = 140$ m.

From Figure 3.10 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic field decrease to zero and have values near zero where z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 3.11.



(c) (d)

Figure 3.11: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = -0.001$, (a) $a = -0.0001 \;\; m^{-1}$ (b) $a = -0.001 \;\; m^{-1}$ (c) $a = -0.005 \; m^{-1}$ and (d) $a = -0.01 \; m^{-1}$.

From Figure 3.11 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and $-0.01 \; m^{-1}$, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area where the values of magnetic field is low.

Consider for the case of an exponentially increasing conductivity $\sigma(r,z) = \sigma_0 e^{(az+br)}$ when $a > 0$ and $b = 0.001$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 3.12 and 3.13.

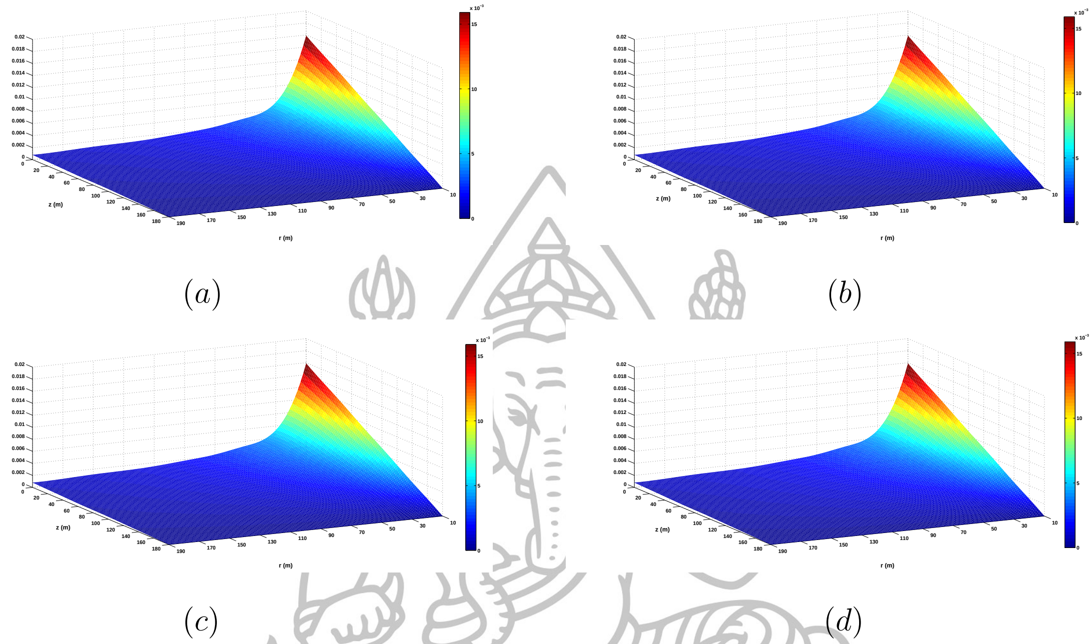
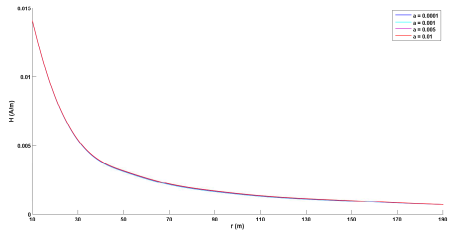
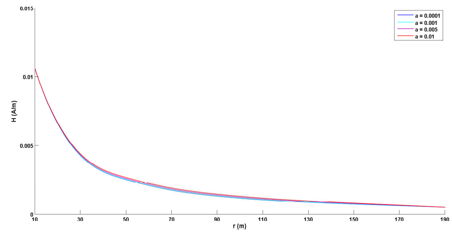


Figure 3.12: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0.001$, a varies and z is fixed (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

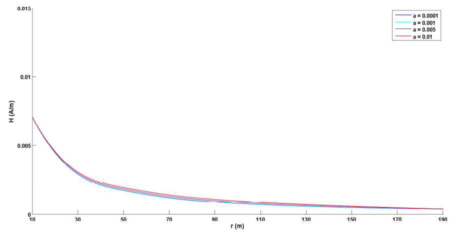
From Figure 3.12 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1}, respectively, we can see that the values of magnetic fields decrease exponentially as r and z increase. The values of magnetic fields increase where a increases.



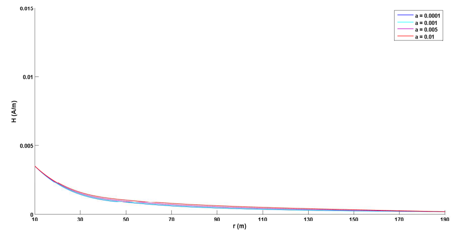
(a)



(b)



(c)



(d)

Figure 3.13: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0.001$, a varies from 0.0001, 0.001, 0.005 and 0.01 m^{-1} and z is fixed. (a) $z = 20\ m$ (b) $z = 60\ m$ (c) $z = 100\ m$ and (d) $z = 140\ m$.

From Figure 3.13 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic fields decrease to zero and has value near zero where z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 3.14.

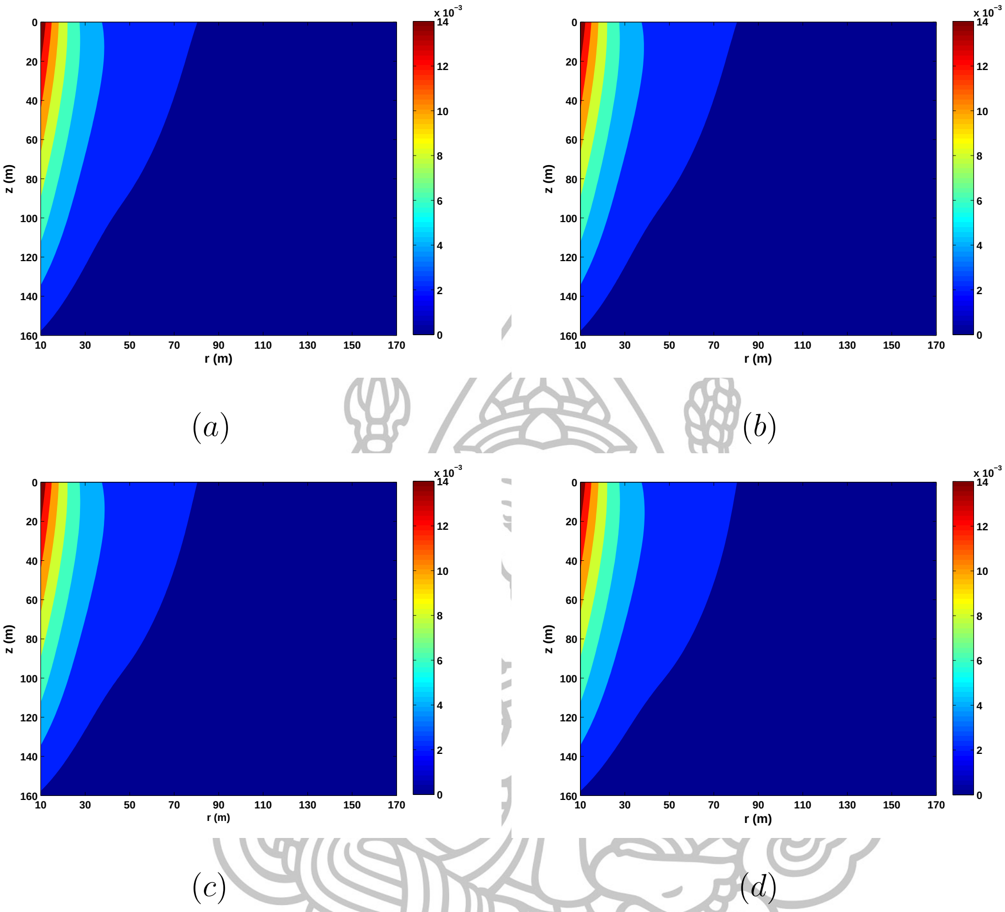


Figure 3.14: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0.001$, (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

From Figure 3.14 (a) to (d), when $a = 0.01, 0.05, 0.1, 0.2$ and 0.3 m^{-1}, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area where the values of magnetic field is low.

3.2 Summarize

In this chapter, we present a mathematical model by using the Magnetometric Resistivity Method with 2−dimensional continuously conductivity model as $\sigma(r,z) = \sigma_0 e^{(az+br)}$. The relationship between magnetic field and electric field are used by considering Maxwell's equations. The magnetic field intensity is obtained by solving partial differential equation. The solution are obtained by using Finite Difference Method. MATLAB program is used to calculate and plot graph for the value of magnetic field intensity. The behavior of magnetic field intensity will be performed at different depths and locations. In our research, the behavior of magnetic field decreases to zero when the depth of soil increases. As well as the case of increasing the space between source - receiver, the magnetic field decreases to zero too. The values of a and b are important role for the conduction of the ground and effect to the magnetic field quantities as well. For the high conductive ground (a and $b > 0$), the response of magnetic field will be very strong. In the opposite direction (a and $b < 0$), the response field will be very weak.

Chapter 4

Finite Element Method for Magnetic Field Response

In this chapter, the numerical method for finding the approximate solution is presented by using Finite Element Method (FEM).

The FEM is a numerical method for finding approximate solutions to the differential equations. Finite Element Method basically involves the following steps [2,10,12,13] :
(1) Variational formulation of the given problem.
(2) Discretization using the FEM : Discretize the domain Ω into a finite number of element, then find H.
(3) Solution of the discrete problem : Approximate using the Finite Element approximations.
(4) Solve a system of equations for variable H.

Using equation (3.1), our problem can be written as

$$\Delta H(r,z) - a\frac{\partial H(r,z)}{\partial z} + (\frac{1}{r} - b)\frac{\partial H(r,z)}{\partial r} - (\frac{b}{r} + \frac{1}{r^2})H(r,z) = 0,\ for(r,z) \in \Omega, \quad (4.1)$$

where $\Omega \in [10,190] \times [0,180]$ and $X \in [r,z]$ with the boundary conditions

$$\begin{aligned} H(r,z) &= 0 && on \quad \partial\Omega_1,\\ H(r,z) &= -4.4 \times 10^{-5} z + 0.0008 && on \quad \partial\Omega_2,\\ H(r,z) &= H(\bar{n}-1,0), \bar{n} = 1,2,\ldots,10 && on \quad \partial\Omega_3,\\ H(r,z) &= -8.83 \times 10^{-5} z + 0.0159 && on \quad \partial\Omega_4. \end{aligned} \quad (4.2)$$

Solution of equation (4.1) can be determined by first deriving the residual error

$$R(X) = L[H] - f(X),$$

where L is linear differential operator, R is residual error, H is magnetic field and $f(X)$ is known function.

$$L[H] := \Delta H(r,z) - a\frac{\partial H(r,z)}{\partial z} + (\frac{1}{r} - b)\frac{\partial H(r,z)}{\partial r} - (\frac{b}{r} + \frac{1}{r^2})H(r,z).$$

So from equation (4.1) residual error can be written as

$$R(X) = \Delta H(r,z) - a\frac{\partial H(r,z)}{\partial z} + (\frac{1}{r} - b)\frac{\partial H(r,z)}{\partial r} - (\frac{b}{r} + \frac{1}{r^2})H(r,z). \quad (4.3)$$

The method of weighted residual is applied to our problem. We multiply (4.3) by a test function or a weight function v and integrate the weighted residual error over Ω. Setting the total weighted residual error to zero, we shall find $H \in \tilde{V}$ such that

$$\int_\Omega R(X) v \; d\Omega = 0, \quad \forall v \in V, \quad (4.4)$$

where $V = \{v \in \mathbf{H^1}(\Omega) : v$ is a continuous function on $\Omega, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial z}$ are piecewise continuous on Ω and $v = 0$ on $\partial\Omega\}$, $\tilde{V} = \{H \in \mathbf{H^1}(\Omega) : H$ is a continuous function on $\Omega, \frac{\partial H}{\partial r}$ and $\frac{\partial H}{\partial z}$ are piecewise continuous on Ω and H satisfies conditions (4.2)$\}$,

$\mathbf{H^1}(\Omega) = \{v \in \mathbf{L_2}(\Omega) : \nabla v \in \mathbf{L_2}(\Omega)\ \}$ and $\mathbf{L_2}(\Omega) = \{v : v$ is defined on Ω and $\int_\Omega v^2 dX < \infty\}$. So equation (4.4) can be written as

$$\int_\Omega \left[\Delta H(r,z) - a\frac{\partial H(r,z)}{\partial z} + (\frac{1}{r} - b)\frac{\partial H(r,z)}{\partial r} - (\frac{b}{r} + \frac{1}{r^2})H(r,z) \right] v\ d\Omega = 0.$$

$$\int_\Omega v\Delta H(r,z)\ d\Omega - \int_\Omega av\frac{\partial H(r,z)}{\partial z}\ d\Omega + \int_\Omega (\frac{1}{r} - b)v\frac{\partial H(r,z)}{\partial r}\ d\Omega - \int_\Omega (\frac{b}{r} + \frac{1}{r^2})vH(r,z)\ d\Omega = 0. \tag{4.5}$$

Consider the first term of above equation

$$v\Delta H(r,z) = v\nabla \cdot \nabla H(r,z),$$

using the vector product and the divergence theorem

$$v\nabla \cdot \nabla H(r,z) = \nabla \cdot (v\nabla H) - \nabla H \cdot \nabla v,$$

Then we have

$$\int_\Omega v\Delta H(r,z)\ d\Omega = \oint_{\partial\Omega} v\frac{\partial H(r,z)}{\partial \eta} d\Omega - \int_\Omega \nabla H \cdot \nabla v\ d\Omega.$$

The equation (4.5) now becomes

$$\oint_{\partial\Omega} v\frac{\partial H(r,z)}{\partial \eta} d\Omega - \int_\Omega \nabla H \cdot \nabla v\ d\Omega - \int_\Omega av\frac{\partial H(r,z)}{\partial z}\ d\Omega + \int_\Omega (\frac{1}{r} - b)v\frac{\partial H(r,z)}{\partial r}\ d\Omega - \int_\Omega (\frac{b}{r} + \frac{1}{r^2})vH(r,z)\ d\Omega = 0.$$

Since $v \in V$, $v = 0$ on $\partial\Omega$, we have

$$\int_\Omega \nabla H \cdot \nabla v\ d\Omega + \int_\Omega av\frac{\partial H}{\partial z}\ d\Omega - \int_\Omega (\frac{1}{r} - b)v\frac{\partial H}{\partial r}\ d\Omega + \int_\Omega (\frac{b}{r} + \frac{1}{r^2})vH\ d\Omega = 0.$$

Thus, variational statement can be written as

Find $H \in \tilde{V}$ such that

$$\int_\Omega \nabla H \cdot \nabla v\ d\Omega + \int_\Omega av\frac{\partial H}{\partial z}\ d\Omega - \int_\Omega (\frac{1}{r} - b)v\frac{\partial H}{\partial r}\ d\Omega + \int_\Omega (\frac{b}{r} + \frac{1}{r^2})vH\ d\Omega = 0,\ \forall v \in V. \tag{4.6}$$

We shall now construct a finite-dimensional subspace V_h of V and $\tilde{V_h}$ of $\tilde{V}$. $V_h \subset V \subseteq \mathbf{H^1}(\Omega)$ and $\tilde{V_h} \subset \tilde{V} \subseteq \mathbf{H^1}(\Omega)$. The problem the becomes :

Find $H_h(X) \in \tilde{V_h}$ such that

$$\int_\Omega \nabla H_h \cdot \nabla v \, d\Omega + \int_\Omega av\frac{\partial H_h}{\partial z}\, d\Omega - \int_\Omega (\frac{1}{r} - b)v\frac{\partial H_h}{\partial r}\, d\Omega + \int_\Omega (\frac{b}{r} + \frac{1}{r^2})vH_h \, d\Omega = 0, \; \forall v \in V_h. \tag{4.7}$$

Let $\{\varphi_i(X)\}_{i=1}^n$ be the basis function of both $\tilde{V_h}$ and V_h such that

$$\varphi_j(X_i) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Then our solution H_h and v can be expressed as

$$H_h(X) = \sum_{j=1}^{n} H_j \varphi_j(X)$$

and

$$v(X) = \sum_{i=1}^{n} \beta_i \varphi_i(X),$$

where $H_j = H_h(X_j)$ and $\beta_i = v(X_i)$.

The Finite Element approximation (4.7) becomes

$$\sum_{j=1}^{n} \left[\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, d\Omega + \int_\Omega a\varphi_i \frac{\partial \varphi_j}{\partial z}\, d\Omega - \int_\Omega (\frac{1}{r} - b)\varphi_i \frac{\partial \varphi_j}{\partial r}\, d\Omega + \int_\Omega (\frac{b}{r} + \frac{1}{r^2})\varphi_i\varphi_j \, d\Omega \right] H_j = 0, \tag{4.8}$$

for $i = 1, 2, \ldots, n$.

Equation (4.8) can be written in matrix form as

$$A\bar{H} = F,$$

where $\bar{H} = [H_1, H_2, ..., H_n]^T$, $F = [f_1, f_2, ..., f_n]^T$, and $A = A_1 + A_2 + A_3 + A_4$.

Here $A_1 = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, d\Omega$, $A_2 = \int_\Omega a\varphi_i \frac{\partial \varphi_j}{\partial z}\, d\Omega$, $A_3 = -\int_\Omega (\frac{1}{r} - b)\varphi_i \frac{\partial \varphi_j}{\partial r}\, d\Omega$

and $A_4 = \int_\Omega (\frac{b}{r} + \frac{1}{r^2})\varphi_i\varphi_j \, d\Omega$.

4.1 Rectangular Elements

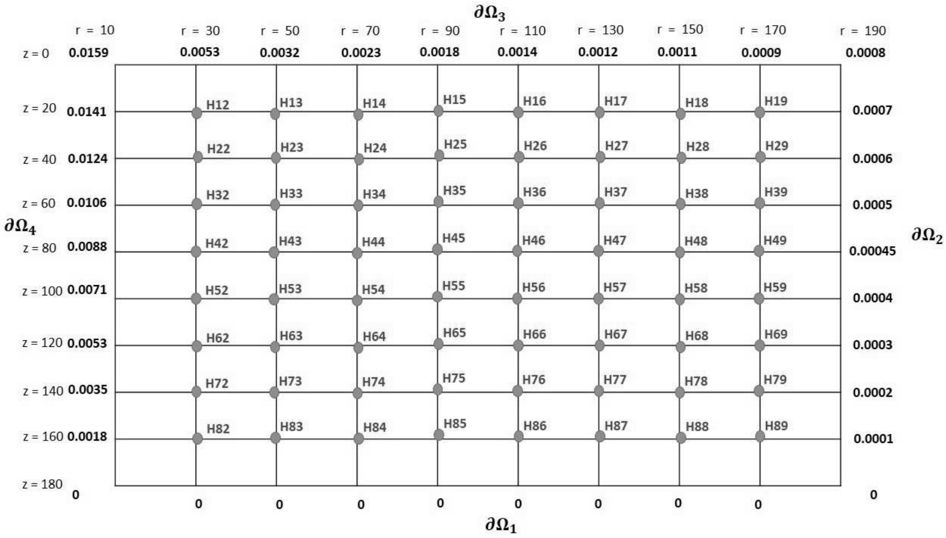
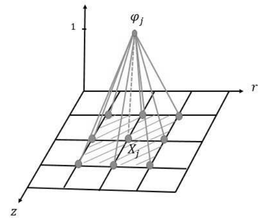


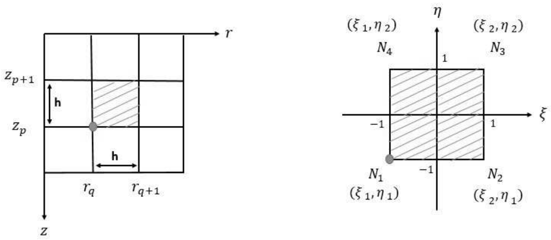
Figure 4.1: Discretizing the domain Ω using a rectangular uniform grid.

We divide Ω into rectangular elements, Ω_k, $(\bar{n}-1)\times(m-1) = N_e$ elements, where $k = 1, 2, \ldots, N_e$, $\bar{n}$ is the partition in r direction and m is the partition in z direction. We denote $H(X_i)$, $i = 1, 2, \ldots, 100$ for H_n, $n = 1, 2, \ldots, 100$ and nodes X_i, $i = 1, 2, \ldots, 100$ for (r_i, z_i), $i = 1, 2, \ldots, 10$.

The support of φ_j consists of rectangles with the common node X_j.



We transform each element to a reference element, $\tilde{\Omega}$, by using the following transformation as,



where N_i is basis function at node i, $i = 1, 2, 3, 4$.

