REGULARITY OF SOME TRANSFORMATION SEMIGROUPS

By

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ให้ X เป็นเซตที่ไม่ใช่เซตว่าง และให้ $\Im = \{Y_i : i \in I\}$ เป็นวงศ์ของเซตย่อยของ X ที่ไม่เป็นเซตว่าง พร้อมด้วยสมบัติที่ว่า $X = \bigcup_{i\in I} Y_i$ และ $Y_i \cap Y_j = \varnothing$ สำหรับทุกๆ $i, j \in I$ โดยที่ $i \neq j$ ให้ $\varnothing \neq J \subseteq I$ และให้

$$T_{\Im}^{(J)}(X) = \{\alpha \in T(X) : \forall i \in I, \exists j \in J, Y_i\alpha \subseteq Y_j\}$$

แล้ว $T_{\Im}^{(J)}(X)$ เป็นกึ่งกรุปย่อยของกึ่งกรุป $T(X, Y^{(J)})$ ของฟังก์ชันบน X ซึ่งมีเรนจ์เป็นเซตย่อยของ $Y^{(J)}$ เมื่อ $Y^{(J)} = \bigcup_{i\in J} Y_i$ ต่อไปสำหรับแต่ละ $\alpha \in T_{\Im}^{(J)}(X)$ เราให้ $\chi^{(\alpha)} : I \to J$ เป็นฟังก์ชันที่นิยามโดย $i\chi^{(\alpha)} = j \Leftrightarrow Y_i\alpha \subseteq Y_j$ และนิยามความสัมพันธ์สมภาค χ และ $\tilde{\chi}$ บน $T_{\Im}^{(J)}(X)$ ดังนี้ $(\alpha,\beta) \in \chi \Leftrightarrow \chi^{(\alpha)} = \chi^{(\beta)}$ และ $(\alpha,\beta) \in \tilde{\chi} \Leftrightarrow \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J$ ในวิทยานิพนธ์นี้ เราเริ่มต้นด้วยการศึกษาความปรกติของกึ่งกรุปผลหาร $T_{\Im}^{(J)}(X)/\chi$ และ $T_{\Im}^{(J)}(X)/\tilde{\chi}$ และกึ่งกรุป $T_{\Im}^{(J)}(X)$ สำหรับแต่ละ $\alpha \in T_{\Im}(X) := T_{\Im}^{(I)}(X)$ เราจะเห็นว่าชั้นสมมูล $[\alpha]$ ของ α ภายใต้ χ เป็นกึ่งกรุปของ $T_{\Im}(X)$ ก็ต่อเมื่อ $\chi^{(\alpha)}$ เป็นนิจพลในกึ่งกรุปของการแปลงแบบเต็ม $T(I)$ ให้ $I_{\Im}(X)$, $S_{\Im}(X)$ และ $B_{\Im}(X)$ เป็นเซตของฟังก์ชัน α ใน $T_{\Im}(X)$ โดยที่ $\chi^{(\alpha)}$ เป็นฟังก์ชันหนึ่งต่อหนึ่ง ฟังก์ชันทั่วถึง และฟังก์ชันหนึ่งต่อหนึ่งแบบทั่วถึง ตามลำดับ แล้วจะได้ว่า $I_{\Im}(X)$, $S_{\Im}(X)$ และ $B_{\Im}(X)$ เป็นกึ่งกรุปย่อยของ $T_{\Im}(X)$ สุดท้ายเราศึกษาความปรกติของกึ่งกรุป $[\alpha]$, $I_{\Im}(X), S_{\Im}(X)$ และ $B_{\Im}(X)$ ทั้งสี่นี้

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ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

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Let X be a nonempty set, and let $\Im=\{Y_i : i \in I\}$ be a family of nonempty subsets of X with the properties that $X=\bigcup_{i\in I} Y_i$ and $Y_i \cap Y_j = \varnothing$ for all $i, j \in I$ with $i \neq j$. Let $\varnothing \neq J \subseteq I$, and let

$$T_{\Im}^{(J)}(X)=\{\alpha \in T(X) : \forall i \in I, \exists j \in J, Y_i\alpha \subseteq Y_j\}.$$

Then $T_{\Im}^{(J)}(X)$ is a subsemigroup of the semigroup $T(X, Y^{(J)})$ of functions on X having ranges contained in $Y^{(J)}$, where $Y^{(J)}=\bigcup_{i\in J} Y_i$. For each $\alpha \in T_{\Im}^{(J)}(X)$, let $\chi^{(\alpha)} : I \to J$ be defined by $i\chi^{(\alpha)}=j \Leftrightarrow Y_i\alpha \subseteq Y_j$. Next, we define two congruence relations χ and $\tilde{\chi}$ on $T_{\Im}^{(J)}(X)$ as follows: $(\alpha,\beta)\in\chi \Leftrightarrow \chi^{(\alpha)}=\chi^{(\beta)}$ and $(\alpha,\beta)\in\tilde{\chi}$ $\Leftrightarrow \chi^{(\alpha)}|_J=\chi^{(\beta)}|_J$. We begin this thesis by studying the regularity of the quotient semigroups $T_{\Im}^{(J)}(X)/\chi$ and $T_{\Im}^{(J)}(X)/\tilde{\chi}$ and the semigroup $T_{\Im}^{(J)}(X)$. For each $\alpha \in T_{\Im}(X) := T_{\Im}^{(I)}(X)$, we see that the equivalence class $[\alpha]$ of α under χ is a subsemigroup of $T_{\Im}(X)$ if and only if $\chi^{(\alpha)}$ is an idempotent in the full transformation semigroup $T(I)$. Let $I_{\Im}(X)$, $S_{\Im}(X)$ and $B_{\Im}(X)$ be the sets of functions α in $T_{\Im}(X)$ such that $\chi^{(\alpha)}$ is injective, surjective and bijective respectively. Then $I_{\Im}(X)$, $S_{\Im}(X)$ and $B_{\Im}(X)$ are subsemigroups of $T_{\Im}(X)$. We end this thesis by investigating the regularity of the four semigroups $[\alpha]$, $I_{\Im}(X), S_{\Im}(X)$ and $B_{\Im}(X)$.