Let ξ and η are our new variable in the coordinate (ξ, η).
We are going to transform (r, z) to (ξ, η) such that

$$r = r_q + \frac{h}{2}(1+\xi), \ q = 1, 2, \ldots, n-1,$$

$$z = z_p + \frac{h}{2}(1+\eta), \ p = 1, 2, \ldots, m-1.$$

The differential form can be determined as

$$dr = \frac{h}{2}d\xi, \qquad d\xi = \frac{2}{h}dr,$$

$$dz = \frac{h}{2}d\eta, \qquad d\eta = \frac{2}{h}dz,$$

$$drdz = \frac{h^2}{4}d\xi d\eta.$$

The relationship between coordinate (r, z) to the basis functions in coordinate (ξ, η) are

$$r(\xi, \eta) = N_1 r_1 + N_2 r_2 + N_3 r_3 + N_4 r_4,$$

$$z(\xi, \eta) = N_1 z_1 + N_2 z_2 + N_3 z_3 + N_4 z_4.$$

The basis functions can be written in the form of ξ and η as

$$N_1(\xi, \eta) = \frac{(\xi_2 - \xi)(\eta_2 - \eta)}{(\xi_2 - \xi_1)(\eta_2 - \eta_1)} = \frac{1}{4}(1-\xi)(1-\eta),$$

$$N_2(\xi,\eta) = \frac{(\xi-\xi_1)(\eta_2-\eta)}{(\xi_2-\xi_1)(\eta_2-\eta_1)} = \frac{1}{4}(1+\xi)(1-\eta),$$

$$N_3(\xi,\eta) = \frac{(\xi-\xi_1)(\eta-\eta_1)}{(\xi_2-\xi_1)(\eta_2-\eta_1)} = \frac{1}{4}(1+\xi)(1+\eta),$$

$$N_4(\xi,\eta) = \frac{(\xi_2-\xi)(\eta-\eta_1)}{(\xi_2-\xi_1)(\eta_2-\eta_1)} = \frac{1}{4}(1-\xi)(1+\eta).$$

We now consider the member of A.

$$A_1 = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \; d\Omega = \iint_\Omega (\frac{\partial\varphi_i}{\partial r}\frac{\partial\varphi_j}{\partial r} + \frac{\partial\varphi_i}{\partial z}\frac{\partial\varphi_j}{\partial z}) \; drdz, \tag{4.9}$$

After the transformation, we have

$$A_1 = \sum_{k=1}^{\tilde{N}_e} \iint_{\tilde{\Omega}_k} \left[(\frac{2}{h}\frac{\partial\tilde{\varphi}_i}{\partial\xi})(\frac{2}{h}\frac{\partial\tilde{\varphi}_j}{\partial\xi}) + (\frac{2}{h}\frac{\partial\tilde{\varphi}_i}{\partial\eta})(\frac{2}{h}\frac{\partial\tilde{\varphi}_j}{\partial\eta}) \right] (\frac{h^2}{4}) d\xi d\eta$$

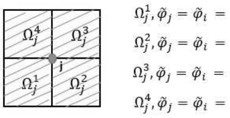
$$= \sum_{k=1}^{\tilde{N}_e} \int_{-1}^{1}\int_{-1}^{1} (\frac{\partial\tilde{\varphi}_i}{\partial\xi}\frac{\partial\tilde{\varphi}_j}{\partial\xi} + \frac{\partial\tilde{\varphi}_i}{\partial\eta}\frac{\partial\tilde{\varphi}_j}{\partial\eta}) \; d\xi d\eta, \tag{4.10}$$

where $\tilde{N}_e$ is number of reference element.

For a fixed j, we have 9 cases for i, $i, j = 1, 2, \ldots, 100$.

Interior nodes

case $1 : i = j$



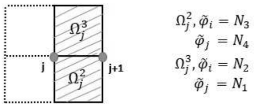
$$A_1 = \int_{-1}^{1}\int_{-1}^{1} [(\frac{\partial N_3}{\partial\xi}\frac{\partial N_3}{\partial\xi} + \frac{\partial N_3}{\partial\eta}\frac{\partial N_3}{\partial\eta}) + (\frac{\partial N_4}{\partial\xi}\frac{\partial N_4}{\partial\xi} + \frac{\partial N_4}{\partial\eta}\frac{\partial N_4}{\partial\eta})$$

$$+ (\frac{\partial N_1}{\partial\xi}\frac{\partial N_1}{\partial\xi} + \frac{\partial N_1}{\partial\eta}\frac{\partial N_1}{\partial\eta}) + (\frac{\partial N_2}{\partial\xi}\frac{\partial N_2}{\partial\xi} + \frac{\partial N_2}{\partial\eta}\frac{\partial N_2}{\partial\eta})] \; d\xi d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} ([(\frac{1}{4}(1+\eta))^2 + (\frac{1}{4}(1+\xi))^2] + [(-\frac{1}{4}(1+\eta))^2 + (\frac{1}{4}(1-\xi))^2]$$

$$+ [(-\frac{1}{4}(1-\eta))^2 + (-\frac{1}{4}(1-\xi))^2] + [(\frac{1}{4}(1-\eta))^2 + (-\frac{1}{4}(1+\xi))^2]) \; d\xi d\eta = \frac{8}{3}\,.$$

case $2 : i = j + 1$

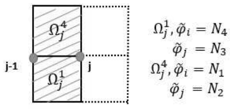


$$A_1 = \int_{-1}^{1}\int_{-1}^{1}[(\frac{\partial N_3}{\partial \xi}\frac{\partial N_4}{\partial \xi} + \frac{\partial N_3}{\partial \eta}\frac{\partial N_4}{\partial \eta}) + (\frac{\partial N_2}{\partial \xi}\frac{\partial N_1}{\partial \xi} + \frac{\partial N_2}{\partial \eta}\frac{\partial N_1}{\partial \eta})]\ d\xi d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1}([(\frac{1}{4}(1+\eta))(-\frac{1}{4}(1+\eta)) + (\frac{1}{4}(1+\xi))(\frac{1}{4}(1-\xi))]$$

$$+ [(\frac{1}{4}(1-\eta))(-\frac{1}{4}(1-\eta)) + (-\frac{1}{4}(1+\xi))(-\frac{1}{4}(1-\xi))])\ d\xi d\eta = -\frac{1}{3}\,.$$

case $3 : i = j - 1$

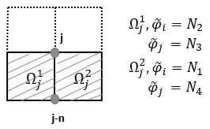


$$A_1 = \int_{-1}^{1}\int_{-1}^{1}[(\frac{\partial N_4}{\partial \xi}\frac{\partial N_3}{\partial \xi} + \frac{\partial N_4}{\partial \eta}\frac{\partial N_3}{\partial \eta}) + (\frac{\partial N_1}{\partial \xi}\frac{\partial N_2}{\partial \xi} + \frac{\partial N_1}{\partial \eta}\frac{\partial N_2}{\partial \eta})]\ d\xi d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1}([(-\frac{1}{4}(1+\eta))(\frac{1}{4}(1+\eta)) + (\frac{1}{4}(1-\xi))(\frac{1}{4}(1+\xi))]$$

$$+ [(-\frac{1}{4}(1-\eta))(\frac{1}{4}(1-\eta)) + (-\frac{1}{4}(1-\xi))(-\frac{1}{4}(1+\xi))])\ d\xi d\eta = -\frac{1}{3}\,.$$

case $4 : i = j - n$



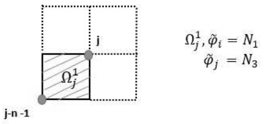
$$A_1 = \int_{-1}^{1}\int_{-1}^{1}[(\frac{\partial N_2}{\partial \xi}\frac{\partial N_3}{\partial \xi} + \frac{\partial N_2}{\partial \eta}\frac{\partial N_3}{\partial \eta}) + (\frac{\partial N_1}{\partial \xi}\frac{\partial N_4}{\partial \xi} + \frac{\partial N_1}{\partial \eta}\frac{\partial N_4}{\partial \eta})]\ d\xi d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1}([(\frac{1}{4}(1-\eta))(\frac{1}{4}(1+\eta)) + (-\frac{1}{4}(1+\xi))(\frac{1}{4}(1+\xi))]$$

$$+ [(-\frac{1}{4}(1-\eta))(-\frac{1}{4}(1+\eta)) + (-\frac{1}{4}(1-\xi))(\frac{1}{4}(1-\xi))])\ d\xi d\eta$$

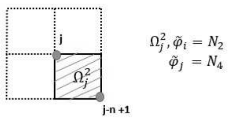
$$= -\frac{1}{3}\,.$$

case 5 : $i = j - n - 1$



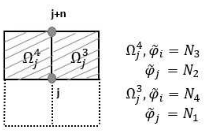
$$\begin{aligned}
A_1 &= \int_{-1}^{1}\int_{-1}^{1} [(\frac{\partial N_1}{\partial \xi}\frac{\partial N_3}{\partial \xi} + \frac{\partial N_1}{\partial \eta}\frac{\partial N_3}{\partial \eta})]\ d\xi d\eta \\
&= \int_{-1}^{1}\int_{-1}^{1} ([(-\frac{1}{4}(1-\eta))(\frac{1}{4}(1+\eta)) + (-\frac{1}{4}(1-\xi))(\frac{1}{4}(1+\xi))])\ d\xi d\eta \\
&= -\frac{1}{3}\,.
\end{aligned}$$

case 6 : $i = j - n + 1$



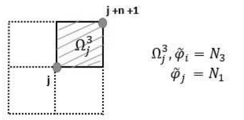
$$\begin{aligned}
A_1 &= \int_{-1}^{1}\int_{-1}^{1} [(\frac{\partial N_2}{\partial \xi}\frac{\partial N_4}{\partial \xi} + \frac{\partial N_2}{\partial \eta}\frac{\partial N_4}{\partial \eta})]\ d\xi d\eta \\
&= \int_{-1}^{1}\int_{-1}^{1} ([(\frac{1}{4}(1-\eta))(-\frac{1}{4}(1+\eta)) + (-\frac{1}{4}(1+\xi))(\frac{1}{4}(1-\xi))])\ d\xi d\eta \\
&= -\frac{1}{3}\,.
\end{aligned}$$

case 7 : $i = j + n$



$$\begin{aligned}
A_1 &= \int_{-1}^{1}\int_{-1}^{1} [(\frac{\partial N_4}{\partial \xi}\frac{\partial N_1}{\partial \xi} + \frac{\partial N_4}{\partial \eta}\frac{\partial N_1}{\partial \eta}) + (\frac{\partial N_3}{\partial \xi}\frac{\partial N_2}{\partial \xi} + \frac{\partial N_3}{\partial \eta}\frac{\partial N_2}{\partial \eta})]\ d\xi d\eta \\
&= \int_{-1}^{1}\int_{-1}^{1} ([(-\frac{1}{4}(1+\eta))(-\frac{1}{4}(1-\eta)) + (\frac{1}{4}(1-\xi))(-\frac{1}{4}(1-\xi))] \\
&+ [(\frac{1}{4}(1+\eta))(\frac{1}{4}(1-\eta)) + (\frac{1}{4}(1+\xi))(-\frac{1}{4}(1+\xi))])\ d\xi d\eta \\
&= -\frac{1}{3}\,.
\end{aligned}$$

case 8 : $i = j + n + 1$

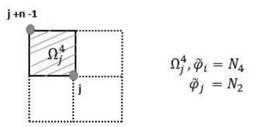


$$A_1 = \int_{-1}^{1}\int_{-1}^{1} [(\frac{\partial N_3}{\partial \xi}\frac{\partial N_1}{\partial \xi} + \frac{\partial N_3}{\partial \eta}\frac{\partial N_1}{\partial \eta})] \ d\xi d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} ([(\frac{1}{4}(1+\eta))(-\frac{1}{4}(1-\eta)) + (\frac{1}{4}(1+\xi))(-\frac{1}{4}(1-\xi))]) \ d\xi d\eta$$

$$= -\frac{1}{3}\,.$$

case 9 : $i = j + n - 1$



$$A_1 = \int_{-1}^{1}\int_{-1}^{1} [(\frac{\partial N_4}{\partial \xi}\frac{\partial N_2}{\partial \xi} + \frac{\partial N_4}{\partial \eta}\frac{\partial N_2}{\partial \eta})] \ d\xi d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} ([(-\frac{1}{4}(1+\eta))(\frac{1}{4}(1-\eta)) + (\frac{1}{4}(1-\xi))(-\frac{1}{4}(1+\xi))]) \ d\xi d\eta$$

$$= -\frac{1}{3}\,.$$

This gives the matrix A_1 has the form,

$$A_1 = (\frac{1}{6}) \begin{bmatrix} C & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B & D & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & D & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B & D & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & D & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B & D & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B & D & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & D & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & C \end{bmatrix}_{100\times 100} ,$$

where

$$C = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 8 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 8 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 8 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 8 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 8 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}_{10\times 10},$$

$$B = \begin{bmatrix} -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \end{bmatrix}_{10\times 10},$$

$$D = \begin{bmatrix} 8 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 16 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 16 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 16 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 16 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 16 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 16 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 16 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 16 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8 \end{bmatrix}_{10\times 10}.$$

The matrix A_2 can be determined as

$$A_2 = \int_\Omega a\varphi_i \frac{\partial \varphi_j}{\partial z}\, d\Omega = \iint_\Omega (a\varphi_i \frac{\partial \varphi_j}{\partial z})\, drdz, \tag{4.11}$$

After the transformation, we have

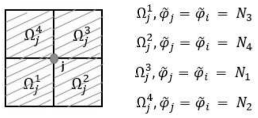
$$\begin{aligned} A_2 &= \sum_{k=1}^{\tilde{N}_e} \iint_{\tilde{\Omega}_k} \left[a\tilde{\varphi}_i (\frac{2}{h}\frac{\partial \tilde{\varphi}_j}{\partial \eta}) \right] (\frac{h^2}{4}) d\xi d\eta \\ &= \frac{ah}{2} \sum_{k=1}^{\tilde{N}_e} \int_{-1}^{1}\int_{-1}^{1} (\tilde{\varphi}_i \frac{\partial \tilde{\varphi}_j}{\partial \eta})\, d\xi d\eta, \end{aligned} \tag{4.12}$$

where $\tilde{N}_e$ is number of reference element.

For a fixed j, we have 9 cases for i, $i, j = 1, 2, \ldots, 100$.

Interior nodes

case $1 : i = j$

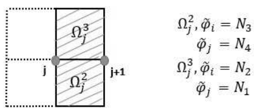


$$A_2 = \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_3\frac{\partial N_3}{\partial \eta}) + (N_4\frac{\partial N_4}{\partial \eta}) + (N_1\frac{\partial N_1}{\partial \eta}) + (N_2\frac{\partial N_2}{\partial \eta})]\ d\xi d\eta$$

$$= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1+\xi))) + ((\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1-\xi)))$$

$$+ ((\frac{1}{4}(1-\xi)(1-\eta))(-\frac{1}{4}(1-\xi))) + ((\frac{1}{4}(1+\xi)(1-\eta))(-\frac{1}{4}(1+\xi)))]\ d\xi d\eta = 0\ .$$

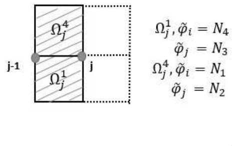
case $2 : i = j + 1$



$$A_2 = \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_3\frac{\partial N_4}{\partial \eta}) + (N_2\frac{\partial N_1}{\partial \eta})]\ d\xi d\eta$$

$$= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\xi))) + ((\frac{1}{4}(1+\xi)(1-\eta))(-\frac{1}{4}(1-\xi)))]\ d\xi d\eta = 0\ .$$

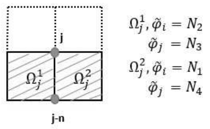
case $3 : i = j - 1$



$$A_2 = \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_4\frac{\partial N_3}{\partial \eta}) + (N_1\frac{\partial N_2}{\partial \eta})]\ d\xi d\eta$$

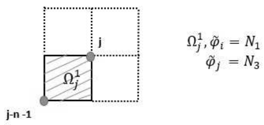
$$= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1+\xi))) + ((\frac{1}{4}(1-\xi)(1-\eta))(-\frac{1}{4}(1+\xi)))]\ d\xi d\eta = 0\ .$$

case 4 : $i = j - n$



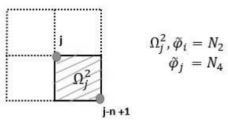
$$
\begin{aligned}
A_2 &= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_2\frac{\partial N_3}{\partial \eta}) + (N_1\frac{\partial N_4}{\partial \eta})]\ d\xi d\eta \\
&= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1+\xi))) + ((\frac{1}{4}(1-\xi)(1-\eta))(\frac{1}{4}(1-\xi)))]\ d\xi d\eta \\
&= \frac{ah}{3}\ .
\end{aligned}
$$

case 5 : $i = j - n - 1$



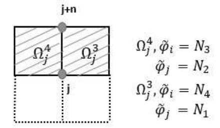
$$
\begin{aligned}
A_2 &= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_1\frac{\partial N_3}{\partial \eta})]\ d\xi d\eta \\
&= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(\frac{1}{4}(1-\xi)(1-\eta))(\frac{1}{4}(1+\xi))]\ d\xi d\eta = \frac{ah}{12}\ .
\end{aligned}
$$

case 6 : $i = j - n + 1$



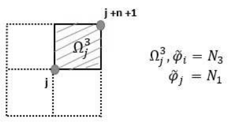
$$
\begin{aligned}
A_2 &= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_2\frac{\partial N_4}{\partial \eta})]\ d\xi d\eta \\
&= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1-\xi))]\ d\xi d\eta = \frac{ah}{12}\ .
\end{aligned}
$$

case $7 : i = j + n$



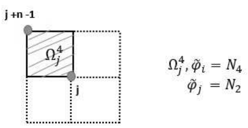
$$\begin{aligned}
A_2 &= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_3\frac{\partial N_2}{\partial \eta}) + (N_4\frac{\partial N_1}{\partial \eta})]\ d\xi d\eta \\
&= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{4}(1+\xi)(1+\eta))(-\frac{1}{4}(1+\xi))) + ((\frac{1}{4}(1-\xi)(1+\eta))(-\frac{1}{4}(1-\xi)))]\ d\xi d\eta \\
&= -\frac{ah}{3}\,.
\end{aligned}$$

case $8 : i = j + n + 1$



$$\begin{aligned}
A_2 &= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_3\frac{\partial N_1}{\partial \eta})]\ d\xi d\eta \\
&= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(\frac{1}{4}(1+\xi)(1+\eta))(-\frac{1}{4}(1-\xi))]\ d\xi d\eta = -\frac{ah}{12}\,.
\end{aligned}$$

case $9 : i = j + n - 1$



$$\begin{aligned}
A_2 &= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(N_4\frac{\partial N_2}{\partial \eta})]\ d\xi d\eta \\
&= \frac{ah}{2}\int_{-1}^{1}\int_{-1}^{1}[(\frac{1}{4}(1-\xi)(1+\eta))(-\frac{1}{4}(1+\xi))]\ d\xi d\eta = -\frac{ah}{12}\,.
\end{aligned}$$

This gives the matrix A_2 has the form,

$$A_2 = (\frac{ah}{12})\begin{bmatrix} -C & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -C & D & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & D & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C & D & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C & D & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -C & D & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -C & D & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -C & D & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C & D & C \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C & C \end{bmatrix}_{100\times 100},$$

where

$$C = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}_{10\times 10},$$

and D is zeros matrix size 10×10.

The matrix A_3 can be determined as

$$A_3 = -\int_{\Omega} [(\frac{1}{r} - b)\varphi_i \frac{\partial \varphi_j}{\partial r}]\, d\Omega = -\iint_{\Omega} [(\frac{1}{r} - b)\varphi_i \frac{\partial \varphi_j}{\partial r}]\, drdz, \tag{4.13}$$

After the transformation, we have

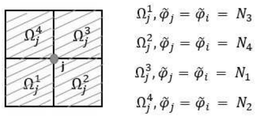
$$\begin{aligned} A_3 &= -\sum_{k=1}^{\tilde{N}_e} \iint_{\tilde{\Omega}_k} \left[\left(\frac{1}{\left(r_q + \frac{h}{2}(1+\xi)\right)} - b \right) \tilde{\varphi}_i (\frac{2}{h}\frac{\partial \tilde{\varphi}_j}{\partial \xi}) \right] (\frac{h^2}{4}) d\xi d\eta \\ &= -\frac{h}{2} \sum_{k=1}^{\tilde{N}_e} \int_{-1}^{1}\int_{-1}^{1} (\left(\frac{1}{\left(r_q + \frac{h}{2}(1+\xi)\right)} - b \right) \tilde{\varphi}_i \frac{\partial \tilde{\varphi}_j}{\partial \xi})\ d\xi d\eta, \end{aligned} \tag{4.14}$$

where $\tilde{N}_e$ is number of reference element and q is index of i.

For a fixed j, we have 9 cases for i, $i, j = 1, 2, \ldots, 100$.