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Thesis Advisor's signature ....

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Chapter 1

Literature Review and Preliminaries

We call an element a of a semigroup S a *regular element* of S if there is an element b of S such that $aba = a$. A semigroup whose every element is regular is called a *regular semigroup*. For any nonempty set X, it is well-known that the semigroup $T(X)$ of all functions from X to itself under the composition, called a *full transformation semigroup*, is regular (see [2], Exercise 1.9, page 33). That is, for every $\alpha \in T(X)$, there exists a $\beta \in T(X)$ such that $\alpha\beta\alpha = \alpha$, in fact, for each $\alpha \in T(X)$, the function β on X defined by $x\beta = a_x$ if $x \in X\alpha$ and $x\beta = a$ otherwise, where a is a point in X which is fixed and for each $x \in X\alpha$, a_x is a point in $x\alpha^{-1}$ which is also fixed, satisfies $\alpha\beta\alpha = \alpha$. There have been several research works on studying subsemigroups of the full transformation semigroups. We mention some of them as follows.

For each nonempty set X and nonempty subset Y of X, let

$$T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

and

$$\overline{T}(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

The two subsemigroups $T(X, Y)$ and $\overline{T}(X, Y)$ of the full transformation semigroup $T(X)$ were considered by J. S. V. Symons in [19] and K. D. Magill Jr. in [14] respectively. Since then, they have been studied by many people (see [6], [16], [17], [18] for some related works). In [16], Nenthein, Youngkong, and Kemprasit studied the regularity of the semigroups $T(X, Y)$ and $\overline{T}(X, Y)$. The authors provided some characterizations of regular elements in these two semigroups and made use of those results to deduce the regularity of them. The following are what they obtained.

Theorem 1. ([16], Theorem 2.1, page 308) *Let X and Y be nonempty sets such that $Y \subseteq X$. For any $\alpha \in T(X, Y)$, the following are equivalent:*

(1) *α is regular;*

(2) *$X\alpha = Y\alpha$;*

(3) *$x \ker \alpha \cap Y \neq \emptyset$ for all $x \in X$;*

(4) $x\alpha^{-1} \cap Y \neq \emptyset$ *for all* $x \in X\alpha$,

where $x \ker \alpha := \{z \in X : x\alpha = z\alpha\}$ *for each* $x \in X$.

Theorem 2. ([16], Corollary 2.2, page 309) *Let* X *and* Y *be nonempty sets such that* $Y \subseteq X$. *Then* $T(X, Y)$ *is regular if and only if* $Y = X$ *or* $|Y| = 1$.

Theorem 3. ([16], Theorem 2.3, page 309) *Let* X *and* Y *be nonempty sets such that* $Y \subseteq X$. *For any* $\alpha \in \overline{T}(X, Y)$, *the following are equivalent:*

(1) α *is regular;*

(2) $X\alpha \cap Y = Y\alpha$;

(3) $x \ker \alpha \cap Y \neq \emptyset$ *for all* $x \in X$ *with* $x\alpha \in Y$;

(4) $x\alpha^{-1} \cap Y \neq \emptyset$ *for all* $x \in X\alpha \cap Y$,

where for each $x \in X$ *the set* $x \ker \alpha$ *is defined as in Theorem* 1.

Theorem 4. ([16], Corollary 2.4, page 310) *Let* X *and* Y *be nonempty sets such that* $Y \subseteq X$. *Then* $\overline{T}(X, Y)$ *is regular if and only if* $Y = X$ *or* $|Y| = 1$.

Remark 5. *In the proof of Theorem* 2 *mentioned above, the authors showed that if* $|Y| > 1$ *and* $Y \neq X$, *then* $T(X, Y)$ *is not regular by defining a function* $\alpha \in T(X, Y)$ *which is not regular as follows:* $x\alpha = a$ *if* $x \in Y$ *and* $x\alpha = b$ *otherwise, where* a *and* b *are two fixed different points in* Y.

In [18], Sanwong and Sommanee also studied the regularity of the semigroup $T(X, Y)$. The author obtained the same result as that of Nenthein et al. mentioned above that $T(X, Y)$ is regular if and only if $X = Y$ or $|Y| = 1$. They also found that the set of all regular elements of $T(X, Y)$ is exactly the set

$$F(X, Y) = \{\alpha \in T(X, Y) : X\alpha = Y\alpha\}$$

and forms a regular subsemigroup of $T(X, Y)$. Observe that the studies of semigroups mentioned above deal with functions on a set without any mathematical structures. However, there have also been a bunch of research works on semigroup regularity dealing with functions on a set along with a mathematical structure, for instance, a vector space and a partially ordered set (see [1], [7],[9], [12], [15], [20], [21] for some references). Another interesting one is a set together with an equivalence relation.