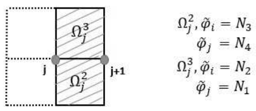
Interior nodes

case $1 : i = j$



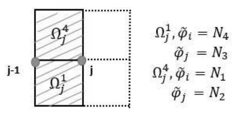
$$
\begin{aligned}
A_3 &= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)N_3\frac{\partial N_3}{\partial \xi}) + ((\frac{1}{(r_{i}+\frac{h}{2}(1+\xi))} - b)N_4\frac{\partial N_4}{\partial \xi}) \\
&+ ((\frac{1}{(r_{i}+\frac{h}{2}(1+\xi))} - b)N_1\frac{\partial N_1}{\partial \xi}) + ((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)N_2\frac{\partial N_2}{\partial \xi})]\ d\xi d\eta \\
&= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1+\eta))) \\
&+ ((\frac{1}{(r_{i}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1+\eta))(-\frac{1}{4}(1+\eta))) + ((\frac{1}{(r_{i}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1-\eta))(-\frac{1}{4}(1-\eta))) \\
&+ ((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1-\eta)))]\ d\xi d\eta = -\frac{h}{2}[p + q + \hat{r} + s]\ .
\end{aligned}
$$

case $2 : i = j + 1$



$$
\begin{aligned}
A_3 &= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)N_3\frac{\partial N_4}{\partial \xi}) + ((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)N_2\frac{\partial N_1}{\partial \xi})]\ d\xi d\eta \\
&= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(-\frac{1}{4}(1+\eta))) \\
&+ ((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1-\eta))(-\frac{1}{4}(1-\eta)))]\ d\xi d\eta = -\frac{h}{2}[\hat{m} + \hat{n}]\ .
\end{aligned}
$$

case $3 : i = j - 1$

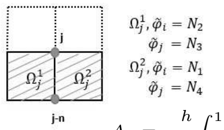


$$A_3 = -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)N_4\frac{\partial N_3}{\partial \xi})+((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)N_1\frac{\partial N_2}{\partial \xi})]\ d\xi d\eta$$

$$= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1+\eta)))$$

$$+((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)(\frac{1}{4}(1-\xi)(1-\eta))(\frac{1}{4}(1-\eta)))]\ d\xi d\eta = -\frac{h}{2}[\hat{a}+\hat{b}]\ .$$

case $4 : i = j - n$

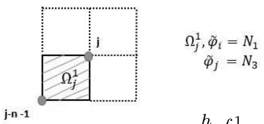


$$A_3 = -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))}-b)N_2\frac{\partial N_3}{\partial \xi})+((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)N_1\frac{\partial N_4}{\partial \xi})]\ d\xi d\eta$$

$$= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))}-b)(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1+\eta)))$$

$$+((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)(\frac{1}{4}(1-\xi)(1-\eta))(-\frac{1}{4}(1+\eta)))]\ d\xi d\eta = -\frac{h}{2}[c+d]\ .$$

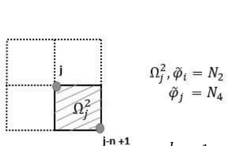
case $5 : i = j - n - 1$



$$A_3 = -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)N_1\frac{\partial N_3}{\partial \xi})]\ d\xi d\eta$$

$$= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_i+\frac{h}{2}(1+\xi))}-b)(\frac{1}{4}(1-\xi)(1-\eta))(\frac{1}{4}(1+\eta)))]\ d\xi d\eta = -\frac{h}{2}[f]\ .$$

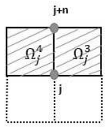
case $6 : i = j - n + 1$



$$A_3 = -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))}-b)N_2\frac{\partial N_4}{\partial \xi})]\ d\xi d\eta$$

$$= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))}-b)(\frac{1}{4}(1+\xi)(1-\eta))(-\frac{1}{4}(1+\eta)))]\ d\xi d\eta = -\frac{h}{2}[e]\ .$$

case $7: i = j + n$



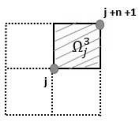
$\Omega_j^4, \tilde{\varphi}_i = N_3$
$\tilde{\varphi}_j = N_2$
$\Omega_j^3, \tilde{\varphi}_i = N_4$
$\tilde{\varphi}_j = N_1$

$$A_3 = -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)N_3\frac{\partial N_2}{\partial \xi}) + ((\frac{1}{(r_i+\frac{h}{2}(1+\xi))} - b)N_4\frac{\partial N_1}{\partial \xi})]\ d\xi d\eta$$

$$= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\eta)))$$

$$+ ((\frac{1}{(r_i+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1+\eta))(-\frac{1}{4}(1-\eta)))]\ d\xi d\eta = -\frac{h}{2}[g+i]\ .$$

case $8: i = j + n + 1$

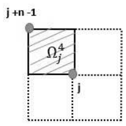


$\Omega_j^3, \tilde{\varphi}_i = N_3$
$\tilde{\varphi}_j = N_1$

$$A_3 = -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)N_3\frac{\partial N_1}{\partial \xi})]\ d\xi d\eta$$

$$= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(-\frac{1}{4}(1-\eta)))]\ d\xi d\eta = -\frac{h}{2}[l]\ .$$

case $9: i = j + n - 1$



$\Omega_j^4, \tilde{\varphi}_i = N_4$
$\tilde{\varphi}_j = N_2$

$$A_3 = -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_i+\frac{h}{2}(1+\xi))} - b)N_4\frac{\partial N_2}{\partial \xi})]\ d\xi d\eta$$

$$= -\frac{h}{2}\int_{-1}^{1}\int_{-1}^{1}[((\frac{1}{(r_i+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1-\eta)))]\ d\xi d\eta = -\frac{h}{2}[k]\ .$$

This gives the matrix A_3 has the form,

$$A_3 = (-\frac{h}{2})\begin{bmatrix} C & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G & D & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G & D & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G & D & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & D & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & D & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & D & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & D & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & E \end{bmatrix}_{100\times100},$$

where

$$C = \begin{bmatrix} \hat{r} & \hat{b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{n} & (\hat{r}+s) & \hat{b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{n} & (\hat{r}+s) & \hat{b} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{n} & (\hat{r}+s) & \hat{b} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{n} & (\hat{r}+s) & \hat{b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{n} & (\hat{r}+s) & \hat{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{n} & (\hat{r}+s) & \hat{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{n} & (\hat{r}+s) & \hat{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{n} & (\hat{r}+s) & \hat{b} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{n} & s \end{bmatrix}_{10\times10},$$

$$B = \begin{bmatrix} d & \hat{f} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e & (c+d) & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & (c+d) & f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & (c+d) & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & (c+d) & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & (c+d) & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e & (c+d) & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e & (c+d) & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & (c+d) & f \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & c \end{bmatrix}_{10\times10},$$

$$G = \begin{bmatrix} i & \hat{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ l & (g+i) & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l & (g+i) & k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l & (g+i) & k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & (g+i) & k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l & (g+i) & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & l & (g+i) & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & l & (g+i) & k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l & (g+i) & k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l & g \end{bmatrix}_{10\times10},$$

$$D = \begin{bmatrix} (q+\hat{r}) & (\hat{a}+\hat{b}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{m}+\hat{n}) & U & (\hat{a}+\hat{b}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{m}+\hat{n}) & (p+s) \end{bmatrix}_{10\times 10},$$

$$U = (q + \hat{r} + p + s),$$

$$E = \begin{bmatrix} q & \hat{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{m} & (q+p) & \hat{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{m} & (q+p) & \hat{a} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{m} & (q+p) & \hat{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{m} & (q+p) & \hat{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{m} & (q+p) & \hat{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{m} & (q+p) & \hat{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{m} & (q+p) & \hat{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{m} & (q+p) & \hat{a} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{m} & p \end{bmatrix}_{10\times 10}.$$

such that

$$p = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1+\eta)))d\xi d\eta,$$

$$q = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1+\eta))(-\frac{1}{4}(1+\eta)))d\xi d\eta,$$

$$\hat{r} = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1-\eta))(-\frac{1}{4}(1-\eta)))d\xi d\eta,$$

$$s = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1-\eta)))]\ d\xi d\eta,$$

$$\hat{m} = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(-\frac{1}{4}(1+\eta)))d\xi d\eta,$$

$$\hat{n} = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1-\eta))(-\frac{1}{4}(1-\eta)))\ d\xi d\eta,$$

$$\hat{a} = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i}+\frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1+\eta)))\ d\xi d\eta,$$

$$\hat{b} = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_i + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1-\eta))(\frac{1}{4}(1-\eta)))\ d\xi d\eta,$$

$$c = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1} + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1+\eta)))\ d\xi d\eta,$$

$$d = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_i + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1-\eta))(-\frac{1}{4}(1+\eta)))\ d\xi d\eta,$$

$$f = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_i + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1-\eta))(\frac{1}{4}(1+\eta)))\ d\xi d\eta,$$

$$e = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1} + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1-\eta))(-\frac{1}{4}(1+\eta)))\ d\xi d\eta,$$

$$g = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1} + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\eta)))\ d\xi d\eta,$$

$$i = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_i + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1+\eta))(-\frac{1}{4}(1-\eta)))\ d\xi d\eta,$$

$$l = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_{i-1} + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1+\xi)(1+\eta))(-\frac{1}{4}(1-\eta)))\ d\xi d\eta,$$

$$k = \int_{-1}^{1}\int_{-1}^{1}((\frac{1}{(r_i + \frac{h}{2}(1+\xi))} - b)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1-\eta)))\ d\xi d\eta.$$

The matrix A_4 can be determined as

$$A_4 = \int_{\Omega}[(\frac{1}{r^2} + \frac{b}{r})\varphi_i\varphi_j]\ d\Omega = \iint_{\Omega}[(\frac{1}{r^2} + \frac{b}{r})\varphi_i\varphi_j]\ drdz, \tag{4.15}$$

After the transformation, we have

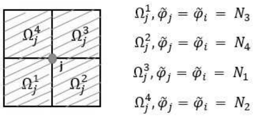
$$\begin{aligned} A_4 &= \sum_{k=1}^{\tilde{N}_e}\iint_{\tilde{\Omega}_k}\left[\left(\frac{1}{\left(r_q + \frac{h}{2}(1+\xi)\right)^2} + \frac{b}{\left(r_q + \frac{h}{2}(1+\xi)\right)}\right)\tilde{\varphi_i}\tilde{\varphi_j}\right](\frac{h^2}{4})d\xi d\eta \\ &= \frac{h^2}{4}\sum_{k=1}^{\tilde{N}_e}\int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{1}{\left(r_q + \frac{h}{2}(1+\xi)\right)^2} + \frac{b}{\left(r_q + \frac{h}{2}(1+\xi)\right)}\right)\tilde{\varphi_i}\tilde{\varphi_j}\right]\ d\xi d\eta, \end{aligned} \tag{4.16}$$

where $\tilde{N}_e$ is number of reference element and q is index of i.

For a fixed j, we have 9 cases for i, $i, j = 1, 2, \ldots, 100$.

Interior nodes

case $1 : i = j$



$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_3N_3)+(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)N_4N_4)$$

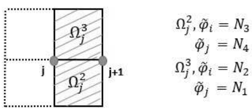
$$+(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)N_1N_1)+(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_2N_2)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))^2)$$

$$+(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1+\eta))^2)+(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1-\eta))^2)$$

$$+(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1-\eta))^2)]\ d\xi d\eta = \frac{h^2}{4}[p+q+\hat{r}+s]\ .$$

case $2 : i = j+1$

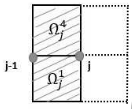


$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_3N_4)+(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_2N_1)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\xi)(1+\eta)))$$

$$+(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1-\xi)(1-\eta)))]\ d\xi d\eta = \frac{h^2}{4}[\hat{a}+\hat{b}]\ .$$

case $3 : i = j - 1$



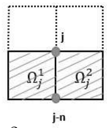
$\Omega_j^1, \tilde{\varphi}_i = N_4$
$\tilde{\varphi}_j = N_3$
$\Omega_j^4, \tilde{\varphi}_i = N_1$
$\tilde{\varphi}_j = N_2$

$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)N_4N_3)+(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)N_2N_1)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\xi)(1+\eta)))$$

$$+(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1-\xi)(1-\eta)))]\ d\xi d\eta = \frac{h^2}{4}[c+d]\ .$$

case $4 : i = j - n$



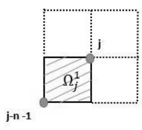
$\Omega_j^1, \tilde{\varphi}_i = N_2$
$\tilde{\varphi}_j = N_3$
$\Omega_j^2, \tilde{\varphi}_i = N_1$
$\tilde{\varphi}_j = N_4$

$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_2N_3)+(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)N_1N_4)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1-\eta))(\frac{1}{4}(1+\xi)(1+\eta)))$$

$$+(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1-\eta))(\frac{1}{4}(1-\xi)(1+\eta)))]\ d\xi d\eta = \frac{h^2}{4}[e+f]\ .$$

case $5 : i = j - n - 1$

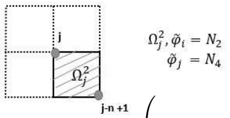


$\Omega_j^1, \tilde{\varphi}_i = N_1$
$\tilde{\varphi}_j = N_3$

$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)N_3N_1)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\xi)(1-\eta)))]\ d\xi d\eta = \frac{h^2}{4}[i]\ .$$

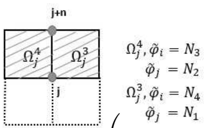
case $6 : i = j - n + 1$



$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_4N_2)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1+\xi)(1-\eta)))]\ d\xi d\eta = \frac{h^2}{4}[g]\ .$$

case $7 : i = j + n$

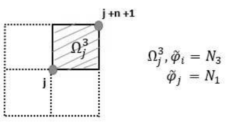


$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_3N_2)+(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)N_4N_1)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1+\xi)(1-\eta)))$$

$$+(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1-\xi)(1-\eta)))]\ d\xi d\eta = \frac{h^2}{4}[k+l]\ .$$

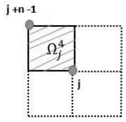
case $8 : i = j + n + 1$



$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)N_3N_1)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\xi)(1-\eta)))]\ d\xi d\eta = \frac{h^2}{4}[\hat{n}]\ .$$

case $9 : i = j + n - 1$



$\Omega_j^4, \tilde{\varphi}_i = N_4$
$\tilde{\varphi}_j = N_2$

$$A_4 = \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)N_4N_2)]\ d\xi d\eta$$

$$= \frac{h^2}{4}\int_{-1}^{1}\int_{-1}^{1}[(\left(\frac{1}{\left(r_i+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_i+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1+\xi)(1-\eta)))]\ d\xi d\eta = \frac{h^2}{4}[\hat{m}]\ .$$

This gives the matrix A_4 has the form,

$$A_4 = (\frac{h^2}{4})\begin{bmatrix} C & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G & D & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G & D & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G & D & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & D & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & D & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & D & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & D & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & E \end{bmatrix}_{100\times100},$$

where

$$C = \begin{bmatrix} \hat{r} & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{b} & (\hat{r}+s) & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{b} & (\hat{r}+s) & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{b} & (\hat{r}+s) & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{b} & (\hat{r}+s) & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{b} & (\hat{r}+s) & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{b} & (\hat{r}+s) & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{b} & (\hat{r}+s) & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{b} & (\hat{r}+s) & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{b} & s \end{bmatrix}_{10\times10},$$

$$B = \begin{bmatrix} f & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g & (e+f) & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g & (e+f) & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & (e+f) & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g & (e+f) & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & (e+f) & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & (e+f) & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g & (e+f) & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & (e+f) & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & e \end{bmatrix}_{10\times 10},$$

$$G = \begin{bmatrix} l & \hat{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{n} & (l+k) & \hat{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{n} & (l+k) & \hat{m} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{n} & (l+k) & \hat{m} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{n} & (l+k) & \hat{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{n} & (l+k) & \hat{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{n} & (l+k) & \hat{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{n} & (l+k) & \hat{m} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{n} & (l+k) & \hat{m} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{n} & k \end{bmatrix}_{10\times 10},$$

$$D = \begin{bmatrix} (q+\hat{r}) & (c+d) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\hat{a}+\hat{b}) & U & (c+d) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\hat{a}+\hat{b}) & U & (c+d) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\hat{a}+\hat{b}) & U & (c+d) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\hat{a}+\hat{b}) & U & (c+d) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\hat{a}+\hat{b}) & U & (c+d) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\hat{a}+\hat{b}) & U & (c+d) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\hat{a}+\hat{b}) & U & (c+d) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{a}+\hat{b}) & U & (c+d) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{a}+\hat{b}) & (p+s) \end{bmatrix}_{10\times 10},$$

where $U = (q + \hat{r} + p + s)$

$$E = \begin{bmatrix} q & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{a} & (q+p) & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{a} & (q+p) & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{a} & (q+p) & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{a} & (q+p) & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{a} & (q+p) & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{a} & (q+p) & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{a} & (q+p) & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{a} & (q+p) & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{a} & p \end{bmatrix}_{10\times 10}.$$

such that

$$p=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1+\eta)\right)^2\right)\,d\xi d\eta,$$

$$q=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1-\xi)(1+\eta)\right)^2\right)\,d\xi d\eta,$$

$$\hat{r}=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1-\xi)(1-\eta)\right)^2\right)\,d\xi d\eta,$$

$$s=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1-\eta)\right)^2\right)\,d\xi d\eta,$$

$$\hat{a}=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1+\eta)\right)\left(\frac{1}{4}(1-\xi)(1+\eta)\right)\right)\,d\xi d\eta,$$

$$\hat{b}=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1-\eta)\right)\left(\frac{1}{4}(1-\xi)(1-\eta)\right)\right)\,d\xi d\eta,$$

$$c=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1+\eta)\right)\left(\frac{1}{4}(1-\xi)(1+\eta)\right)\right)\,d\xi d\eta,$$

$$d=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1-\eta)\right)\left(\frac{1}{4}(1-\xi)(1-\eta)\right)\right)\,d\xi d\eta,$$

$$e=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1-\eta)\right)\left(\frac{1}{4}(1+\xi)(1+\eta)\right)\right)\,d\xi d\eta,$$

$$f=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1-\xi)(1-\eta)\right)\left(\frac{1}{4}(1-\xi)(1+\eta)\right)\right)\,d\xi d\eta,$$

$$i=\int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)\left(\frac{1}{4}(1+\xi)(1+\eta)\right)\left(\frac{1}{4}(1-\xi)(1-\eta)\right)\right)\,d\xi d\eta,$$

$$g = \int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1+\xi)(1-\eta))\right)\ d\xi d\eta,$$

$$k = \int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1+\xi)(1-\eta))\right)\ d\xi d\eta,$$

$$l = \int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1-\xi)(1-\eta))\right)\ d\xi d\eta,$$

$$\hat{n} = \int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i-1}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1+\xi)(1+\eta))(\frac{1}{4}(1-\xi)(1-\eta))\right)\ d\xi d\eta,$$

$$\hat{m} = \int_{-1}^{1}\int_{-1}^{1}\left(\left(\frac{1}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)^2}+\frac{b}{\left(r_{i}+\frac{h}{2}(1+\xi)\right)}\right)(\frac{1}{4}(1-\xi)(1+\eta))(\frac{1}{4}(1+\xi)(1-\eta))\right)\ d\xi d\eta.$$

The system becomes

$$A\bar{H} = 0,$$

$$\left[A_{i,j} = A_{1(i,j)} + A_{2(i,j)} + A_{3(i,j)} + A_{4(i,j)}\right]_{100\times 100}\begin{bmatrix} H_1 \\ H_2 \\ . \\ . \\ . \\ H_{n-1} \\ H_n \end{bmatrix}_{100\times 1} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}_{100\times 1}.$$

Observe that the boundary conditions (4.2) . $H_1, H_2, \ldots, H_{10}$ and $H_{10C+1}, H_{10(C+1)}$, $n = 1, 2, \ldots, 100$ and $C = 1, 2, \ldots, 8$ and $H_{91}, H_{92}, \ldots, H_{100}$ are known value. So we don't have to solve for H is known value.

The system becomes

$$A\bar{H} = F,$$

$$\begin{bmatrix} A_{i,j} \end{bmatrix}_{64\times 64} \begin{bmatrix} H_{M1} \\ H_{M2} \\ . \\ . \\ . \\ H_{M8} \end{bmatrix}_{64\times 1} = \begin{bmatrix} f_{M1} \\ f_{M2} \\ . \\ . \\ . \\ f_{M8} \end{bmatrix}_{64\times 1},$$

where

$$H_{MC} = \begin{bmatrix} H_{10C+2} \\ H_{10C+3} \\ . \\ . \\ . \\ H_{10C+9} \end{bmatrix}_{8\times 1} \quad and \quad f_{MC} = \begin{bmatrix} f_{10C+2} \\ f_{10C+3} \\ . \\ . \\ . \\ f_{10C+9} \end{bmatrix}_{8\times 1},$$

$C = 1, 2, \ldots, 8.$

Here

$f_{12} = f_{12} - A_{12,2}H_2 - A_{12,3}H_3 - A_{12,1}H_1 - A_{12,11}H_{11} - A_{12,21}H_{21},$

$f_{19} = f_{19} - A_{19,9}H_9 - A_{19,10}H_{10} - A_{19,8}H_8 - A_{19,20}H_{20} - A_{19,30}H_{30},$

$f_{82} = f_{82} - A_{82,71}H_{71} - A_{82,81}H_{81} - A_{82,91}H_{91} - A_{82,92}H_{92} - A_{82,93}H_{93},$

$f_{89} = f_{89} - A_{89,80}H_{80} - A_{89,90}H_{90} - A_{89,100}H_{100} - A_{89,99}H_{99} - A_{89,98}H_{98},$

Let $i = 13, 14, 15, 16, 17, 18$

$f_i = f_i - A_{i,(i-10)}H_{(i-10)} - A_{i,(i-11)}H_{(i-11)} - A_{i,(i-9)}H_{(i-9)},$

Let $i = 22, 32, 42, 52, 62, 72$

$f_i = f_i - A_{i,(i-1)}H_{(i-1)} - A_{i,(i+9)}H_{(i+9)} - A_{i,(i-11)}H_{(i-11)},$

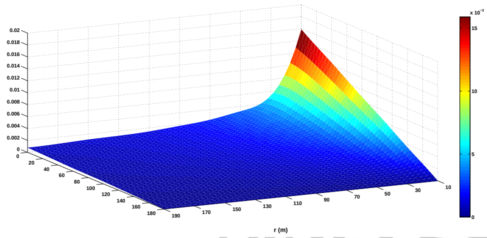
Let $i = 29, 39, 49, 59, 69, 79$

$f_i = f_i - A_{i,(i+1)}H_{(i+1)} - A_{i,(i-9)}H_{(i-9)} - A_{i,(i+11)}H_{(i+11)}.$

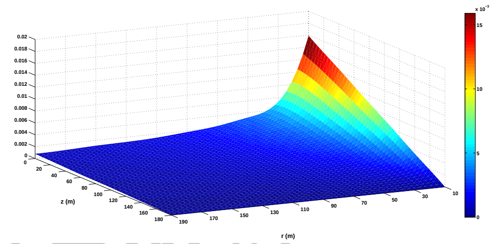
4.1.1 Numerical Experiments

Numerical results of the magnetic field intensity from equation (4.1) are obtained by using the Finite Element Method form rectangular elements. There is a source providing a DC voltage of direct current $I = 1A$ and receiver on the ground surface which picks up the signal from $r = 10$ to $r = 190$ m. The depth z start from the ground surface $z = 0$ to $z = 180$ m. The grid size $h = 20$ m. a and b are given constants . The magnetic field intensity is computed by using MATLAB programing.

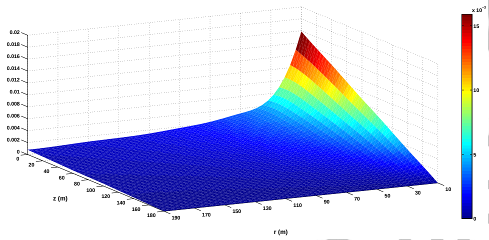
Consider for the case of an exponentially decreasing conductivity $\sigma(r, z) = \sigma_0 e^{(az+br)}$ when $a < 0$ and $b = 0$, the graphs of the relationship between magnetic field intensity and spacing of source-receiver at various depths are plotted as shown in Figure 4.2 and 4.3.