Let X be a nonempty set, and let $\mathcal{E}$ an equivalence relation on X. Let

$$T_{\mathcal{E}}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \mathcal{E} \Rightarrow (x\alpha, y\alpha) \in \mathcal{E}\}.$$

This semigroup was observed in [8] by Huisheng. The author proved that $T_{\mathcal{E}}(X)$ is exactly the semigroup of all continuous functions on X equipped with the topology having the family of all equivalence classes as a base. The regularity of this semigroup was studied in [10] by the same author in 2005. He obtained that a function α in $T_{\mathcal{E}}(X)$ is regular if and only if for each equivalence class A, there exists an equivalence class B such that $A \cap X\alpha \subseteq B\alpha$. It was remarked that if the equivalence relation $\mathcal{E}$ on the set X is neither $\{(x, x) : x \in X\}$ nor $X \times X$, then the semigroup $T_{\mathcal{E}}(X)$ is not regular. To see this explicitly, the author defined a function $\alpha \in T_{\mathcal{E}}(X)$ which is not regular as follows: fix an $A \in X/\mathcal{E}$ such that $A \neq X$ and $|A| > 1$, choose $a, b \in A$ with $a \neq b$, and let $\alpha : X \to X$ be defined by $x\alpha = a$ if $x \in A$ and $x\alpha = b$ otherwise. The nonregularity of α was deduced from the fact that $A \cap X\alpha$ is not a subset of $B\alpha$ for all $B \in X/\mathcal{E}$. From this, the following result was stated.

Theorem 6. ([10], Proposition 2.4, page 111) *For any nonempty set X and equivalence relation $\mathcal{E}$ on X, $T_{\mathcal{E}}(X)$ is regular if and only if $\mathcal{E} = \{(x, x) : x \in X\}$ or $\mathcal{E} = X \times X$.*

There have been a number of works extending the results of Huisheng mentioned above (see [4], [11], [13] for some references). In this research, we deal with another approach of the setting of Huisheng and extend it to a more general setting.

Chapter 2

Theoretical Background

In this section, we provide some elementary definitions and facts about relations, functions and semigroups which are necessary for understanding the main results presented in Chapter 3. These can be seen in some texts in set theory and semigroup theory (see [5] and [3] for some examples).

2.1 Relations and functions

Definition 2.1.1. Let A and B be sets, and let $r \subseteq A \times B$. Then r is called a *relation from A to B*. For each $(a, b) \in A \times B$, if $(a, b) \in r$, we denote this by arb. A relation from A to itself is called a *relation on A*.

Definition 2.1.2. Let A and B be sets, and let r be a relation from A to B. The set $D_r = \{a \in A : \exists b \in B, (a, b) \in r\}$ is called the *domain* of r and the set $R_r = \{b \in B : \exists a \in A, (a, b) \in r\}$ is called the *range* of r.

Definition 2.1.3. Let A, B and C be sets. Let r be a relation from A to B, and let s be a relation from B to C. The relation $r \circ s = \{(a, c) \in A \times C : \exists b \in B, (a, b) \in r \wedge (b, c) \in s\}$ is called the *composition of r and s*. We may write just rs in place of $r \circ s$.

Definition 2.1.4. Let A be set, and let $\sim$ be a relation on A. The relation $\sim$ is said to be an *equivalence relation on A* if the following properties are satisfied:

(1) *Reflexivity* : for all $a \in A, a \sim a$;

(2) *Symmetry* : for all $a, b \in A$, if $a \sim b$, then $b \sim a$;

(3) *Transitivity* : for all $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Definition 2.1.5. Let $\sim$ be an equivalence relation on a set A. For every $a \in A$, the set $[a] = \{b \in A : a \sim b\}$ is called the *equivalence class of a under $\sim$*. Let $A/\sim = \{[a] : a \in A\}$.

Theorem 2.1.6. *If $\sim$ is an equivalence relation on a set A, then the following hold:*

(1) *for all* $a \in A$, $a \in [a]$;

(2) *for all* $a, b \in A$, $a \sim b$ *if and only if* $[a] = [b]$;

(3) *for all* $a, b \in A$, *either* $[a] \cap [b] = \emptyset$ *or* $[a] = [b]$.

Definition 2.1.7. Let X be a nonempty set. A family $\{A_i : i \in I\}$ of nonempty subsets of X is called a *partition of* X if the following properties are satisfied:

(1) $\bigcup_{i \in I} A_i = X$;

(2) for all $i, j \in I$, either $A_i \cap A_j = \emptyset$ or $A_i = A_j$.

Theorem 2.1.8. *Let* A *be a nonempty set. For any equivalence relation* $\sim$ *on the set* A*, the family* $A/\sim$ *is a partition of* A*. On the other hand, any partition* $\mathscr{F}$ *of* A *determines an equivalence relation* $\sim$ *on* A *such that* $A/\sim = \mathscr{F}$.

Definition 2.1.9. Let A and B be sets, and let f be a relation from A to B. We call f a *function from* A *to* B and write $f : A \to B$ if for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$. If f is a function from A to B and $(a, b) \in f$, then we call b the *image of* a *under* f and denote b by $(a)f$ or just by af if no confusion seems to occur. That is, we write $(a)f = b$ or $af = b$ whenever $(a, b) \in f$.

Definition 2.1.10. Let $f : A \to B$. We call f an *injective function* from A to B if for every $a, b \in A$, $af = bf$ implies $a = b$. If the range of f is exactly B, that is, for any $y \in B$ there is an $a \in A$ such that $af = y$, we call f a *surjective function* from A onto B. We call f a *bijective function* from A onto B if f is both injective and surjective.

Definition 2.1.11. Let $f : A \to B$, and let $C \subseteq A$. We call the function $g : C \to B$ defined by $xg = xf$ for all $x \in C$ the *restriction of* f *to* C and denote g by $f|_C$.

Definition 2.1.12. Let $f : A \to B$, let $C \subseteq A$, and let $D \subseteq B$. We call the sets

$$Cf = \{y \in B : \exists x \in C, xf = y\}$$

and

$$Df^{-1} = \{x \in A : xf \in D\}$$

the *image of* C *under* f and the *inverse image of* D *under* f respectively. If $D = \{x\}$, we denote $\{x\}f^{-1}$ by just xf^{-1}.