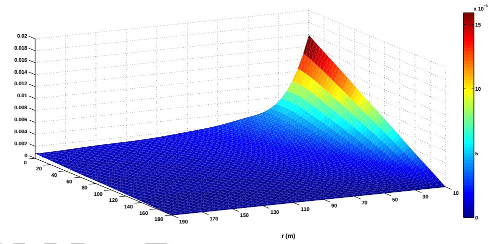
(a)



(b)



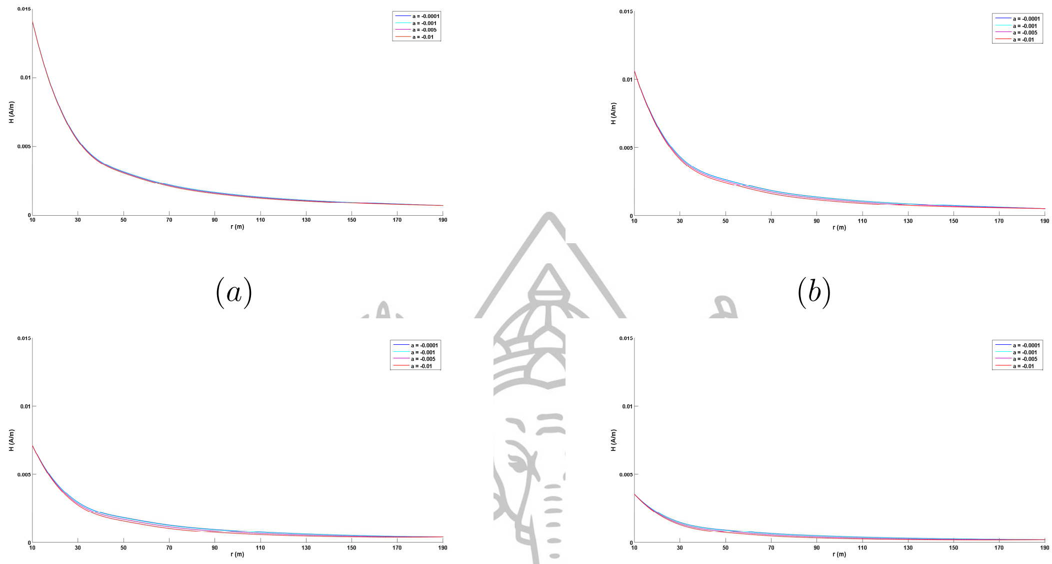
(c)



(d)

Figure 4.2: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0$, a is varied and z is fixed (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

From Figure 4.2 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and $-0.01\ m^{-1}$, respectively, we can see that the values of magnetic field decrease exponentially as r and z increase. The values of magnetic field decrease as a decreases.



(a) (b)

(c) (d)

Figure 4.3: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0$, a varies from -0.0001, -0.001, -0.005 and $-0.01\ m^{-1}$ and z is fixed. (a) $z = 20\ m$ (b) $z = 60\ m$ (c) $z = 100\ m$ and (d) $z = 140\ m$.

From Figure 4.3 (a) to (d) represents the values of magnetic field which are plotted against r whereas a varies and z is fixed at $20, 60, 100$ and $140\ m$, respectively. We can see that the values of magnetic fields where $a = -0.0001, -0.001, -0.005$ and $-0.01\ m^{-1}$ decrease exponentially as r increases and it has similar maner when z increases. Because the values of magnetic field decrease to zero and have values near zero when z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.4.

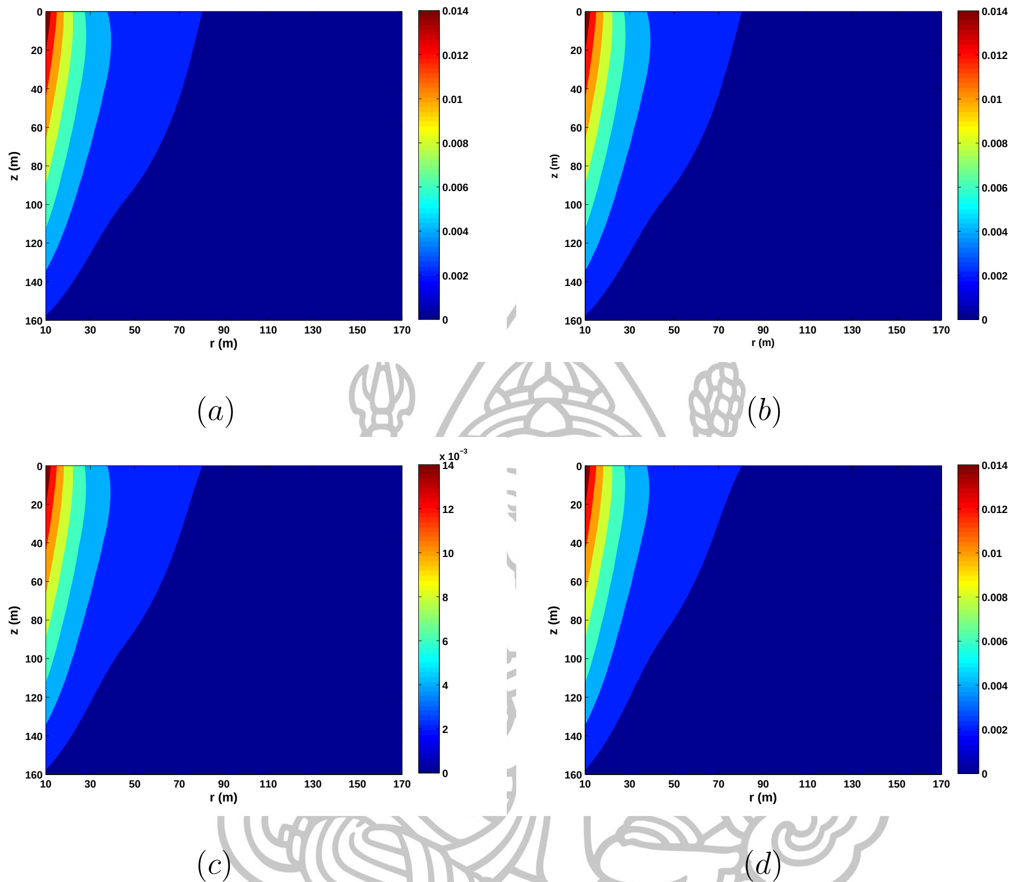
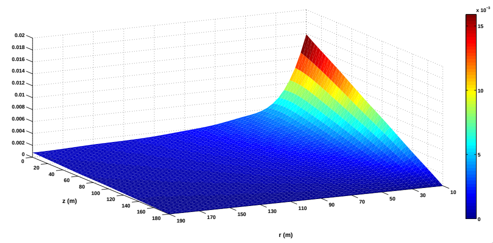


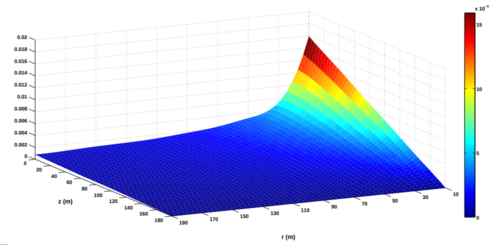
Figure 4.4: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0$, (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

From Figure 4.4 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1}, respectively, the red color shows the area when the values of magnetic field is high and the blue color shows the area when the values of magnetic field is low.

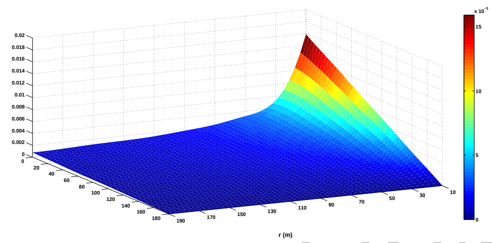
Consider for the case of an exponentially increasing conductivity $\sigma(r,z) = \sigma_0 e^{(az+br)}$ when $a > 0$ and $b = 0$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 4.5 and 4.6.



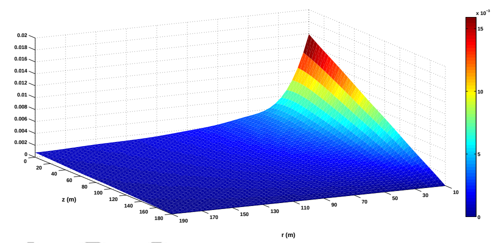
(a)



(b)



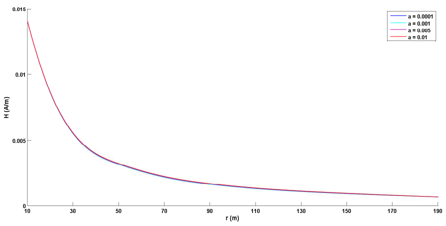
(c)



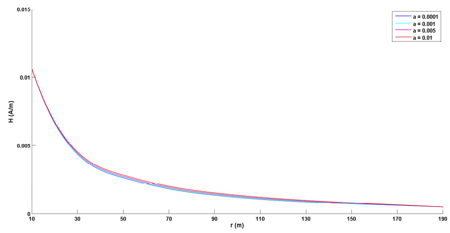
(d)

Figure 4.5: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0$, a varies and z is fixed (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

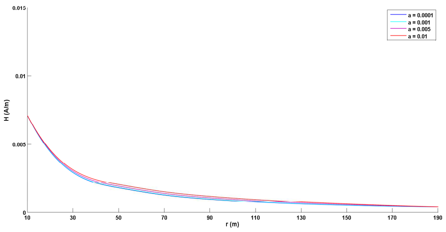
From Figure 4.5 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1}, respectively, we can see that the values of magnetic fields decrease exponentially as r and z increase. The values of magnetic fields increase whereas a increases. The results agree to Tunnurak et al. [12].



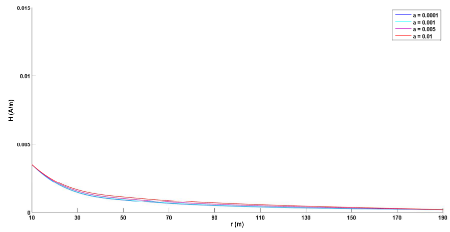
(*a*)



(*b*)



(*c*)

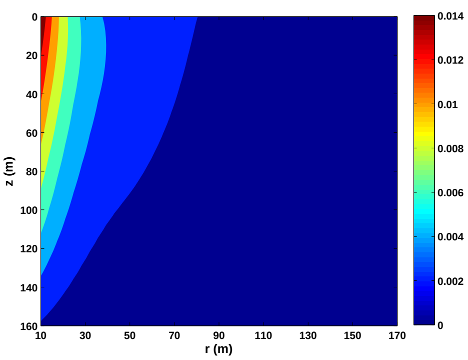


(*d*)

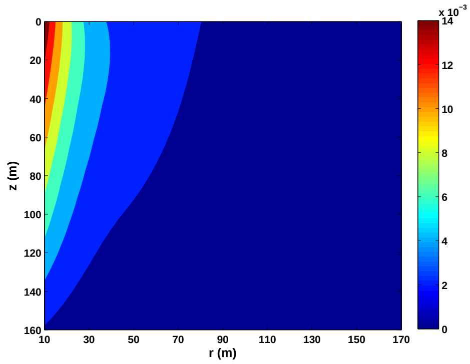
Figure 4.6: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0$, a varies from 0.0001, 0.001, 0.005 and 0.01 m^{-1} and z is fixed. (a) $z = 20$ m (b) $z = 60$ m (c) $z = 100$ m and (d) $z = 140$ m.

From Figure 4.6 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic fields decrease to zero and has value near zero where z increases as a varies.

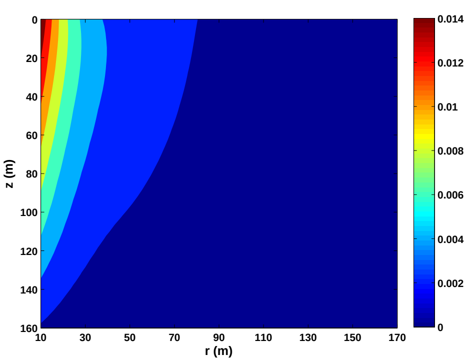
Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.7.



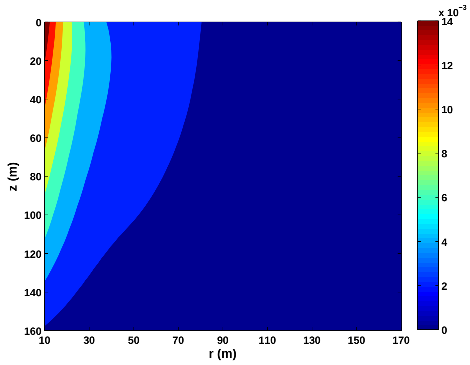
(a)



(b)



(c)



(d)

Figure 4.7: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0$, (a) $a = 0.0001 \ \ m^{-1}$ (b) $a = 0.001 \ \ m^{-1}$ (c) $a = 0.005 \ m^{-1}$ and (d) $a = 0.01 \ m^{-1}$.

From Figure 4.7 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and $0.01 \ m^{-1}$, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area when the values of magnetic field is low. The results agree to Tunnurak et al. [12].

Consider for the case of an exponentially decreasing conductivity $\sigma(r, z) = \sigma_0 e^{(az+br)}$ when $a < 0$ and $b = -0.001$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 4.8 and 4.9.

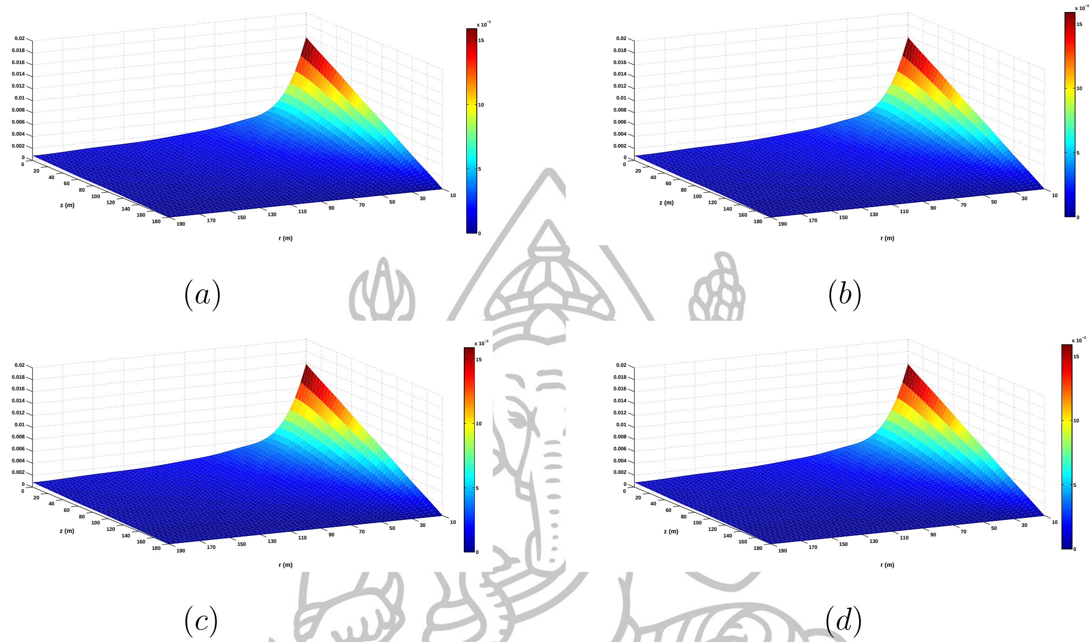


Figure 4.8: Graphs of the magnetic field intensity via distance of receiver from source where $b = -0.001$, a varies and z is fixed (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

From Figure 4.8 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1}, respectively, we can see that the values of magnetic field decrease exponentially as r and z increase. The values of magnetic field decrease where a decreases.

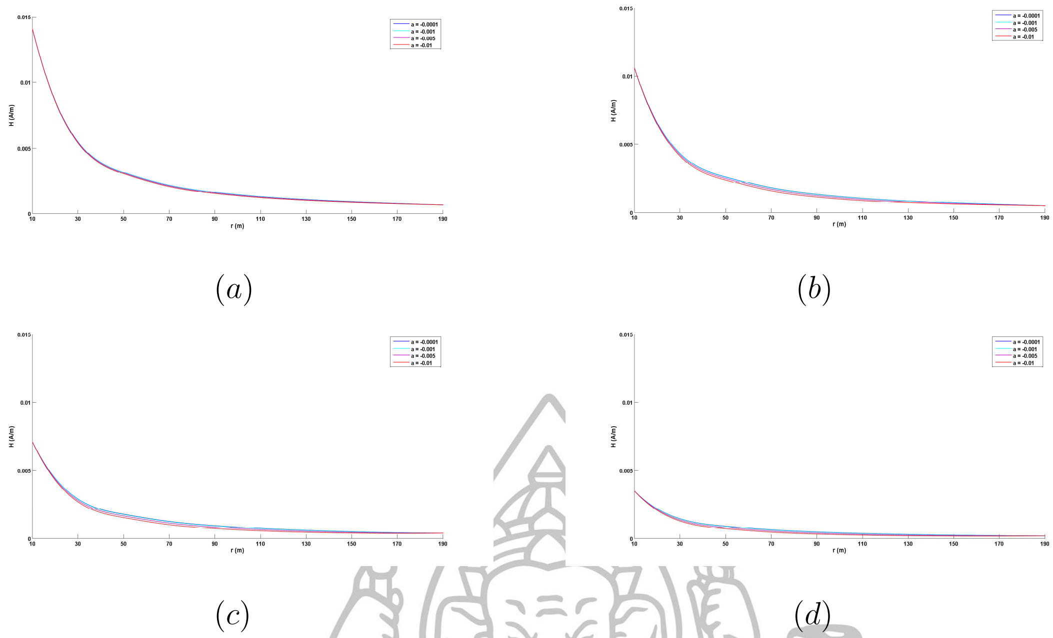
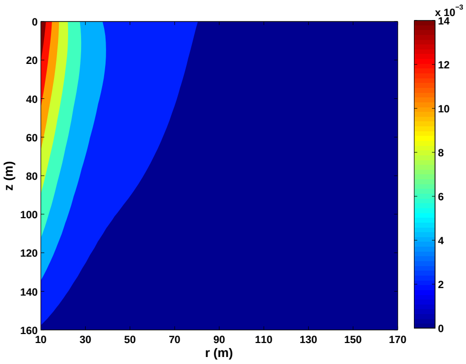


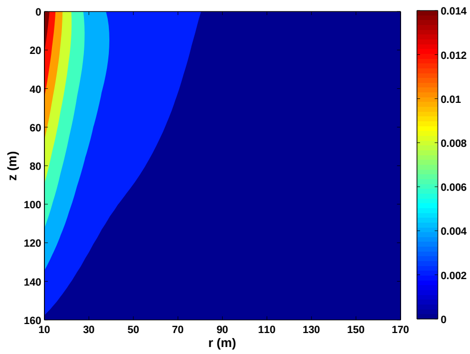
Figure 4.9: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = -0.001$, a varies from -0.0001, -0.001, -0.005 and -0.01 m^{-1} as z is fixed. (a) $z = 20$ m (b) $z = 60$ m (c) $z = 100$ m and (d) $z = 140$ m.

From Figure 4.9 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic field decrease to zero and have values near zero where z increases as a varies.

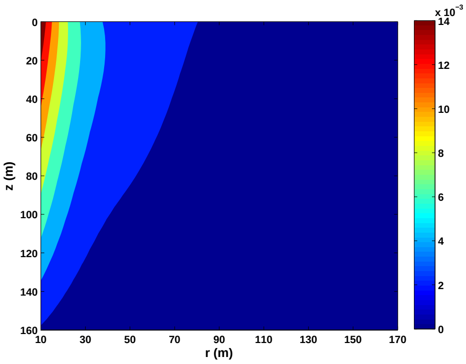
Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.10.



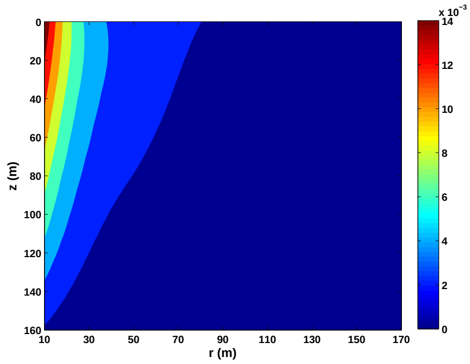
(a)



(b)



(c)



(d)

Figure 4.10: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = -0.001$, (a) $a = -0.0001 \; m^{-1}$ (b) $a = -0.001 \; m^{-1}$ (c) $a = -0.005 \; m^{-1}$ and (d) $a = -0.01 \; m^{-1}$.

From Figure 4.10 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and $-0.01 \; m^{-1}$, respectively, the red color shows the area when the values of magnetic field is high and the blue color shows the area when the values of magnetic field is low.

Consider for the case of an exponentially increasing conductivity $\sigma(r, z) = \sigma_0 e^{(az+br)}$ when $a > 0$ and $b = 0.001$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 4.11 and 4.12.