Definition 2.1.13. Let A and B be sets. We say that A and B *have the same cardinality* or A *has the same cardinality as* B if there is a bijective function from A onto B. In this circumstance, we write $|A| = |B|$. Any set which is either empty or has the same cardinality as the set $J_n := \{1, 2, ..., n\}$ for some positive integer n is called a *finite set*. If A is a finite set having the same cardinality as the set J_n, we write $|A| = n$. In other words, $|A|$ is used to refer to as the number of elements of A when A is finite.

Theorem 2.1.14. *Let A and B be finite sets. Then the following are equivalent:*

(1) *A and B have the same cardinality;*

(2) *every injective function from A to B is surjective;*

(3) *every surjective function from A onto B is injective.*

Theorem 2.1.15. *Let $f : A \to B$ and $g : B \to C$.Then $fg : A \to C$ and $(x)fg = (xf)g$ for all $x \in A$.*

2.2 Semigroups

Definition 2.2.1. Let S be a nonempty set. A *binary operation* $*$ on S is a function from $S \times S$ to S. For any $a, b \in S$, we write $a * b$ to refer to as the image $(a, b)*$ of (a, b) under the binary operation $*$.

Definition 2.2.2. A binary operation $*$ on a set S is said to be *associative* if $(x * y) * z = x * (y * z)$ for all $x, y, z \in S$.

Definition 2.2.3. Let S be a nonempty set, and let $*$ be an associative binary operation on S. We call the pair $(S, *)$ a *semigroup*. We may sometimes write just the underlying set S as a semigroup if no confusion of which associative binary operation being considered seems to occur. And in this situation, we write just ab to stand for the image of (a, b) under the binary operation on the semigroup S.

Definition 2.2.4. A semigroup S is called a *monoid* if there exists an element e of S such that $ea = a = ae$ for all $a \in S$.

Let $(S, *)$ be a semigroup, and let A be a nonempty subset of S, if the restriction $*|_{A\times A}$ of $*$ to $A \times A$ has the range contained in A, then we have immediately that $(A, *|_{A\times A})$ is a semigroup.

Definition 2.2.5. Let $(S, *)$ be a semigroup, and let A be a nonempty subset of S, if the restriction $*|_{A\times A}$ of $*$ to $A \times A$ has the range contained in A, then we call the semigroup $(A, *|_{A\times A})$ a *subsemigroup of* $(S, *)$.

Definition 2.2.6. Let S and T be semigroups. A function $\psi : S \to T$ is called a *homomorphism* if for all $a, b \in S$, $(ab)\psi = a\psi b\psi$. If a function $\psi : S \to T$ is a bijective homomorphism, we call ψ an *isomorphism*.

Definition 2.2.7. Let S be a semigroup, and let $a \in S$. If $a^2 = a$, then we call a an *idempotent* of S. We call a a regular element of S if there is an element b of S such that $aba = a$.

We see that an indempotent of a semigroup S is necessarily a regular element of S.

Definition 2.2.8. Let S be a semigroup. We call S a *regular semigroup* if every $a \in S$ is regular. The set of all regular elements of S is denoted by $\mathcal{R}(S)$.

Definition 2.2.9. Let S be a semigroup. An equivalence relation $\sim$ on S is called a *congruence relation* if for all $x, y, z, w \in S$, $x \sim y$ and $z \sim w$ implies $xz \sim yw$.

Theorem 2.2.10. *Let S be a semigroup, and let $\sim$ be a congruence relation on S. Then the relation $\star$ from $S/\sim \times S/\sim$ to $S/\sim$ defined by*

$$\star = \{(([a],[b]),[ab]) : a, b \in S\}$$

is a function from $S/\sim \times S/\sim$ to $S/\sim$.

Corollary 2.2.11. *Let S be a semigroup, and let $\sim$ be a congruence relation on S. Then $(S/\sim, \star)$, where $\star$ is the binary operation on $S/\sim$ defined in Theorem* 2.2.10*, is a semigroup.*

Chapter 3

Regularity of Transformation Semigroups Defined by a Partition

In this chapter, we present the main results of this research. Our main results are divided into two sections as follows.

3.1 Semigroups of transformations defined by a partition

In this section, we define a new subsemigroup of the full transformation semigroup $T(X)$, where X is a fixed nonempty set, and then study its regularity. Our setting is an extension of that of Huisheng defined in [8].

Definition 3.1.1. Let X be a nonempty set, and let $\mathscr{F} = \{Y_i : i \in I\}$ a family of nonempty subsets of X. We call $\mathscr{F}$ a *partition of* X if $X = \bigcup_{i \in I} Y_i$ and $Y_i \cap Y_j = \emptyset$ for all i, j with $i \neq j$. Let $\Sigma_X = \{X\}$ and $\Lambda_X = \{\{a\} : a \in X\}$. Each of these two partitions is called a *trivial* partition of X.

Notice that the definition of a partition of a set provided above is stronger than the one stated in Definition 2.1.7. Throughout the rest of this chapter, let X be a nonempty set which is arbitrarily fixed. For any $\alpha \in T(X)$, if $X\alpha = \{a_i : i \in J\}$ with $a_i \neq a_j$ for all $i \neq j$, then we write α in a form of matrix by

$$\alpha = \begin{pmatrix} a_i\alpha^{-1} \\ a_i \end{pmatrix}.$$

This notation was introduced by Clifford and Preston in [3] (see page 241). We also make use of the following notation: if $\{Y_i : i \in I\}$ is a partition of the set X, then for each $\alpha \in T(X)$, we write

$$\alpha = \begin{pmatrix} Y_i \\ \alpha_i \end{pmatrix},$$

where for each $i \in I$, α_i is the restriction of α to Y_i. Next, we consider another approach of the setting of Huisheng originally defined in [8]. Here, we begin with a fixed partition of the set X. It is well-known that any partition of a set induces naturally an equivalence relation on that set (see Theorem 2.1.8). Let $\mathscr{F} = \{Y_i : i \in I\}$ be an arbitrarily fixed partition of X, and let

$$T_{\mathscr{F}}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in I, Y_i\alpha \subseteq Y_j\}.$$

The semigroup $T_{\mathscr{F}}(X)$ can be generalized by fixing, in addition to the partition $\mathscr{F}$ of X, a nonempty subset J of the index set I as follows. Let $J \subseteq I$ with $J \neq \emptyset$, and let

$$T_{\mathscr{F}}^{(J)}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in J, Y_i\alpha \subseteq Y_j\}.$$

Let $Y^{(J)} = \bigcup\limits_{i\in J} Y_i$. We can easily see that $T_{\mathscr{F}}^{(J)}(X) \subseteq T\left(X, Y^{(J)}\right)$. Furthermore, $T_{\mathscr{F}}^{(J)}(X)$ is subsemigroup of $T\left(X, Y^{(J)}\right)$. Indeed, for any $\alpha, \beta \in T_{\mathscr{F}}^{(J)}(X)$ and $i \in I$, there are $j, k \in J$ such that $Y_i\alpha \subseteq Y_j$ and $Y_j\beta \subseteq Y_k$, which yields that $Y_i\alpha\beta = (Y_i\alpha)\beta \subseteq Y_j\beta \subseteq Y_k$.

Proposition 3.1.2. (1) *$T_{\mathscr{F}}^{(J)}(X) = T\left(X, Y^{(J)}\right)$ if and only if $|J| = 1$ or $\mathscr{F} = \Lambda_X$.*

(2) *$T_{\mathscr{F}}^{(J)}(X) = T(X)$ if and only if $J = I$ and $\mathscr{F}$ is trivial.*

Proof. (1) Suppose that $|J| \geq 2$, and that $\mathscr{F} \neq \Lambda_X$. Then there are $\nu, \mu \in J$ and $i \in I$ such that $\nu \neq \mu$ and $|Y_i| \geq 2$. Next, we fix $a \in Y_\mu$, $b \in Y_\nu$ and $c \in Y_i$ and then define a map $\alpha : X \to X$ as follows:

$$\alpha = \begin{pmatrix} Y_i \setminus \{c\} & (X \setminus Y_i) \cup \{c\} \\ a & b \end{pmatrix}.$$

Clearly, $\alpha \in T\left(X, Y^{(J)}\right) \setminus T_{\mathscr{F}}^{(J)}(X)$. We are now going to prove the necessity. Suppose that $|J| = 1$ or $\mathscr{F} = \Lambda_X$. We have already had that $T_{\mathscr{F}}^{(J)}(X) \subseteq T\left(X, Y^{(J)}\right)$. According to the assumption that $|J| = 1$ or $\mathscr{F} = \Lambda_X$, there are two cases to be considered.

<u>Case 1</u> $|J| = 1$, says $J = \{j\}$: We have for every $\alpha \in T\left(X, Y^{(J)}\right)$ and $i \in I$ that $Y_i\alpha \subseteq X\alpha \subseteq Y^{(J)} = Y_j$, which yields $\alpha \in T_{\mathscr{F}}^{(J)}(X)$. So $T\left(X, Y^{(J)}\right) \subseteq T_{\mathscr{F}}^{(J)}(X)$.

<u>Case 2</u> $\mathscr{F} = \Lambda_X := \{\{x\} : x \in X\}$: For each $x \in X$, let $Y_x = \{x\}$. Then $\mathscr{F} = \{Y_x : x \in X\}$ and $Y^{(J)} = \bigcup\limits_{x\in J} Y_x = \bigcup\limits_{x\in J} \{x\} = J$. So, for every $\alpha \in T\left(X, Y^{(J)}\right)$ and $x \in X$, we have that $Y_x\alpha = \{x\}\alpha = \{x\alpha\} = Y_{x\alpha}$ and $x\alpha \in Y^{(J)} = J$. Thus $T\left(X, Y^{(J)}\right) \subseteq T_{\mathscr{F}}^{(J)}(X)$.

(2) Assume that $T_{\mathscr{F}}^{(J)}(X) = T(X)$. Then $T_{\mathscr{F}}^{(J)}(X) = T\left(X, Y^{(J)}\right) = T(X)$. Since $T\left(X, Y^{(J)}\right) = T(X)$, it follows that $X = Y^{(J)}$. Hence $J = I$. And since $T_{\mathscr{F}}^{(J)}(X) = T\left(X, Y^{(J)}\right)$, we have by (1) that $|I| = |J| = 1$ or $\mathscr{F} = \Lambda_X$, which yields that $\mathscr{F}$ is trivial. To prove the necessity, suppose that $J = I$ and $\mathscr{F}$ is

trivial. Then by the assumption that $\mathscr{F}$ is trivial, we have two cases to consider. Case 1 $\mathscr{F} = \Sigma_X$: We have in this case that $|J| = 1 = |I|$. Since $|J| = 1$, we get that by (1) that $T_{\mathscr{F}}^{(J)}(X) = T\left(X, Y^{(J)}\right)$. And since $|I| = |J|$, we obtain that $X = Y^{(J)}$, which implies that $T\left(X, Y^{(J)}\right) = T(X)$. Accordingly, $T_{\mathscr{F}}^{(J)}(X) = T(X)$. Case 2 $\mathscr{F} = \Lambda_X$: In this case, we have that $J = I = X$. For each $x \in X$, let $Y_x = \{x\}$. Then $\mathscr{F} = \{Y_x : x \in X\}$. Thus, for each $\alpha \in T(X)$, we have that $Y_x\alpha = \{x\}\alpha = \{x\alpha\} = Y_{x\alpha}$ for every $x \in X$, which yields that $\alpha \in T_{\mathscr{F}}^{(J)}(X)$. Therefore, $T_{\mathscr{F}}^{(J)}(X) = T(X)$. $\square$