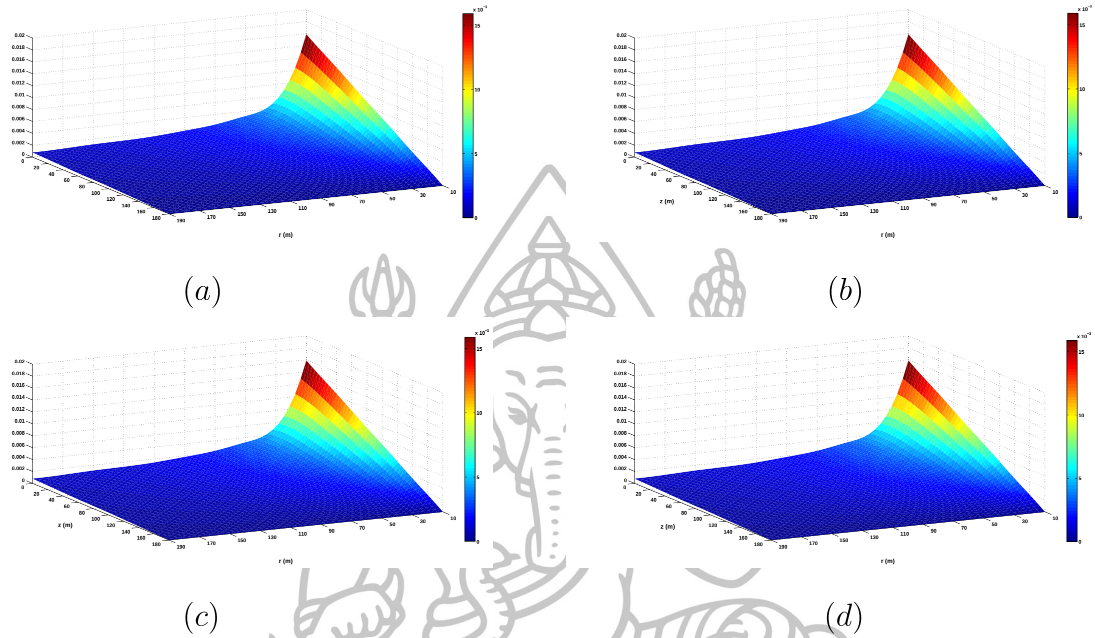


Figure 4.11: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0.001$, a varies and z is fixed (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

From Figure 4.11 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1}, respectively, we can see that the values of magnetic fields decrease exponentially as r and z increase. The values of magnetic fields increase when a increases.

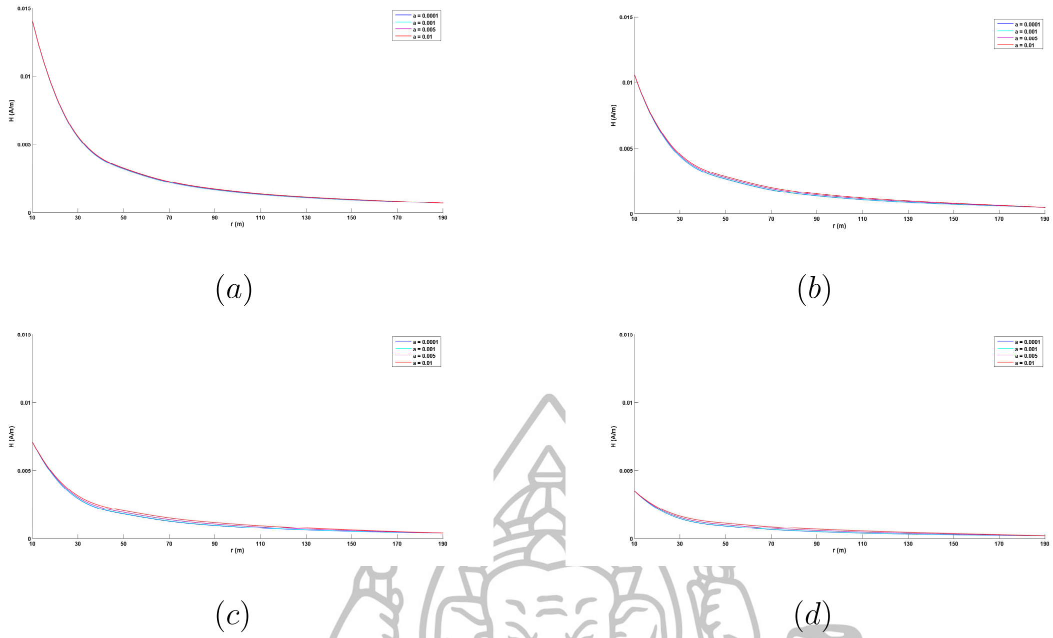


Figure 4.12: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0.001$, a varies from 0.0001, 0.001, 0.005 and 0.01 m^{-1} and z is fixed. (a) $z = 20$ m (b) $z = 60$ m (c) $z = 100$ m and (d) $z = 140$ m.

From Figure 4.12 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic fields decrease to zero and has value near zero where z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.13.

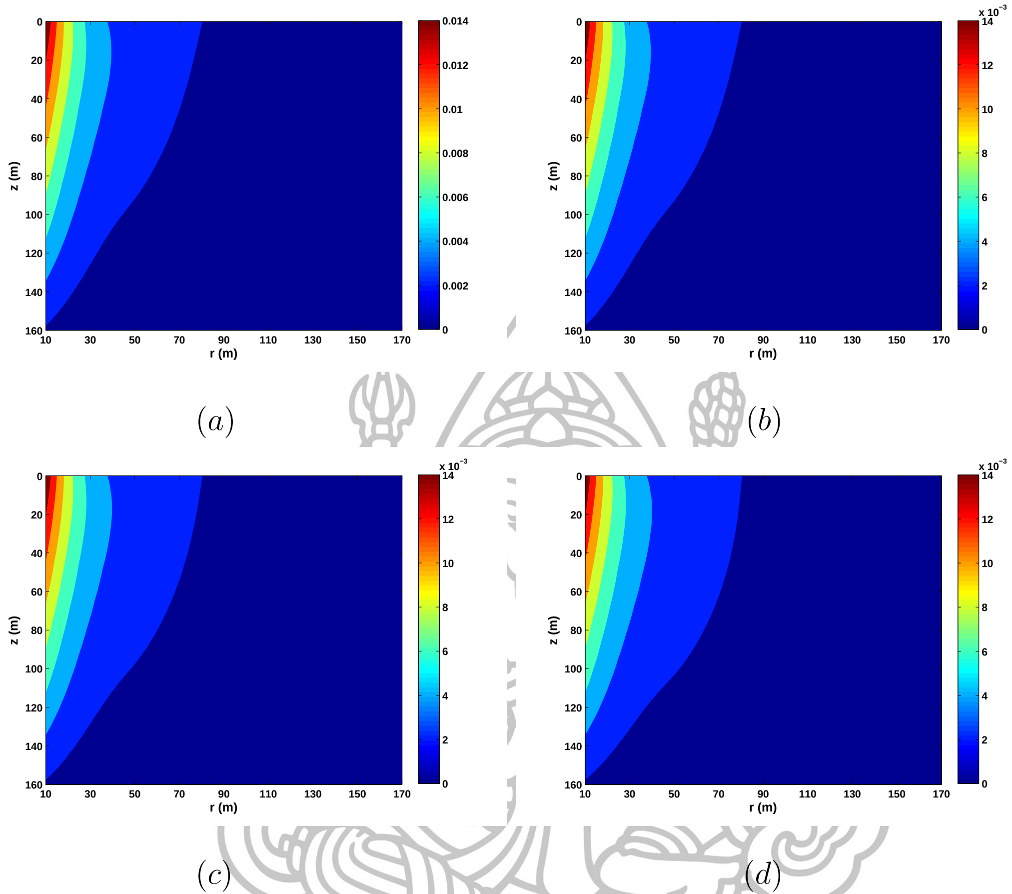


Figure 4.13: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0.001$, (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

From Figure 4.13 (a) to (d), when $a = 0.01, 0.05, 0.1, 0.2$ and 0.3 m^{-1}, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area where the values of magnetic field is low.

4.1.2 Summarize

In this section, we present a mathematical model by using the Magnetometric Resistivity Method with 2−dimensional continuously conductivity model as $\sigma(r, z) = \sigma_0 e^{(az+br)}$. The relationship between magnetic field and electric field is considered by using the Maxwell's equations. The magnetic field intensity is obtained by solving partial differential equation. The solution are obtained by using rectangular Finite Element Method. MATLAB program is used to calculate and plot graph for the value of magnetic field intensity. The behavior of magnetic field intensity will be performed at different depths and locations. In our research, the behavior of magnetic field decreases to zero when we increases the depth. As well as the case of increasing the space between source - receiver, the magnetic field decreases to zero too. The values of a and b are important role for the conduction of the ground and effect to the magnetic field quantities as well. For the high conductive ground (a and $b > 0$), the response of magnetic field will be very strong. In the opposite direction (a and $b < 0$), the response field will be very weak.

4.2 Triangular Elements

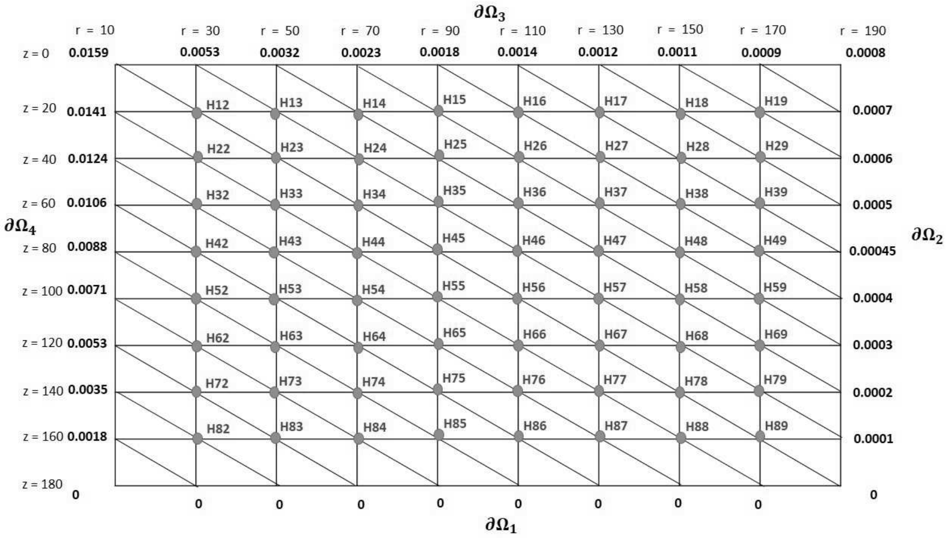
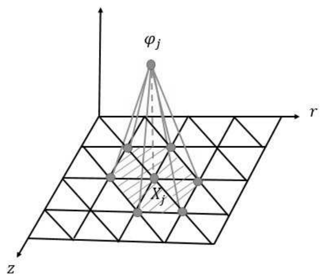


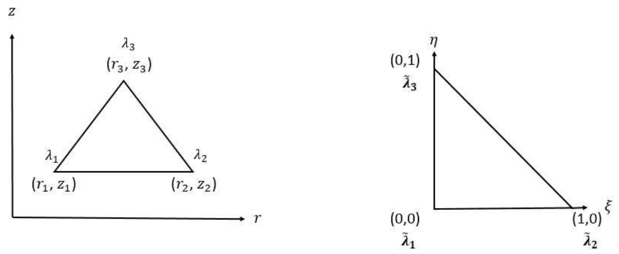
Figure 4.14: Discretizing the domain Ω using a triangular uniform grid.

We divide Ω into triangular elements, Ω_k, $2[(\bar{n}-1)\times(m-1)] = N_e$ elements, where $k = 1, 2, \ldots, N_e$, $\bar{n}$ is the partition in r direction and m is the partition in z direction. We denote $H(X_i)$, $i = 1, 2, \ldots, 100$ for H_i, $i = 1, 2, \ldots, 100$ and nodes X_i, $i = 1, 2, \ldots, 100$ for (r_i, z_i), $i = 1, 2, \ldots, 10$.

The support of φ_i consists of triangles with the common nodes X_j,



We transfrom each element Ω_k into reference element $\tilde{\Omega}$, by using the following transformation as



where λ_i is basis function at node i, $i = 1, 2, 3$.
Let ξ and η are our new variable in the coordinate (ξ, η).
The relationship between coordinate (r, z) to the basis functions in coordinate (ξ, η) are

$$r(\xi, \eta) = r_1\tilde{\lambda_1} + r_2\tilde{\lambda_2} + r_3\tilde{\lambda_3},$$

$$z(\xi, \eta) = z_1\tilde{\lambda_1} + z_2\tilde{\lambda_2} + z_3\tilde{\lambda_3},$$

where,

$$r(0, 0) = r_1, r(1, 0) = r_2, r(0, 1) = r_3,$$

$$z(0, 0) = z_1, r(1, 0) = z_2, r(0, 1) = z_3.$$

The basis functions can be written in the form of ξ and η as

$$\tilde{\lambda_1} = 1 - \xi - \eta,$$

$$\tilde{\lambda_2} = \xi,$$

$$\tilde{\lambda_3} = \eta.$$

Thus,

$$r(\xi, \eta) = r_1(1 - \xi - \eta) + r_2(\xi) + r_3(\eta) = r_1 + (r_2 - r_1)\xi + (r_3 - r_1)\eta.$$

and

$$z(\xi,\eta) = z_1(1-\xi-\eta) + z_2(\xi) + z_3(\eta) = z_1 + (z_2 - z_1)\xi + (z_3 - z_1)\eta.$$

The Jacobian matrix can be written as

$$\begin{bmatrix} dr \\ dz \end{bmatrix} = \begin{bmatrix} \dfrac{\partial r}{\partial \xi} & \dfrac{\partial r}{\partial \eta} \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial z}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} r_2 - r_1 & r_3 - r_1 \\ z_2 - z_1 & z_3 - z_1 \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}.$$

So, $drdz =\mid J \mid d\xi d\eta$.

We have two types of elements in Ω namely Ω^1 and Ω^2

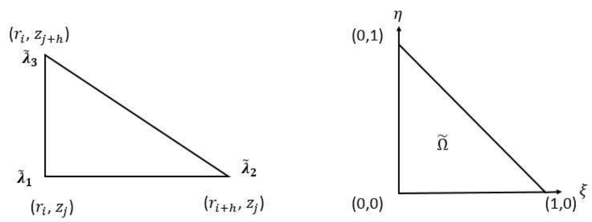


Figure 4.15: Ω^1.

such that

$$r = r_i + h\xi \qquad and \qquad z = z_j + h\eta$$

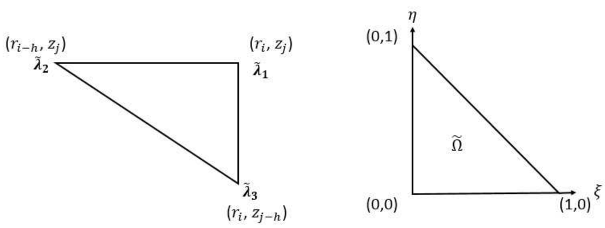


Figure 4.16: Ω^2.

such that

$$r = r_i - h\xi \qquad and \qquad z = z_j - h\eta.$$

Therefore, the Jacobian matrix of Ω^1

$$J = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix},$$

$$\mid J \mid = h^2,$$

and the Jacobian matrix of Ω^2 similar Ω^1.

So,

$$drdz = \mid J \mid d\xi d\eta = h^2 d\xi d\eta.$$

We now consider the member of A.

$$A_1 = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, d\Omega = \iint_\Omega (\frac{\partial\varphi_i}{\partial r}\frac{\partial\varphi_j}{\partial r} + \frac{\partial\varphi_i}{\partial z}\frac{\partial\varphi_j}{\partial z}) \, drdz, \tag{4.17}$$

After the transformation, we have

$$A_1 = \sum_{k=1}^{\tilde{N}_e} \iint_{\tilde{\Omega}_k} \left[(\frac{1}{h}\frac{\partial\tilde{\varphi}_i}{\partial\xi})(\frac{1}{h}\frac{\partial\tilde{\varphi}_j}{\partial\xi}) + (\frac{1}{h}\frac{\partial\tilde{\varphi}_i}{\partial\eta})(\frac{1}{h}\frac{\partial\tilde{\varphi}_j}{\partial\eta}) \right] (h^2) d\xi d\eta$$

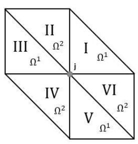
$$= \sum_{k=1}^{\tilde{N}_e} \int_0^1 \int_0^{1-\eta} (\frac{\partial\tilde{\varphi}_i}{\partial\xi}\frac{\partial\tilde{\varphi}_j}{\partial\xi} + \frac{\partial\tilde{\varphi}_i}{\partial\eta}\frac{\partial\tilde{\varphi}_j}{\partial\eta}) \, d\xi d\eta, \tag{4.18}$$

where $\tilde{N}_e$ is number of reference element.

For a fixed j, we have 7 cases for i, $i, j = 1, 2, \ldots, 100$.

Interior nodes

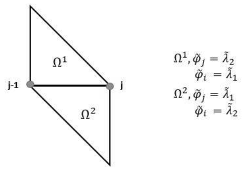
case 1 : $i = j$



$$A_1 = \int_0^1 \int_0^{1-\eta} 2[((\frac{\partial\tilde{\lambda}_1}{\partial\xi})^2 + (\frac{\partial\tilde{\lambda}_1}{\partial\eta})^2) + ((\frac{\partial\tilde{\lambda}_2}{\partial\xi})^2 + (\frac{\partial\tilde{\lambda}_2}{\partial\eta})^2) + ((\frac{\partial\tilde{\lambda}_3}{\partial\xi})^2 + (\frac{\partial\tilde{\lambda}_3}{\partial\eta})^2)] d\xi d\eta$$

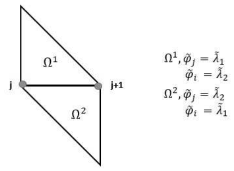
$$= 4 \, .$$

case $2 : i = j - 1$



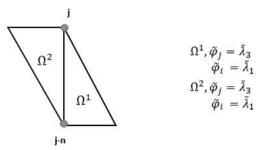
$$A_1 = \int_0^1 \int_0^{1-\eta} 2(\frac{\partial \tilde{\lambda}_1}{\partial \xi}\frac{\partial \tilde{\lambda}_2}{\partial \xi} + \frac{\partial \tilde{\lambda}_1}{\partial \eta}\frac{\partial \tilde{\lambda}_2}{\partial \eta}) d\xi d\eta = -1 \ .$$

case $3 : i = j + 1$



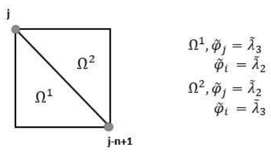
$$A_1 = \int_0^1 \int_0^{1-\eta} 2(\frac{\partial \tilde{\lambda}_2}{\partial \xi}\frac{\partial \tilde{\lambda}_1}{\partial \xi} + \frac{\partial \tilde{\lambda}_2}{\partial \eta}\frac{\partial \tilde{\lambda}_1}{\partial \eta}) d\xi d\eta = -1 \ .$$

case $4 : i = j - n$



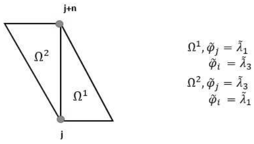
$$A_1 = \int_0^1 \int_0^{1-\eta} 2(\frac{\partial \tilde{\lambda}_1}{\partial \xi}\frac{\partial \tilde{\lambda}_3}{\partial \xi} + \frac{\partial \tilde{\lambda}_1}{\partial \eta}\frac{\partial \tilde{\lambda}_3}{\partial \eta}) d\xi d\eta = -1 \ .$$

case $5 : i = j - n + 1$



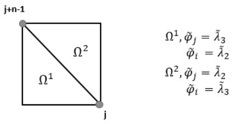
$$A_1 = \int_0^1 \int_0^{1-\eta} 2(\frac{\partial \tilde{\lambda}_2}{\partial \xi}\frac{\partial \tilde{\lambda}_3}{\partial \xi} + \frac{\partial \tilde{\lambda}_2}{\partial \eta}\frac{\partial \tilde{\lambda}_3}{\partial \eta}) d\xi d\eta = 0 \ .$$

case 6 : $i = j + n$



$$A_1 = \int_0^1 \int_0^{1-\eta} 2(\frac{\partial \tilde{\lambda}_3}{\partial \xi}\frac{\partial \tilde{\lambda}_1}{\partial \xi} + \frac{\partial \tilde{\lambda}_3}{\partial \eta}\frac{\partial \tilde{\lambda}_1}{\partial \eta}) d\xi d\eta = -1 \ .$$

case 7 : $i = j + n - 1$



$$A_1 = \int_0^1 \int_0^{1-\eta} 2(\frac{\partial \tilde{\lambda}_3}{\partial \xi}\frac{\partial \tilde{\lambda}_2}{\partial \xi} + \frac{\partial \tilde{\lambda}_3}{\partial \eta}\frac{\partial \tilde{\lambda}_2}{\partial \eta}) d\xi d\eta = 0 \ .$$

This gives the matrix A_1 has the form,

$$A_1 = (\frac{1}{2}) \begin{bmatrix} C & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B & D & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & D & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B & D & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & D & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B & D & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B & D & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & D & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & C \end{bmatrix}_{100\times100} ,$$

where

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}_{10\times10} ,$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}_{10\times 10},$$

$$D = \begin{bmatrix} 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 8 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 8 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 8 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 8 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 8 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 8 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix}_{10\times 10}.$$

The matrix A_2 can be determined as

$$A_2 = \int_{\Omega} a\varphi_i \frac{\partial \varphi_j}{\partial z}\, d\Omega = \iint_{\Omega} (a\varphi_i \frac{\partial \varphi_j}{\partial z})\, drdz, \tag{4.19}$$

After the transformation, we have

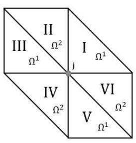
$$\begin{aligned} A_2 &= \sum_{k=1}^{\tilde{N}_e} \iint_{\tilde{\Omega}_k} \left[a\tilde{\varphi}_i (\frac{1}{h}\frac{\partial \tilde{\varphi}_j}{\partial \eta}) \right] (h^2) d\xi d\eta \\ &= ah \sum_{k=1}^{\tilde{N}_e} \int_0^1 \int_0^{1-\eta} (\tilde{\varphi}_i \frac{\partial \tilde{\varphi}_j}{\partial \eta})\, d\xi d\eta, \end{aligned} \tag{4.20}$$

where $\tilde{N}_e$ is number of reference element.

For a fixed j, we have 7 cases for i, $i, j = 1, 2, \ldots, 100$.

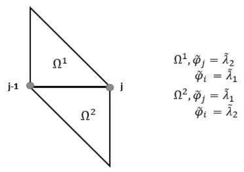
Interior nodes

case 1 : $i = j$



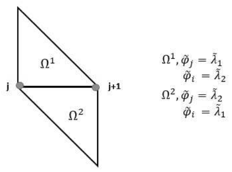
$$A_2 = \int_0^1 \int_0^{1-\eta} 2ah[(\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_1}{\partial \eta}) + (\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_3}{\partial \eta}) + (\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_2}{\partial \eta})] d\xi d\eta$$
$$= 0 \ .$$

case 2 : $i = j - 1$



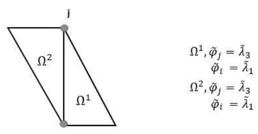
$$A_2 = \int_0^1 \int_0^{1-\eta} ah[(\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_2}{\partial \eta}) + (\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_1}{\partial \eta})] d\xi d\eta$$
$$= \int_0^1 \int_0^{1-\eta} ah[-\xi] d\xi d\eta = -\frac{ah}{6} \ .$$

case 3 : $i = j + 1$



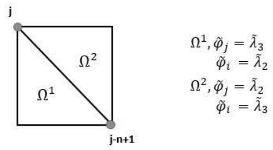
$$A_2 = \int_0^1 \int_0^{1-\eta} ah[(\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_1}{\partial \eta}) + (\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_2}{\partial \eta})] d\xi d\eta$$
$$= \int_0^1 \int_0^{1-\eta} ah[-\xi] d\xi d\eta = -\frac{ah}{6} \ .$$

case 4 : $i = j - n$



$$A_2 = \int_0^1 \int_0^{1-\eta} ah[(\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_3}{\partial \eta}) + (\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_1}{\partial \eta})] d\xi d\eta$$
$$= \int_0^1 \int_0^{1-\eta} ah[(1 - \xi - \eta) - \eta] d\xi d\eta = 0 \ .$$

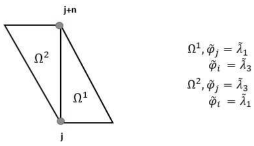
case $5 : i = j - n + 1$



$$A_2 = \int_0^1 \int_0^{1-\eta} ah[(\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_3}{\partial \eta}) + (\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_2}{\partial \eta})] d\xi d\eta$$

$$= \int_0^1 \int_0^{1-\eta} ah[\xi] d\xi d\eta = \frac{ah}{6} .$$

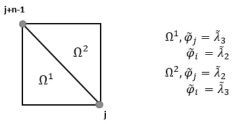
case $6 : i = j + n$



$$A_2 = \int_0^1 \int_0^{1-\eta} ah[(\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_1}{\partial \eta}) + (\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_3}{\partial \eta})] d\xi d\eta$$

$$= \int_0^1 \int_0^{1-\eta} ah[-\eta + (1 - \xi - \eta)] d\xi d\eta = 0 .$$

case $7 : i = j + n - 1$



$$A_2 = \int_0^1 \int_0^{1-\eta} ah[(\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_2}{\partial \eta}) + (\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_3}{\partial \eta})] d\xi d\eta$$

$$= \int_0^1 \int_0^{1-\eta} ah[\xi] d\xi d\eta = \frac{ah}{6} .$$

This gives the matrix A_2 has the form,

$$A_2 = (\frac{ah}{6}) \begin{bmatrix} C & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G & D & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G & D & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G & D & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & D & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & D & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & D & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & D & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & E \end{bmatrix}_{100\times 100},$$

where

$$C = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{10\times 10},$$

$$G = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{10\times 10},$$

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}_{10\times 10},$$

B = -C and E = -G .

The matrix A_3 can be determined as

$$A_3 = -\int_{\Omega} [(\frac{1}{r} - b)\varphi_i \frac{\partial \varphi_j}{\partial r}]\ d\Omega = -\iint_{\Omega} [(\frac{1}{r} - b)\varphi_i \frac{\partial \varphi_j}{\partial r}]\ drdz, \tag{4.21}$$

After the transformation, we have

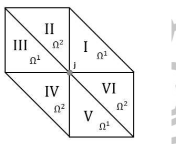
$$\begin{aligned} A_3 &= -\sum_{k=1}^{\tilde{N}_e} \iint_{\tilde{\Omega}_k} \left[(\frac{1}{r} - b)\tilde{\varphi}_i (\frac{1}{h}\frac{\partial \tilde{\varphi}_j}{\partial \xi}) \right]\ (h^2) d\xi d\eta \\ &= -h \sum_{k=1}^{\tilde{N}_e} \int_0^1 \int_0^{1-\eta} \left[(\frac{1}{r} - b)\tilde{\varphi}_i \frac{\partial \tilde{\varphi}_j}{\partial \xi} \right]\ d\xi d\eta. \end{aligned} \tag{4.22}$$

where $\Omega^1 : r = r_i + h\xi$, $\Omega^2 : r = r_i - h\xi$ and $\tilde{N}_e$ is number of reference element.

For a fixed j, we have 7 cases for i, $i, j = 1, 2, \dots, 100$.

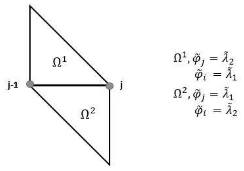
Interior nodes

case $1 : i = j$



$$\begin{aligned} A_3 = \int_0^1 \int_0^{1-\eta} & (-h)[(((\frac{1}{(r_i + h\xi)}) - b)\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_1}{\partial \xi}) + (((\frac{1}{(r_i - h\xi)}) - b)\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_3}{\partial \xi}) \\ & + (((\frac{1}{(r_{i-1} + h\xi)}) - b)\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_2}{\partial \xi}) + (((\frac{1}{(r_i - h\xi)}) - b)\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_1}{\partial \xi}) \\ & + (((\frac{1}{(r_i + h\xi)}) - b)\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_3}{\partial \xi}) + (((\frac{1}{(r_{i+1} - h\xi)}) - b)\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_2}{\partial \xi})]\ d\xi d\eta \\ = h \int_0^1 \int_0^{1-\eta} & [((\frac{1}{(r_i + h\xi)}) - b)(1 - \xi - \eta) - ((\frac{1}{(r_{i-1} + h\xi)}) - b)\xi \\ & + ((\frac{1}{(r_i - h\xi)}) - b)(1 - \xi - \eta) - ((\frac{1}{(r_{i+1} - h\xi)}) - b)\xi]\ d\xi d\eta \\ = h[\hat{a} - d + f - e]\ . & \end{aligned}$$

case $2 : i = j - 1$

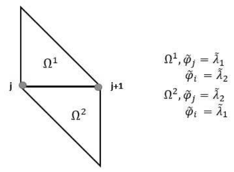


$$A_3 = \int_0^1 \int_0^{1-\eta} (-h)[(((\frac{1}{(r_i + h\xi)}) - b)\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_2}{\partial \xi}) + (((\frac{1}{(r_{i+1} - h\xi)}) - b)\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_1}{\partial \xi})]\ d\xi d\eta$$

$$= h \int_0^1 \int_0^{1-\eta} [((\frac{1}{(r_{i+1} - h\xi)}) - b)\xi - ((\frac{1}{(r_i + h\xi)}) - b)(1 - \xi - \eta)]\ d\xi d\eta$$

$$= h[e - \hat{a}]\ .$$

case $3 : i = j + 1$

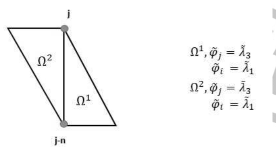


$$A_3 = \int_0^1 \int_0^{1-\eta} (-h)[(((\frac{1}{(r_{i-1} + h\xi)}) - b)\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_1}{\partial \xi}) + (((\frac{1}{(r_{i+1} - h\xi)}) - b)\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_2}{\partial \xi})]\ d\xi d\eta$$

$$= h \int_0^1 \int_0^{1-\eta} [((\frac{1}{(r_{i-1} + h\xi)}) - b)\xi - ((\frac{1}{(r_i - h\xi)}) - b)(1 - \xi - \eta)]\ d\xi d\eta$$

$$= h[\hat{b} - f]\ .$$

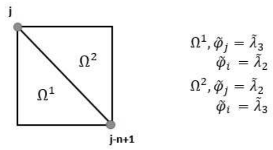
case $4 : i = j - n$



$$A_3 = \int_0^1 \int_0^{1-\eta} (-h)[(((\frac{1}{(r_i + h\xi)}) - b)\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_3}{\partial \xi}) + (((\frac{1}{(r_i - h\xi)}) - b)\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_1}{\partial \xi})]\ d\xi d\eta$$

$$= h \int_0^1 \int_0^{1-\eta} [((\frac{1}{(r_i - h\xi)}) - b)\eta]\ d\xi d\eta = h[g]\ .$$

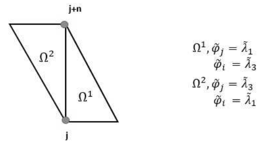
case $5 : i = j - n + 1$



$$A_3 = \int_0^1 \int_0^{1-\eta} (-h)[(((\frac{1}{(r_{i-1} + h\xi)}) - b)\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_3}{\partial \xi}) + (((\frac{1}{(r_i - h\xi)}) - b)\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_2}{\partial \xi})] \; d\xi d\eta$$

$$= h \int_0^1 \int_0^{1-\eta} [((\frac{1}{(r_i - h\xi)}) - b)\eta] \; d\xi d\eta = h[-g] \; .$$

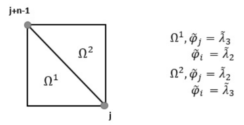
case $6 : i = j + n$



$$A_3 = \int_0^1 \int_0^{1-\eta} (-h)[(((\frac{1}{(r_i + h\xi)}) - b)\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_1}{\partial \xi}) + (((\frac{1}{(r_i - h\xi)}) - b)\tilde{\lambda}_1 \frac{\partial \tilde{\lambda}_3}{\partial \xi})] \; d\xi d\eta$$

$$= h \int_0^1 \int_0^{1-\eta} [((\frac{1}{(r_i + h\xi)}) - b)\eta] \; d\xi d\eta = h[c] \; .$$

case $7 : i = j + n - 1$



$$A_3 = \int_0^1 \int_0^{1-\eta} (-h)[(((\frac{1}{(r_i + h\xi)}) - b)\tilde{\lambda}_3 \frac{\partial \tilde{\lambda}_2}{\partial \xi}) + (((\frac{1}{(r_{i+1} - h\xi)}) - b)\tilde{\lambda}_2 \frac{\partial \tilde{\lambda}_3}{\partial \xi})] \; d\xi d\eta$$

$$= -h \int_0^1 \int_0^{1-\eta} [((\frac{1}{(r_i + h\xi)}) - b)\eta] \; d\xi d\eta = h[-c] \; .$$

This gives the matrix A_3 has the form,

$$A_3 = h\begin{bmatrix} C & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G & D & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G & D & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G & D & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & D & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & D & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & D & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & D & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & E \end{bmatrix}_{100\times100},$$

where

$$C = \begin{bmatrix} \hat{a} & -\hat{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{b} & (\hat{a}+d) & -\hat{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{b} & (\hat{a}+d) & -\hat{a} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{b} & (\hat{a}+d) & -\hat{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{b} & (\hat{a}+d) & -\hat{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{b} & (\hat{a}+d) & -\hat{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{b} & (\hat{a}+d) & -\hat{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{b} & (\hat{a}+d) & -\hat{a} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{b} & (\hat{a}+d) & -\hat{a} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{b} & -d \end{bmatrix}_{10\times10},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -g & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g & g & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g & g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -g & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g & g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g & g \end{bmatrix}_{10\times10},$$

$$G = \begin{bmatrix} c & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & -c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & -c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{10\times10},$$

$$D = \begin{bmatrix}
(\hat{a}-e) & (e-\hat{a}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\hat{b}-f) & U & (e-\hat{a}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (\hat{b}-f) & U & (e-\hat{a}) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\hat{b}-f) & U & (e-\hat{a}) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\hat{b}-f) & U & (e-\hat{a}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (\hat{b}-f) & U & (e-\hat{a}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (\hat{b}-f) & U & (e-\hat{a}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (\hat{b}-f) & U & (e-\hat{a}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{b}-f) & U & (e-\hat{a}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{b}-f) & (f-d)
\end{bmatrix}_{10\times 10},$$

where $\quad U = (\hat{a} - e + f - d)$

$$E = \begin{bmatrix}
-e & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-f & (f-e) & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -f & (f-e) & e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -f & (f-e) & e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -f & (f-e) & e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -f & (f-e) & e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -f & (f-e) & e & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -f & (f-e) & e & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -f & (f-e) & e \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -f & f
\end{bmatrix}_{10\times 10}.$$

such that

$$\hat{a} = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_i + h\xi)}) - b)(1 - \xi - \eta) \; d\xi d\eta,$$

$$d = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_{i-1} + h\xi)}) - b)\xi \; d\xi d\eta,$$

$$f = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_i - h\xi)}) - b)(1 - \xi - \eta) \; d\xi d\eta,$$

$$e = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_{i+1} - h\xi)}) - b)\xi \; d\xi d\eta,$$

$$\hat{b} = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_{i-1} + h\xi)}) - b)\xi \; d\xi d\eta,$$

$$g = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_i - h\xi)}) - b)\eta \; d\xi d\eta,$$

$$c = \int_0^{1-\eta} (\frac{1}{(r_i + h\xi)}) - b)\eta \; d\xi d\eta.$$

The matrix A_4 can be determined as

$$A_4 = \int_\Omega [(\frac{1}{r^2} + \frac{b}{r})\varphi_i \varphi_j] \; d\Omega = \iint_\Omega [(\frac{1}{r^2} + \frac{b}{r})\varphi_i \varphi_j] \; drdz, \tag{4.23}$$

After the transformation, we have

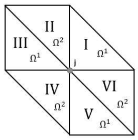
$$A_4 = h^2 \sum_{k=1}^{\tilde{N}_e} \int_0^1 \int_0^{1-\eta} [(\frac{1}{r^2} + \frac{b}{r})\tilde{\varphi}_i \tilde{\varphi}_j] \; d\xi d\eta. \tag{4.24}$$

where $\Omega^1 : r = r_i + h\xi$, $\Omega^2 : r = r_i - h\xi$ and $\tilde{N}_e$ is number of reference element.

For a fixed j, we have 7 cases for i, $i, j = 1, 2, \ldots, 100$.

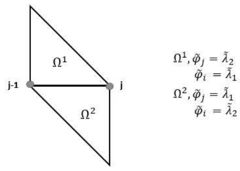
Interior nodes

case $1 : i = j$



$$A_4 = \int_0^1 \int_0^{1-\eta} (h^2)[((\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})\tilde{\lambda}_1\tilde{\lambda}_1) + ((\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})\tilde{\lambda}_3\tilde{\lambda}_3)$$
$$+ ((\frac{1}{(r_{i-1} + h\xi)^2} + \frac{b}{(r_{i-1} + h\xi)})\tilde{\lambda}_2\tilde{\lambda}_2) + ((\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})\tilde{\lambda}_1\tilde{\lambda}_1)$$
$$+ ((\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})\tilde{\lambda}_3\tilde{\lambda}_3) + ((\frac{1}{(r_{i+1} - h\xi)^2} + \frac{b}{(r_i - h\xi)})\tilde{\lambda}_2\tilde{\lambda}_2)] \; d\xi d\eta$$
$$= h^2 \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})(1 - \xi - \eta)^2 + (\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})(\eta)^2$$
$$+ (\frac{1}{(r_{i-1} + h\xi)^2} + \frac{b}{(r_{i-1} + h\xi)})(\xi)^2 + (\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})(1 - \xi - \eta)^2$$
$$+ (\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})(\eta)^2 + (\frac{1}{(r_{i+1} - h\xi)^2} + \frac{b}{(r_{i+1} - h\xi)})(\xi)^2] \; d\xi d\eta$$
$$= (h^2)[\hat{a} + \hat{b} + c + d + f + g] \; .$$

case $2 : i = j - 1$



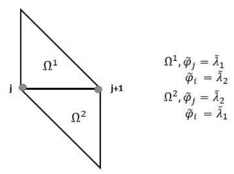
$$A_4 = \int_0^1 \int_0^{1-\eta} (h^2)[((\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})\tilde{\lambda}_1\tilde{\lambda}_2) + ((\frac{1}{(r_{i+1} - h\xi)^2} + \frac{b}{(r_{i+1} - h\xi)})\tilde{\lambda}_2\tilde{\lambda}_1)]\ d\xi d\eta$$

$$= h^2 \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})(1 - \xi - \eta)(\xi)$$

$$+ (\frac{1}{(r_{i+1} - h\xi)^2} + \frac{b}{(r_{i+1} - h\xi)})(\xi)(1 - \xi - \eta)]\ d\xi d\eta$$

$$= (h^2)[k + \hat{l}]\ .$$

case $3 : i = j + 1$

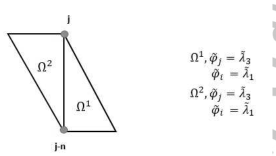


$$A_4 = \int_0^1 \int_0^{1-\eta} (h^2)[((\frac{1}{(r_{i-1} + h\xi)^2} + \frac{b}{(r_{i-1} + h\xi)})\tilde{\lambda}_2\tilde{\lambda}_1) + ((\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})\tilde{\lambda}_1\tilde{\lambda}_2)]\ d\xi d\eta$$

$$= h^2 \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_{i-1} + h\xi)^2} + \frac{b}{(r_{i-1} + h\xi)})(\xi)(1 - \xi - \eta)$$

$$+ (\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})(1 - \xi - \eta)(\xi)]\ d\xi d\eta = (h^2)[\hat{k} + l]\ .$$

case $4 : i = j - n$

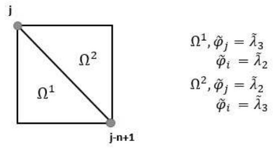


$$A_4 = \int_0^1 \int_0^{1-\eta} (h^2)[((\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})\tilde{\lambda}_1\tilde{\lambda}_3) + ((\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})\tilde{\lambda}_3\tilde{\lambda}_1)]\ d\xi d\eta$$

$$= h^2 \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})(1 - \xi - \eta)(\eta)$$

$$+ (\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})(\eta)(1 - \xi - \eta)]\ d\xi d\eta = (h^2)[\hat{m} + \hat{n}]\ .$$

case 5 : $i = j - n + 1$

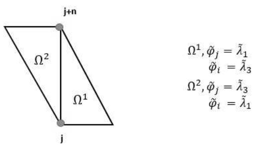


$$A_4 = \int_0^1 \int_0^{1-\eta} (h^2)[((\frac{1}{(r_{i-1}+h\xi)^2} + \frac{b}{(r_{i-1}+h\xi)})\tilde{\lambda}_2\tilde{\lambda}_3) + ((\frac{1}{(r_i-h\xi)^2} + \frac{b}{(r_i-h\xi)})\tilde{\lambda}_3\tilde{\lambda}_2)]\ d\xi d\eta$$

$$= h^2 \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_{i-1}+h\xi)^2} + \frac{b}{(r_{i-1}+h\xi)})(\xi\eta)$$

$$+ (\frac{1}{(r_i-h\xi)^2} + \frac{b}{(r_i-h\xi)})(\eta\xi)]\ d\xi d\eta = (h^2)[s+t]\ .$$

case 6 : $i = j + n$

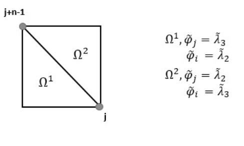


$$A_4 = \int_0^1 \int_0^{1-\eta} (h^2)[((\frac{1}{(r_i+h\xi)^2} + \frac{b}{(r_i+h\xi)})\tilde{\lambda}_3\tilde{\lambda}_1) + ((\frac{1}{(r_i-h\xi)^2} + \frac{b}{(r_i-h\xi)})\tilde{\lambda}_1\tilde{\lambda}_3)]\ d\xi d\eta$$

$$= h^2 \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_i+h\xi)^2} + \frac{b}{(r_i+h\xi)})(\eta)(1-\xi-\eta)$$

$$+ (\frac{1}{(r_i-h\xi)^2} + \frac{b}{(r_i-h\xi)})(1-\xi-\eta)(\eta)]\ d\xi d\eta = (h^2)[\hat{m}+\hat{n}]\ .$$

case 7 : $i = j + n - 1$



$$A_4 = \int_0^1 \int_0^{1-\eta} (h^2)[((\frac{1}{(r_i+h\xi)^2} + \frac{b}{(r_i+h\xi)})\tilde{\lambda}_3\tilde{\lambda}_2) + ((\frac{1}{(r_{i+1}-h\xi)^2} + \frac{b}{(r_{i+1}-h\xi)})\tilde{\lambda}_2\tilde{\lambda}_3)]\ d\xi d\eta$$

$$= h^2 \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_i+h\xi)^2} + \frac{b}{(r_i+h\xi)})(\eta\xi) + (\frac{1}{(r_{i+1}-h\xi)^2} + \frac{b}{(r_{i+1}-h\xi)})(\xi\eta)]\ d\xi d\eta$$

$$= (h^2)[q+\hat{r}]\ .$$

This gives the matrix A_4 has the form,

$$A_4 = h^2 \begin{bmatrix} C & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G & D & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G & D & B & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G & D & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & D & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & D & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G & D & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & D & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & D & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & E \end{bmatrix}_{100\times100},$$

where

$$C = \begin{bmatrix} \hat{a} & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{k} & (\hat{a}+\hat{b}+c) & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{k} & (\hat{a}+\hat{b}+c) & k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{k} & (\hat{a}+\hat{b}+c) & k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{k} & (\hat{a}+\hat{b}+c) & k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{k} & (\hat{a}+\hat{b}+c) & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{k} & (\hat{a}+\hat{b}+c) & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \hat{k} & (\hat{a}+\hat{b}+c) & k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{k} & (\hat{a}+\hat{b}+c) & k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{k} & \hat{b}+c \end{bmatrix}_{10\times10},$$