Form the above proposition, we obtain immediately that $T_{\mathscr{F}}(X) = T(X)$ if and only if $\mathscr{F} = \Sigma_X$ or $\mathscr{F} = \Lambda_X$. By the assumption that $\mathscr{F}$ is a partition of X, we have for each $\alpha \in T_{\mathscr{F}}^{(J)}(X)$ that for every $i \in I$, there is a unique $j_i \in J$ such that $Y_i\alpha \subseteq Y_{j_i}$. Thus we can define a function $\chi^{(\alpha)} : I \to J$ corresponding to α as follows:

$$\chi^{(\alpha)} = \begin{pmatrix} i \\ j_i \end{pmatrix}.$$

The function $\chi^{(\alpha)}$ is called the *character* of α.

Lemma 3.1.3. *For all $\alpha, \beta \in T_{\mathscr{F}}^{(J)}(X)$, $\chi^{(\alpha\beta)} = \chi^{(\alpha)}\chi^{(\beta)}$.*

Proof. Let $\alpha, \beta \in T_{\mathscr{F}}^{(J)}(X)$, and let $i \in I$, $i\chi^{(\alpha)} = j$, and $j\chi^{(\beta)} = k$. Then $Y_i\alpha\beta = (Y_i\alpha)\beta \subseteq Y_j\beta \subseteq Y_k$. Thus $i\chi^{(\alpha\beta)} = k = j\chi^{(\beta)} = i\chi^{(\alpha)}\chi^{(\beta)}$. The proof is complete. $\square$

The notion of character provided above leads us to define two relations χ and $\widetilde{\chi}$ on $T_{\mathscr{F}}^{(J)}(X)$ as follows:

$$(\alpha, \beta) \in \chi \Leftrightarrow \chi^{(\alpha)} = \chi^{(\beta)},$$

and

$$(\alpha, \beta) \in \widetilde{\chi} \Leftrightarrow \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J.$$

It is obvious that $\chi \subseteq \widetilde{\chi}$, and that $\chi = \widetilde{\chi}$ if and only if $I = J$. The relations χ and $\widetilde{\chi}$ are, in fact, congruence relations. We will prove these as follows. Let $\alpha, \beta, \gamma, \lambda \in T_{\mathscr{F}}^{(J)}(X)$. Since $\chi^{(\alpha)} = \chi^{(\alpha)}$, we have $(\alpha, \alpha) \in \chi$. It follows that χ is reflexive. To see that χ is symmetric, we suppose that $(\alpha, \beta) \in \chi$. Then $\chi^{(\alpha)} = \chi^{(\beta)}$. That is, $\chi^{(\beta)} = \chi^{(\alpha)}$, which yields that $(\beta, \alpha) \in \chi$. Thus χ is symmetric. Next, we will show that χ is transitive. Assume that $(\alpha, \beta) \in \chi$ and $(\beta, \gamma) \in \chi$. Then $\chi^{(\alpha)} = \chi^{(\beta)}$ and $\chi^{(\beta)} = \chi^{(\gamma)}$, which implies that $\chi^{(\alpha)} = \chi^{(\gamma)}$. Hence $(\alpha, \gamma) \in \chi$, and so χ is transitive. We now have that χ is an equivalence relation on $T_{\mathscr{F}}^{(J)}(X)$. Next, suppose that $(\alpha, \beta) \in \chi$ and $(\gamma, \lambda) \in \chi$. Then $\chi^{(\alpha)} = \chi^{(\beta)}$ and $\chi^{(\gamma)} = \chi^{(\lambda)}$. Hence, by Lemma 3.1.3, we have that

$$\begin{aligned} \chi^{(\alpha\gamma)} &= \chi^{(\alpha)}\chi^{(\gamma)} \\ &= \chi^{(\beta)}\chi^{(\lambda)} \\ &= \chi^{(\beta\lambda)}, \end{aligned}$$

which means that $(\alpha\gamma, \beta\lambda) \in \chi$. Therefore, χ is a congruence relation on $T^{(J)}_{\mathscr{F}}(X)$. Similarly to χ, it can be shown that $\widetilde{\chi}$ is a congruence relation on $T^{(J)}_{\mathscr{F}}(X)$ as well. Whence, by Corollary 2.2.11, both $T^{(J)}_{\mathscr{F}}(X)/\chi$ and $T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi}$ are semigroups. For each $\alpha \in T^{(J)}_{\mathscr{F}}(X)$, let $[\alpha]$ and $\widetilde{[\alpha]}$ denote the equivalence classes of α under the equivalence relations χ and $\widetilde{\chi}$ respectively.

Theorem 3.1.4. (1) *$T^{(J)}_{\mathscr{F}}(X)/\chi \cong T(I, J)$ by the isomorphism $[\alpha] \mapsto \chi^{(\alpha)}$.*

(2) *$T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi} \cong T(J)$ by the isomorphism $\widetilde{[\alpha]} \mapsto \chi^{(\alpha)}|_J$.*