$$B = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (s+t) & (m+n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (s+t) & (m+n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (s+t) & (m+n) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (s+t) & (m+n) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (s+t) & (m+n) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (s+t) & (m+n) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (s+t) & (m+n) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (s+t) & (m+n) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (s+t) & n \end{bmatrix}_{10\times10},$$

$$G = \begin{bmatrix} m & (q+\hat{r}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (m+n) & (q+\hat{r}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (m+n) & (q+\hat{r}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (m+n) & (q+\hat{r}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (m+n) & (q+\hat{r}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (m+n) & (q+\hat{r}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (m+n) & (q+\hat{r}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (m+n) & (q+\hat{r}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (m+n) & (q+\hat{r}) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}_{10\times10},$$

$$D = \begin{bmatrix}
(\hat{a}+f+g) & (k+\hat{l}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(\hat{k}+l) & U & (k+\hat{l}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (\hat{k}+l) & U & (k+\hat{l}) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\hat{k}+l) & U & (k+\hat{l}) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (\hat{k}+l) & U & (k+\hat{l}) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (\hat{k}+l) & U & (k+\hat{l}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (\hat{k}+l) & U & (k+\hat{l}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (\hat{k}+l) & U & (k+\hat{l}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{k}+l) & U & (k+\hat{l}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\hat{k}+l) & (\hat{b}+c+d)
\end{bmatrix}_{10\times 10},$$

where $\qquad U = (\hat{a}+\hat{b}+c+d)$

$$E = \begin{bmatrix}
f+g & \hat{l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
l & (d+f+g) & \hat{l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & l & (d+f+g) & \hat{l} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & l & (d+f+g) & \hat{l} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & l & (d+f+g) & \hat{l} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & l & (d+f+g) & \hat{l} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & l & (d+f+g) & \hat{l} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & l & (d+f+g) & \hat{l} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & l & (d+f+g) & \hat{l} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l & d
\end{bmatrix}_{10\times 10}.$$

such that

$$\hat{a} = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_i+h\xi)^2}+\frac{b}{(r_i+h\xi)})(1-\xi-\eta)^2\ d\xi d\eta,$$

$$\hat{b} = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_i-h\xi)^2}+\frac{b}{(r_i-h\xi)})(\eta)^2\ d\xi d\eta,$$

$$c = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_{i-1}+h\xi)^2}+\frac{b}{(r_{i-1}+h\xi)})(\xi)^2\ d\xi d\eta,$$

$$d = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_i-h\xi)^2}+\frac{b}{(r_i-h\xi)})(1-\xi-\eta)^2 d\xi d\eta,$$

$$f = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_i+h\xi)^2}+\frac{b}{(r_i+h\xi)})(\eta)^2\xi d\eta,$$

$$g = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_{i+1}-h\xi)^2}+\frac{b}{(r_{i+1}-h\xi)})(\xi)^2\ d\xi d\eta,$$

$$k = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_i+h\xi)^2}+\frac{b}{(r_i+h\xi)})(1-\xi-\eta)(\xi)\ d\xi d\eta,$$

$$\hat{l} = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_{i+1}-h\xi)^2}+\frac{b}{(r_{i+1}-h\xi)})(\xi)(1-\xi-\eta)\ d\xi d\eta,$$

$$\hat{k} = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_{i-1}+h\xi)^2}+\frac{b}{(r_{i-1}+h\xi)})(\xi)(1-\xi-\eta)\ d\xi d\eta,$$

$$l = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_i-h\xi)^2}+\frac{b}{(r_i-h\xi)})(1-\xi-\eta)(\xi)\ d\xi d\eta,$$

$$\hat{m} = \int_0^1\int_0^{1-\eta}(\frac{1}{(r_i+h\xi)^2}+\frac{b}{(r_i+h\xi)})(\eta)(1-\xi-\eta)\ d\xi d\eta,$$

such that

$$\hat{n} = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})(1-\xi-\eta)(\eta)\ d\xi d\eta,$$

$$s = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_{i-1} + h\xi)^2} + \frac{b}{(r_{i-1} + h\xi)})(\xi\eta)\ d\xi d\eta,$$

$$t = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_i - h\xi)^2} + \frac{b}{(r_i - h\xi)})(\eta\xi)\ d\xi d\eta,$$

$$q = \int_0^1 \int_0^{1-\eta} [(\frac{1}{(r_i + h\xi)^2} + \frac{b}{(r_i + h\xi)})(\eta\xi)\ d\xi d\eta,$$

$$\hat{r} = \int_0^1 \int_0^{1-\eta} (\frac{1}{(r_{i+1} - h\xi)^2} + \frac{b}{(r_{i+1} - h\xi)})(\xi\eta)\ d\xi d\eta.$$

The system becomes

$$A\bar{H} = 0,$$

$$\Big[A_{i,j} = A_{1(i,j)} + A_{2(i,j)} + A_{3(i,j)} + A_{4(i,j)}\Big]_{100\times 100} \begin{bmatrix} H_1 \\ H_2 \\ . \\ . \\ . \\ H_{n-1} \\ H_n \end{bmatrix}_{100\times 1} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}_{100\times 1}.$$

Observe that the boundary conditions (4.2) . $H_1, H_2, \ldots, H_{10}$ and $H_{10C+1}, H_{10(C+1)}$, $n = 1, 2, \ldots, 100$ and $C = 1, 2, \ldots, 8$ and $H_{91}, H_{92}, \ldots, H_{100}$ are known value. So we don't have to solve for H is known value.

The system becomes

$$A\bar{H} = F,$$

$$\Big[A_{i,j}\Big]_{64\times 64} \begin{bmatrix} H_{M1} \\ H_{M2} \\ . \\ . \\ . \\ H_{M8} \end{bmatrix}_{64\times 1} = \begin{bmatrix} f_{M1} \\ f_{M2} \\ . \\ . \\ . \\ f_{M8} \end{bmatrix}_{64\times 1},$$

where

$$H_{MC} = \begin{bmatrix} H_{10C+2} \\ H_{10C+3} \\ . \\ . \\ . \\ H_{10C+9} \end{bmatrix}_{8\times 1} \quad and \quad f_{MC} = \begin{bmatrix} f_{10C+2} \\ f_{10C+3} \\ . \\ . \\ . \\ f_{10C+9} \end{bmatrix}_{8\times 1},$$

$C = 1, 2, \ldots, 8.$

Here

$f_{12} = f_{12} - A_{12,2}H_2 - A_{12,1}H_1 - A_{12,11}H_{11},$

$f_{19} = f_{19} - A_{19,8}H_8 - A_{19,9}H_9 - A_{19,20}H_{20} - A_{19,30}H_{30},$

$f_{82} = f_{82} - A_{82,71}H_{71} - A_{82,81}H_{81} - A_{82,92}H_{92},$

$f_{89} = f_{89} - A_{89,99}H_{99} - A_{89,100}H_{100} - A_{89,90}H_{90},$

Let $i = 13, 14, 15, 16, 17, 18$

$f_i = f_i - A_{i,(i-10)}H_{(i-10)} - A_{i,(i-11)}H_{(i-11)},$

Let $i = 22, 32, 42, 52, 62, 72$

$f_i = f_i - A_{i,(i-1)}H_{(i-1)} - A_{i,(i-11)}H_{(i-11)},$

Let $i = 29, 39, 49, 59, 69, 79$

$f_i = f_i - A_{i,(i+1)}H_{(i+1)} - A_{i,(i+11)}H_{(i+11)}\ .$

4.2.1 Numerical Experiments

Numerical results of the magnetic field intensity from equation (4.1) are obtained by using the Finite Element Method form triangular elements. There is a source providing a DC voltage of direct current $I = 1A$ and receiver on the ground surface which picks up the signal from $r = 10$ to $r = 190$ m. The depth z start from the ground surface $z = 0$ to $z = 180$ m. The grid size $h = 20$ m. a and b are given constants . The magnetic field intensity is computed by using MATLAB programing.

Consider for the case of an exponentially decreasing conductivity $\sigma(r, z) = \sigma_0 e^{(az+br)}$ when $a < 0$ and $b = 0$, the graphs of the relationship between magnetic field intensity and spacing of source-receiver at various depths are plotted as shown in Figure 4.17 and 4.18.

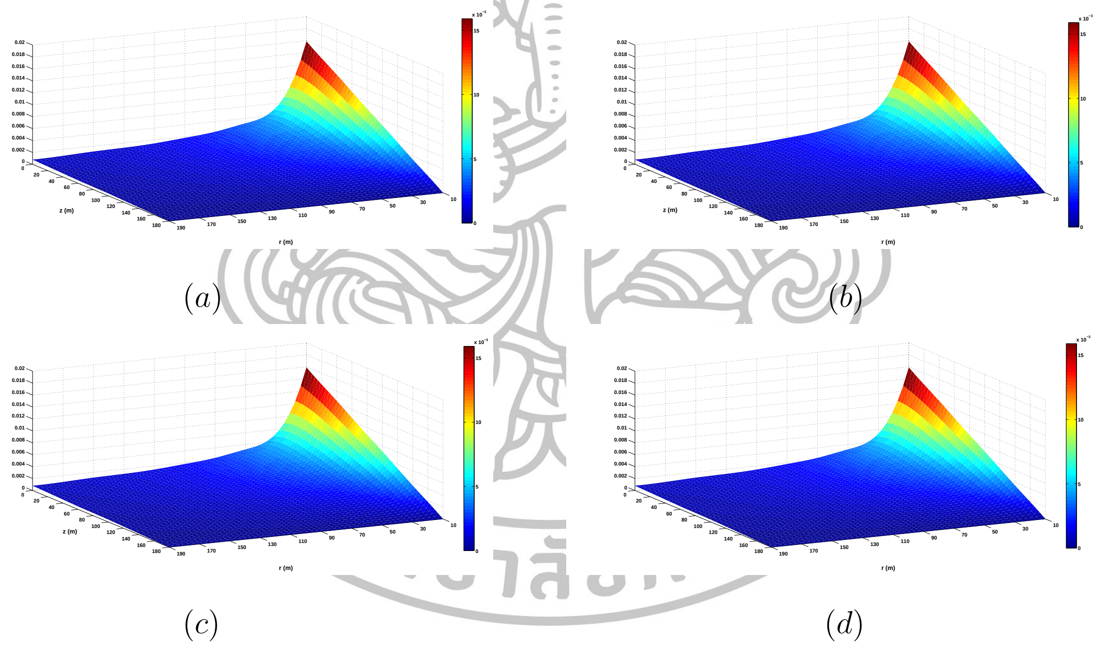
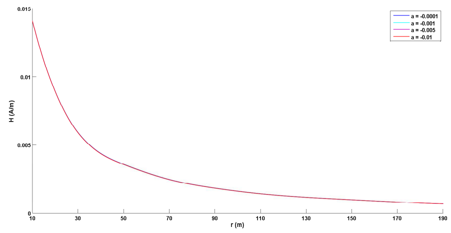
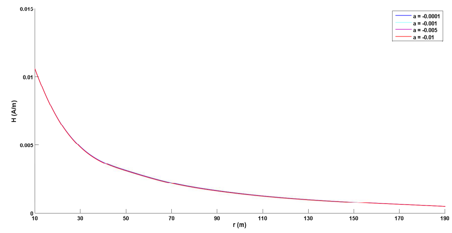


Figure 4.17: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0$, a is varied and z is fixed (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

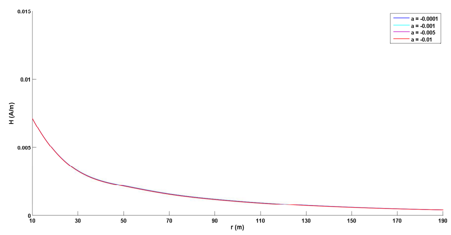
From Figure 4.17 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and $-0.01\ m^{-1}$, respectively, we can see that the values of magnetic field decrease exponentially as r and z increase. The values of magnetic field decrease as a decreases.



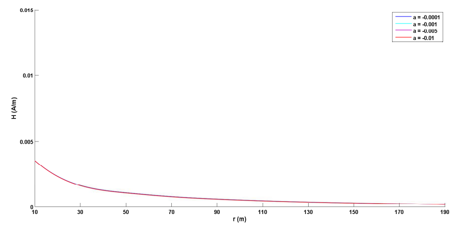
(a)



(b)



(c)

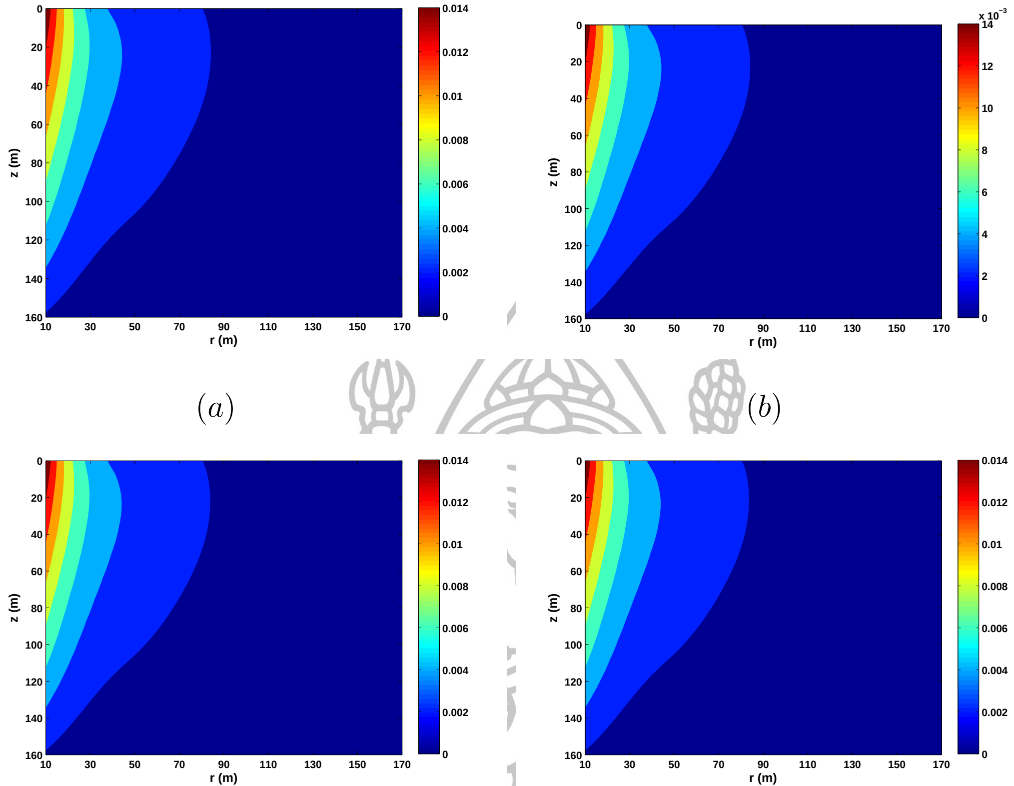


(d)

Figure 4.18: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0$, a varies from -0.0001, -0.001, -0.005 and $-0.01\ m^{-1}$ and z is fixed. (a) $z = 20\ m$ (b) $z = 60\ m$ (c) $z = 100\ m$ and (d) $z = 140\ m$.

From Figure 4.18 (a) to (d) represents the values of magnetic field which are plotted against r whereas a varies and z is fixed at $20, 60, 100$ and $140\ m$, respectively. We can see that the values of magnetic fields where $a = -0.0001, -0.001, -0.005$ and $-0.01\ m^{-1}$ decrease exponentially as r increases and it has similar maner when z increases. Because the values of magnetic field decrease to zero and have values near zero when z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.19.

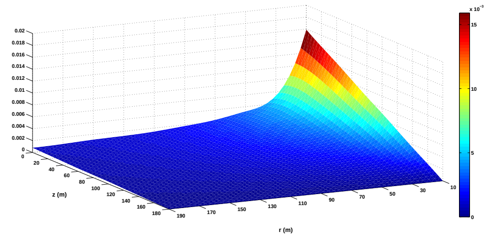


(a) (b) (c) (d)

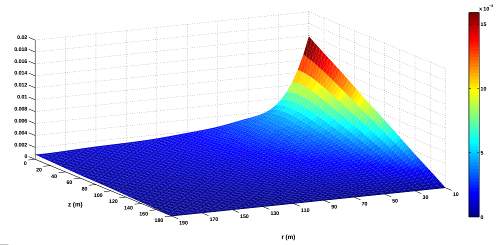
Figure 4.19: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0$, (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

From Figure 4.19 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1}, respectively, the red color shows the area when the values of magnetic field is high and the blue color shows the area when the values of magnetic field is low.

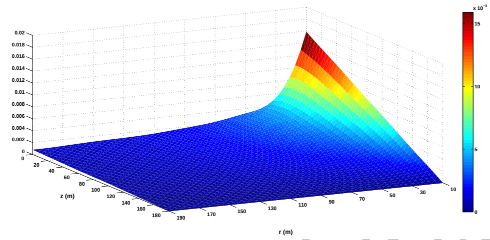
Consider for the case of an exponentially increasing conductivity $\sigma(r,z) = \sigma_0 e^{(az+br)}$ where $a > 0$ and $b = 0$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 4.20 and 4.21.



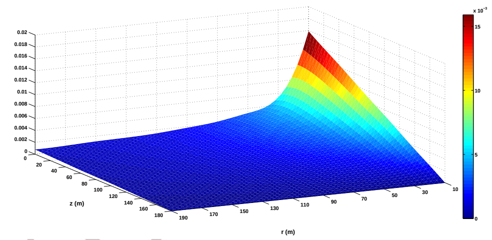
(a)



(b)



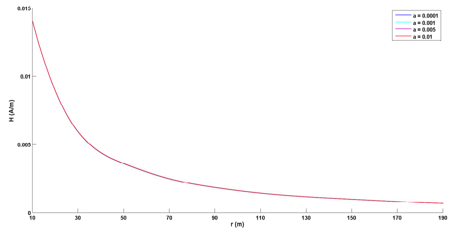
(c)



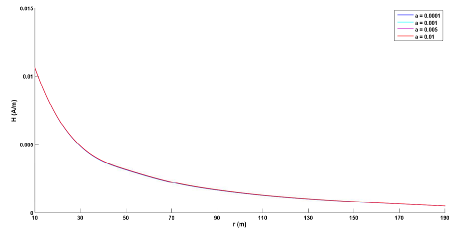
(d)

Figure 4.20: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0$, a varies and z is fixed (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

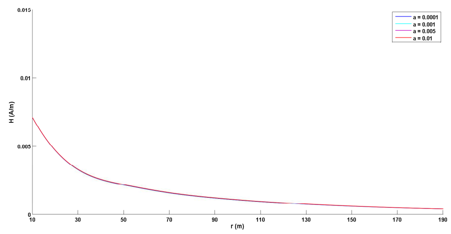
From Figure 4.20 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1}, respectively, we can see that the values of magnetic fields decrease exponentially as r and z increase. The values of magnetic fields increase whereas a increases. The results agree to Tunnurak et al. [12].



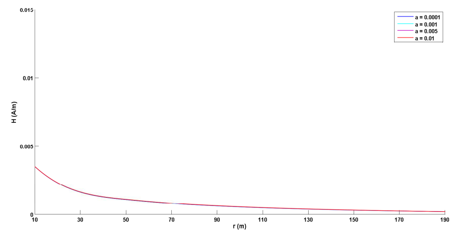
(a)



(b)



(c)

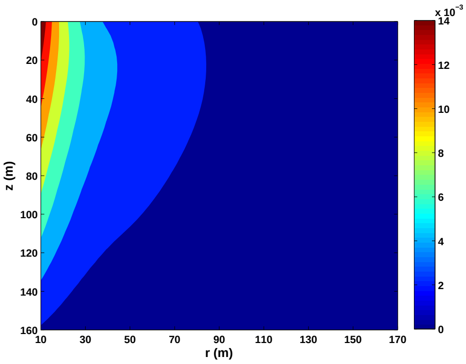


(d)

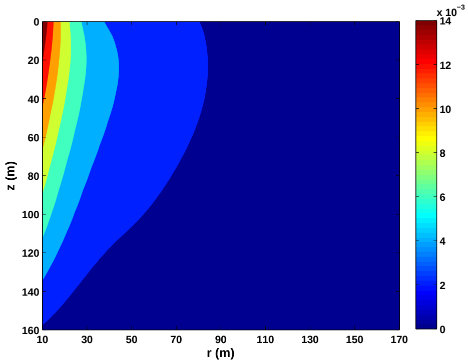
Figure 4.21: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0$, a varies from 0.0001, 0.001, 0.005 and 0.01 m^{-1} and z is fixed. (a) $z = 20\ m$ (b) $z = 60\ m$ (c) $z = 100\ m$ and (d) $z = 140\ m$.

From Figure 4.21 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and $140\ m$, respectively. We can see that the values of magnetic fields where $a = 0.0001, 0.001, 0.005$ and $0.01\ m^{-1}$ decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic fields decrease to zero and has value near zero where z increases as a varies.

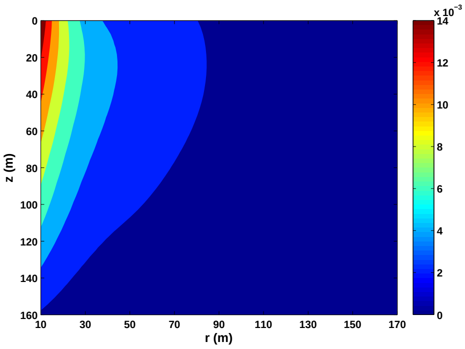
Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.22.



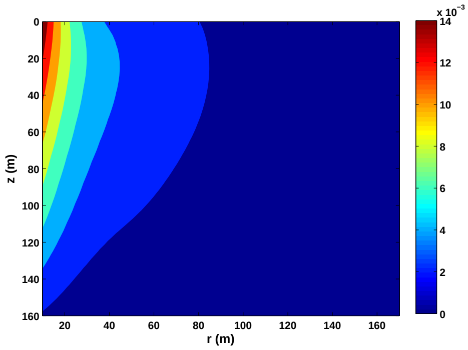
(a)



(b)



(c)

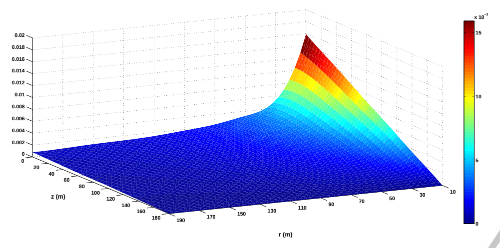


(d)

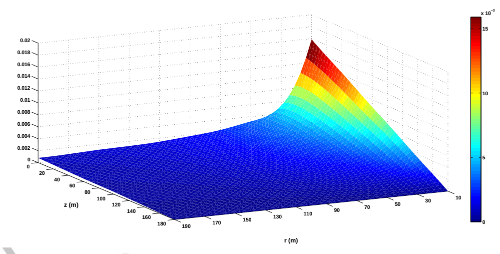
Figure 4.22: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0$, (a) $a = 0.0001 \ m^{-1}$ (b) $a = 0.001 \ m^{-1}$ (c) $a = 0.005 \ m^{-1}$ and (d) $a = 0.01 \ m^{-1}$.

From Figure 4.22 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and $0.01 \ m^{-1}$, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area when the values of magnetic field is low. The results agree to Tunnurak et al. [12].

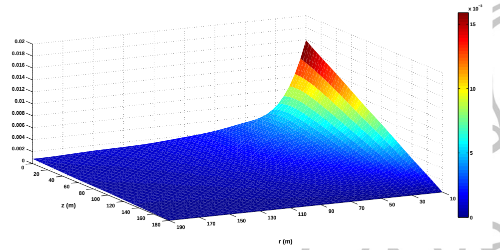
Consider for the case of an exponentially decreasing conductivity $\sigma(r, z) = \sigma_0 e^{(az+br)}$ when $a < 0$ and $b = -0.001$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 4.23 and 4.24.



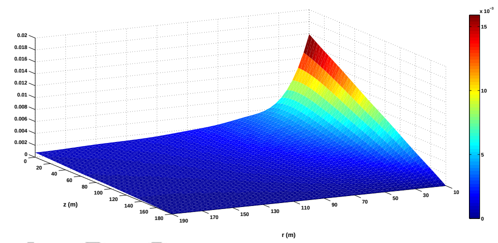
(a)



(b)



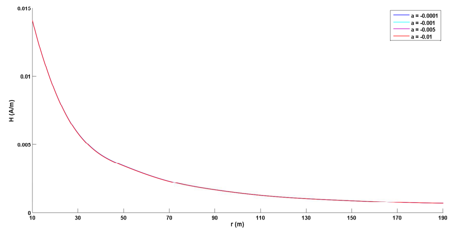
(c)



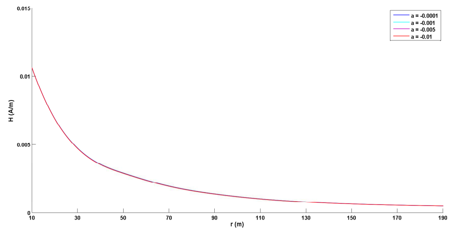
(d)

Figure 4.23: Graphs of the magnetic field intensity via distance of receiver from source where $b = -0.001$, a varies and z is fixed (a) $a = -0.0001$ m^{-1} (b) $a = -0.001$ m^{-1} (c) $a = -0.005$ m^{-1} and (d) $a = -0.01$ m^{-1}.

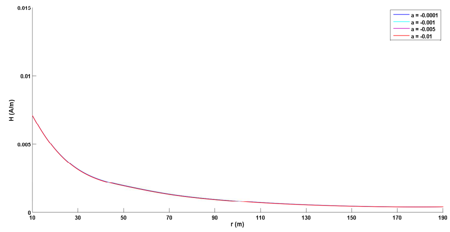
From Figure 4.23 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1}, respectively, we can see that the values of magnetic field decrease exponentially as r and z increase. The values of magnetic field decrease where a decreases.



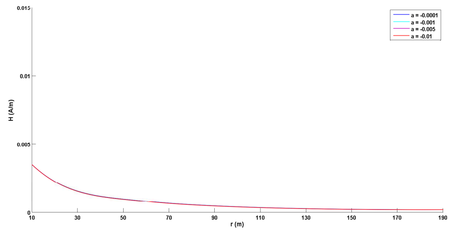
(a)



(b)



(c)



(d)

Figure 4.24: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = -0.001$, a varies from -0.0001, -0.001, -0.005 and -0.01 m^{-1} as z is fixed. (a) $z = 20$ m (b) $z = 60$ m (c) $z = 100$ m and (d) $z = 140$ m.

From Figure 4.24 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = -0.0001, -0.001, -0.005$ and -0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic field decrease to zero and have values near zero where z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.25.

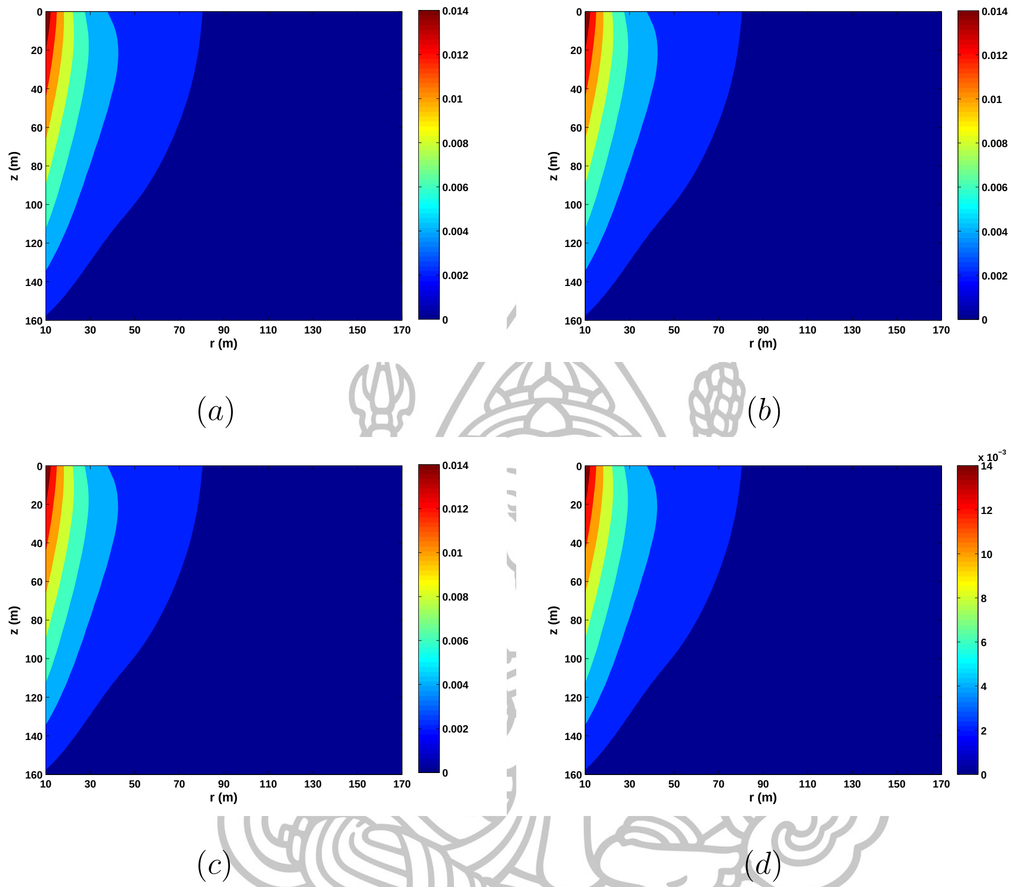


Figure 4.25: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = -0.001$, (a) $a = -0.0001 \quad m^{-1}$ (b) $a = -0.001 \quad m^{-1}$ (c) $a = -0.005\ m^{-1}$ and (d) $a = -0.01\ m^{-1}$.

From Figure 4.25 (a) to (d), when $a = -0.0001, -0.001, -0.005$ and $-0.01\ m^{-1}$, respectively, the red color shows the area when the values of magnetic field is high and the blue color shows the area when the values of magnetic field is low.

Consider for the case of an exponentially increasing conductivity $\sigma(r,z) = \sigma_0 e^{(az+br)}$ when $a > 0$ and $b = 0.001$, the graphs of the relationship between magnetic field intensity and spacing of source - receiver at various depths are plotted as shown in Figure 4.26 and 4.27.

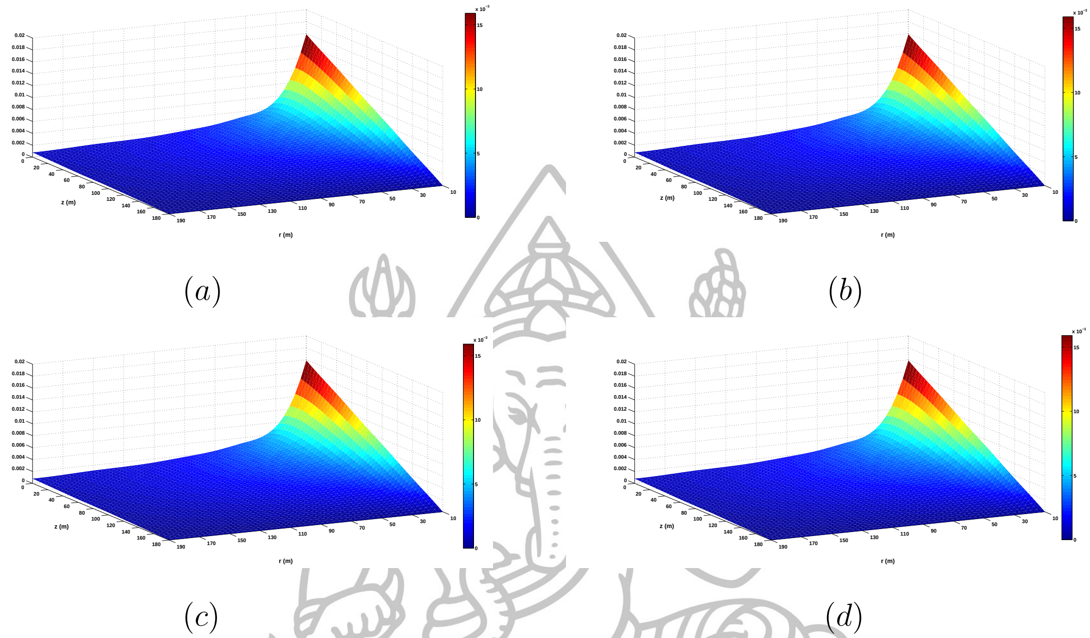
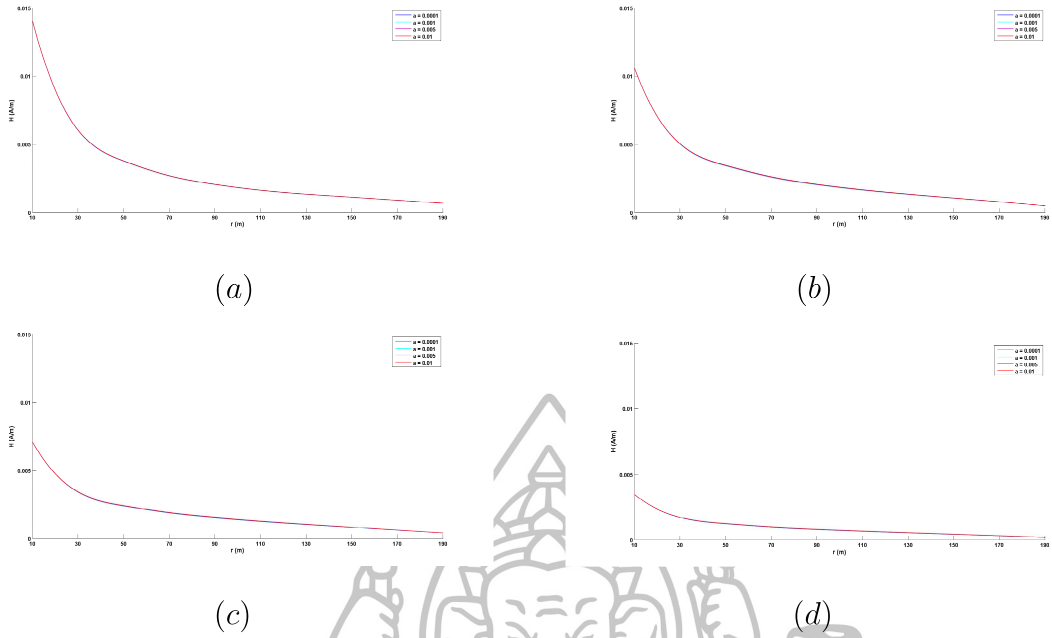


Figure 4.26: Graphs of the magnetic field intensity via distance of receiver from source where $b = 0.001$, a varies and z is fixed (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

From Figure 4.26 (a) to (d), when $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1}, respectively, we can see that the values of magnetic fields decrease exponentially as r and z increase. The values of magnetic fields increase when a increases.



(*a*) (*b*) (*c*) (*d*)

Figure 4.27: Graphs of the relationship between magnetic field intensity and distance of receiver from source where $b = 0.001$, a varies from 0.0001, 0.001, 0.005 and 0.01 m^{-1} and z is fixed. (a) $z = 20$ m (b) $z = 60$ m (c) $z = 100$ m and (d) $z = 140$ m.

From Figure 4.27 (a) to (d) represents the values of magnetic field which are plotted against r where a varies and z is fixed at $20, 60, 100$ and 140 m, respectively. We can see that the values of magnetic fields where $a = 0.0001, 0.001, 0.005$ and 0.01 m^{-1} decrease exponentially as r increases and it has similar maner where z increases. Because the values of magnetic fields decrease to zero and has value near zero where z increases as a varies.

Contour graphs of the relationship between magnetic field and distance of receiver from source at various depth are plotted as shown in Figure 4.28.

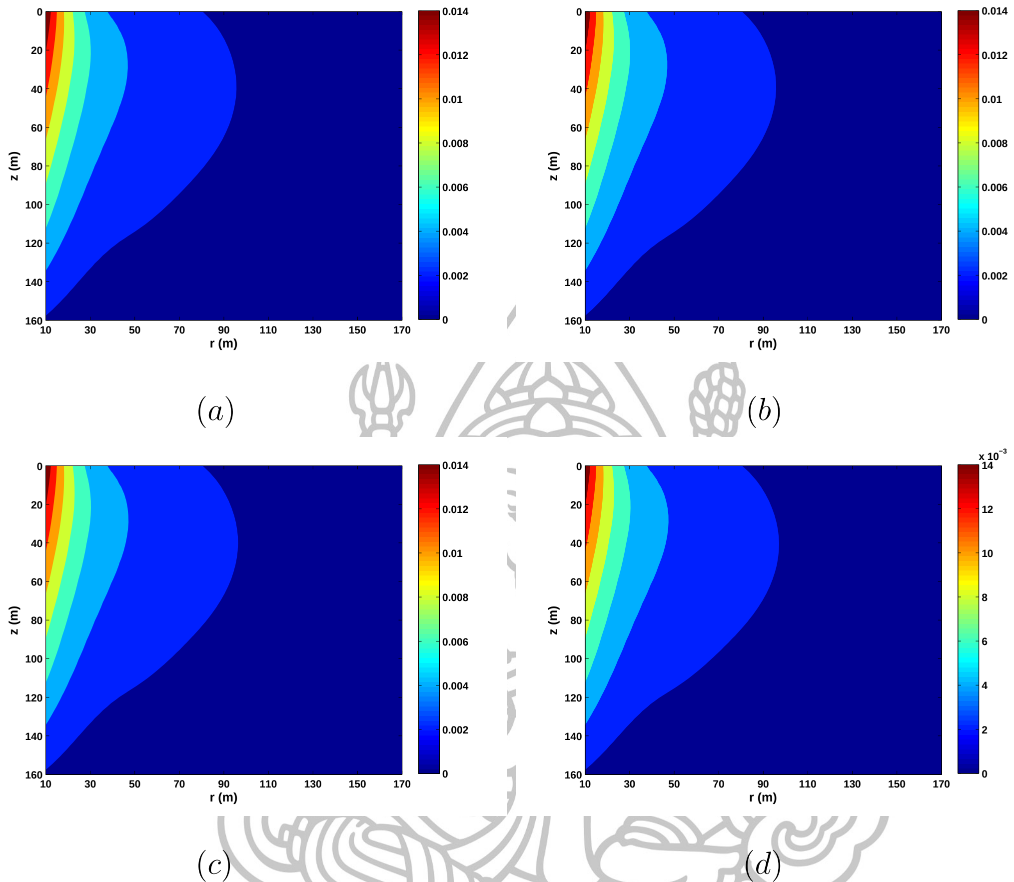


Figure 4.28: Contour graphs of magnetic field at different distances of receiver from source and different depths where $b = 0.001$, (a) $a = 0.0001$ m^{-1} (b) $a = 0.001$ m^{-1} (c) $a = 0.005$ m^{-1} and (d) $a = 0.01$ m^{-1}.

From Figure 4.28 (a) to (d), when $a = 0.01, 0.05, 0.1, 0.2$ and 0.3 m^{-1}, respectively, the red color shows the area where the values of magnetic field is high and the blue color shows the area where the values of magnetic field is low.

4.2.2 Summarize

In this section, we present a mathematical model by using the Magnetometric Resistivity Method with 2−dimensional continuously conductivity model as $\sigma(r, z) = \sigma_0 e^{(az+br)}$. The relationship between magnetic field and electric field is considered by using the Maxwell's equations. The magnetic field intensity is obtained by solving partial differential equation. The solution are obtained by using triangular Finite Element Method. MATLAB program is used to calculate and plot graph for the value of magnetic field intensity. The behavior of magnetic field intensity will be performed at different depths and locations. In our research, the behavior of magnetic field decreases to zero when we increases the depth. As well as the case of increasing the space between source - receiver, the magnetic field decreases to zero too. The values of a and b are important role for the conduction of the ground and effect to the magnetic field quantities as well. For the high conductive ground (a and $b > 0$), the response of magnetic field will be very strong. In the opposite direction (a and $b < 0$), the response field will be very weak.

Chapter 5

Conclusions and Future Works

The aim of this thesis is to present a mathematical model by using the Magnetometric Resistivity Method with a 2−dimensional continuously conductivity model as $\sigma(r,z) = \sigma_0 e^{(az+br)}$. The relationship between magnetic field and electric field are considered by using Maxwell's equations. The magnetic field intensity is obtained by solving partial differential equation. The solution are obtained by using Finite Difference Method and Finite Element Method. MATLAB program is used to calculate and plot graph for the value of magnetic field intensity. The behavior of magnetic field intensity will be performed at different depths and locations. In our research, the behavior of magnetic field decreases to zero when the depth of soil increases. As well as the case of increasing the space between source - receiver, the magnetic field decreases to zero too. The values of a and b are important role for the conduction of the ground and effect to the magnetic field quantities as well. For the high conductive ground (a and $b > 0$), the response of magnetic field will be very strong. In the opposite direction (a and $b < 0$), the response field will be very weak. The comparision of the quantities of magnetic computed by Finite Difference Method and Finite Element Method are similar. Since the magnetic field intensity can be able to inform the location of the ore body under the ground and the behaviour of magnetic field clearly performs very good relation to the conductive ground therefore the research results are very useful in geophysical exploration because normally we can measure magnetic field on the ground surface.

Even though the work presented in this thesis provides interesting idea about the solution to the forward problems for the magnetic field response, the issues that we dealt with suggest numerous avenues for possible extensions and future works. The following outline is a list of interesting future directions that require further investigation:

1. Analytied solution should be developed.
2. Multiplyered earth model should be considered.
3. The inverse problem should be proposed.
4. The difference conductivity model should be considered.

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Appendix A

Publications

- Y. Khonkhem and S. Yooyuanyong, *Finite Difference Magnetic Field Response of a 2−Dimensional Continuously Conductive Ground*, The 20*th* Annual Meeting in Mathematics (*AMM*2015). Silpakorn University, Nakhon Pathom, May 27 − 29, 2015.
- Y. Khonkhem and S. Yooyuanyong, *Finite Difference for Magnetic Field Response from a Two−Dimensional Conductive Ground*, *Applied Mathematical Sciences*, **10**(3) (2016) : 137 − 150.

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ขอมอบเกียรติบัตรเพื่อแสดงว่า

นางสาวเยาวเรศ ขนเข็ม

ได้นำเสนอผลงาน
เรื่อง Finite Difference Magnetic Field Response of a 2-Dimensional Continuously Conductive Ground

ในการประชุมวิชาการทางคณิตศาสตร์ประจำปี ๒๕๕๘ ครั้งที่ ๒๐
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Finite Difference for Magnetic Field Response from a Two−Dimensional Conductive Ground

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Abstract

In our study, mathematical model of finite difference method for the magnetic field response from a two−dimensional continuously conductive ground is presented. The magnetic field at various locations are plotted by assuming the Earth structure having a two−dimensional exponential conductivity profile. There is a source providing a Direct Current (DC) voltage and receiver on the ground surface. Finite difference technique is applied to solve the partial differential equation. MATLAB programing is used to perform both values and graphs of magnetic field at various locations. The results show the intensity of magnetic field for cross−section of the ground structure very well. The behaviour of magnetic field clearly performs the relation to the conductive ground. The research results are very useful in geophysical exploration since normally we can measure magnetic field on the ground surface. The magnetic field can be able to imply the ore under the ground via the technique of inverse problem.

Mathematics Subject Classification: 86A25
Keywords: Finite difference; magnetic; magnetometric

1 Introduction

At present, natural resources are utilized extensively such as minerals, petroleum and groundwater. Geophysical survey is very important to geological structure survey or exploring the natural resources. During the past several

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