Proof. (1) Let $\psi : T^{(J)}_{\mathscr{F}}(X)/\chi \to T(I, J)$ be defined by $[\alpha]\psi = \chi^{(\alpha)}$. For every $[\alpha], [\beta] \in T^{(J)}_{\mathscr{F}}(X)/\chi$, if $[\alpha] = [\beta]$, then we have by the definition of χ that $\chi^{(\alpha)} = \chi^{(\beta)}$. It follows that $[\alpha]\psi = \chi^{(\alpha)} = \chi^{(\beta)} = [\beta]\psi$. So ψ is well-defined. Next, we will show that ψ is injective. To see this, let $[\alpha], [\beta] \in T^{(J)}_{\mathscr{F}}(X)/\chi$, and assume that $[\alpha]\psi = [\beta]\psi$, that is, $\chi^{(\alpha)} = \chi^{(\beta)}$. Then $\alpha\chi\beta$, which implies that $[\alpha] = [\beta]$. Hence ψ is injective. To see that ψ is surjective, let $\gamma \in T(I, J)$. We will find an $\alpha \in T^{(J)}_{\mathscr{F}}(X)$ such that $\chi^{(\alpha)} = \gamma$. For each $i \in I$, fix an $a_i \in Y_i\gamma$, and then define $\alpha \in T(X)$ as follows:

$$\alpha = \begin{pmatrix} Y_i \\ a_i \end{pmatrix}.$$

Since for every $i \in I$, $Y_i\alpha = \{a_i\} \subseteq Y_i\gamma$, it follows that $\alpha \in T^{(J)}_{\mathscr{F}}(X)$ and $\chi^{(\alpha)} = \gamma$. Thus ψ is surjective. Finally, we will to show that ψ is a homomorphism. Let $[\alpha], [\beta] \in T^{(J)}_{\mathscr{F}}(X)/\chi$. Then by Lemma 3.1.3, we have that

$$\begin{aligned} [\alpha]\psi[\beta]\psi &= \chi^{(\alpha)}\chi^{(\beta)} \\ &= \chi^{(\alpha\beta)} \\ &= [\alpha\beta]\psi \\ &= \big([\alpha][\beta]\big)\psi. \end{aligned}$$

Whence the function ψ is a homomorphism, and so it is an isomorphism.

(2) Let $\varphi : T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi} \to T(J)$ be defined by $\widetilde{[\alpha]}\varphi = \chi^{(\alpha)}|_J$. For every $\widetilde{[\alpha]}, \widetilde{[\beta]} \in T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi}$, if $\widetilde{[\alpha]} = \widetilde{[\beta]}$, then by definition of $\widetilde{\chi}$ we have $\chi^{(\alpha)}|_J = \chi^{(\beta)}|_J$, which yields that $\widetilde{[\alpha]}\varphi = \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J = \widetilde{[\beta]}\varphi$. Therefore, φ is well-defined. Next, we will show that φ is injective. Let $\widetilde{[\alpha]}, \widetilde{[\beta]} \in T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi}$, and suppose that $\chi^{(\alpha)}|_J = \chi^{(\beta)}|_J$. Then $\alpha\widetilde{\chi}\beta$, that is, $\widetilde{[\alpha]} = \widetilde{[\beta]}$. Thus φ is injective. To see that φ is surjective, let $\gamma \in T(J)$. For each $i \in J$, fix an $a_i \in Y_{i\gamma}$. Also, we fix an $a \in Y^{(J)}$, and then define $\alpha \in T(X)$ by

$$\alpha = \begin{pmatrix} Y_i & X \setminus Y^{(J)} \\ a_i & a \end{pmatrix}_{i \in J}.$$

Obviously, $\alpha \in T^{(J)}_{\mathscr{F}}(X)$ and $\chi^{(\alpha)}|_J = \gamma$. Finally, we will show that the function φ is a homomorphism. Let $\widetilde{[\alpha]}, \widetilde{[\beta]} \in T^{(J)}_{\mathscr{F}}(X)/\widetilde{\chi}$. Then by Lemma 3.1.3, we obtain that

$$\begin{aligned}\widetilde{[\alpha]}\varphi\widetilde{[\beta]}\varphi &= \chi^{(\alpha)}|_J\chi^{(\beta)}|_J \\ &= \chi^{(\alpha\beta)}|_J \\ &= \left[\widetilde{\alpha}\widetilde{\beta}\right]\varphi \\ &= \left(\widetilde{[\alpha]}\widetilde{[\beta]}\right)\varphi.\end{aligned}$$

Therefore, φ is surjective. Thus we have that φ is an isomorphism. $\square$

From Theorem 2 and Theorem 3.1.4, the following corollary is immediately obtained.

Corollary 3.1.5. (1) *The following are equivalent:*

- (a) *The quotient semigroup $T_{\mathscr{F}}^{(J)}(X)/\chi$ is regular;*
- (b) *the semigroup $T(I, J)$ is regular;*
- (c) $J = I$ *or* $|J| = 1$.

In particular, the quotient semigroup $T_{\mathscr{F}}(X)/\chi$ is regular.

(2) *The quotient semigroup $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi}$ is regular.*

We now turn our attention to investigate the regularity of the semigroup $T_{\mathscr{F}}^{(J)}(X)$.

Theorem 3.1.6. *The semigroup $T_{\mathscr{F}}^{(J)}(X)$ is regular if and only if $\left|T_{\mathscr{F}}^{(J)}(X)\right| = 1$ or $T_{\mathscr{F}}^{(J)}(X) = T(X)$.*

Proof. The necessity is obviously true. To prove the sufficiency, we suppose that $\left|T_{\mathscr{F}}^{(J)}(X)\right| > 1$ and $T_{\mathscr{F}}^{(J)}(X) \neq T(X)$. We divide the proof into two cases.

<u>Case 1</u> $J = I$: In this case, by the assumption that $T_{\mathscr{F}}^{(J)}(X) \neq T(X)$, we have $\mathscr{F}$ is not trivial. Hence there is an $i \in I$ such that $|Y_i| \geq 2$. Let a, b be two different points in Y_i, and let $j \in I$ with $j \neq i$ be fixed. We define a function $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} Y_j & X \setminus Y_j \\ a & b \end{pmatrix}.$$

Clearly, $\alpha \in T_{\mathscr{F}}(X)$. Next, let $\beta \in T_{\mathscr{F}}(X)$. If $i \in j\left(\chi^{(\beta)}\right)^{-1}$, then we obtain for any point $x \in X \setminus Y_j$ that $x\alpha\beta\alpha = a \neq b = x\alpha$. For the case where $i \notin j\left(\chi^{(\beta)}\right)^{-1}$, we have for each point $x \in Y_j$ that $x\alpha\beta\alpha = b \neq a = x\alpha$.

<u>Case 2</u> $J \neq I$: We have in this case that $Y^{(J)} \neq X$, which yields that $T_{\mathscr{F}}^{(J)}(X) \subseteq T\left(X, Y^{(J)}\right) \subsetneq T(X)$. And by the assumption that $\left|T_{\mathscr{F}}^{(J)}(X)\right| > 1$, we have $\left|Y^{(J)}\right| > 1$. Let α be the nonregular element of the semigroup $T\left(X, Y^{(J)}\right)$ defined in Remark 5. It is clear that $\alpha \in T_{\mathscr{F}}^{(J)}(X)$, which yields that $T_{\mathscr{F}}^{(J)}(X)$ is not regular. $\square$

Remark 3.1.7. *Case* 1 *in the proof of the above theorem can actually be deduced from Theorem* 6 *of Huisheng mentioned in the introduction. Here, we prove again by our own way. Our proof is just straightforward from the definition of regularity.*

3.2 Some further subsemigroups of $T(X)$

In this section, we introduce some new subsemigroups of the semigroup $T_{\mathscr{F}}(X)$ and study their regularity.

According to Theorem 3.1.4, for any $\gamma \in T(I)$, the equivalence class of $\alpha \in T_{\mathscr{F}}(X)$ having the character γ may be sometimes denoted for convenience by $\overline{\gamma}$. For each $\gamma \in T(I)$, we see that the set $\overline{\gamma}$ is a subsemigroup of $T_{\mathscr{F}}(X)$ if and only if γ is an idempotent. We will prove this as follows. Suppose that $\overline{\gamma}$ is a subsemigroup of $T_{\mathscr{F}}(X)$, and let α be a fixed element of $\overline{\gamma}$. Then $\alpha^2 \in \overline{\gamma}$, and thus $\chi^{(\alpha^2)} = \gamma$. Hence, by Lemma 3.1.3, we get that $\gamma^2 = \gamma\gamma = \chi^{(\alpha)}\chi^{(\alpha)} = \chi^{(\alpha^{(2)})} = \gamma$. Conversely, assume that $\gamma^2 = \gamma$. Then for any $\alpha, \beta \in \overline{\gamma}$, we have by Lemma 3.1.3 that $\chi^{(\alpha\beta)} = \chi^{(\alpha)}\chi^{(\beta)} = \gamma\gamma = \gamma^2 = \gamma$, which implies that $\alpha\beta \in \overline{\gamma}$. Thus $\overline{\gamma}$ is a subsemigroup of $T_{\mathscr{F}}(X)$.

The following result on characterizations of idempotents in the full transformation semigroup $T(Z)$, for any set Z, is elementary. We state and prove here again for completeness and self-containness of the contents in this thesis.

Proposition 3.2.1. *Let Z be a nonempty set, and let $\gamma \in T(Z)$. Then the following are equivalent:*

(1) *γ is an idempotent;*

(2) *$\gamma|_{Z\gamma}$ is the identity on $Z\gamma$;*

(3) *there is a partition $\{Z_j : j \in E\}$ of the set Z, and there is a subset $\{z_j : \forall j \in E, z_j \in Z_j\}$ of Z such that*

$$\gamma = \begin{pmatrix} Z_j \\ z_j \end{pmatrix}.$$

In this situation, the partition $\{Z_j : j \in E\}$ and the subset $\{z_j : j \in E\}$ of the set Z are uniquely determined by γ.

Proof. (1) $\Leftrightarrow$ (2). The following argument shows that the equivalence (1) $\Leftrightarrow$ (2) holds.

$$\begin{aligned} \gamma \text{ is an idempotent} \;&\Leftrightarrow\; \gamma^2 = \gamma \\ &\Leftrightarrow\; i\gamma^2 = i\gamma \text{ for all } i \in Z \\ &\Leftrightarrow\; (i\gamma)\gamma = i\gamma \text{ for all } i \in Z \\ &\Leftrightarrow\; j\gamma = j \text{ for all } j \in Z\gamma. \end{aligned}$$

(2) $\Rightarrow$ (3). Suppose that $\gamma|_{Z\gamma}$ is the identity on $Z\gamma$. Then for all $y \in Z\gamma$, we have $y\gamma = y$. Let, for each $y \in Z\gamma$, $Z_y = y\gamma^{-1}$. Then $y \in Z_y$ and $\{Z_y | y \in Z\gamma\}$ is a partition of Z. It is clear that

$$\gamma = \begin{pmatrix} Z_y \\ y \end{pmatrix}.$$

(3) $\Rightarrow$ (2). Suppose that there is a partition $\{Z_j | j \in E\}$ of Z, and that there is a subset $\{z_j : \forall j \in E, z_j \in Z_j\}$ of Z such that

$$\gamma = \begin{pmatrix} Z_j \\ z_j \end{pmatrix}.$$

Thus $Z\gamma = \{z_j : j \in E\}$ and $z_j\gamma = z_j$ for all $j \in E$. Therefore, $\gamma|_{Z\gamma}$ is the identity on $Z\gamma$. □

Theorem 3.2.2. *Let $\gamma \in T(I)$ be an idempotent, and let $\{I_j : j \in E\}$ be the partition of I with the element i_j of I_j for all $j \in E$ such that*

$$\gamma = \begin{pmatrix} I_j \\ i_j \end{pmatrix}.$$

For each $j \in E$, let $W_j = \bigcup_{i \in I_j} Y_i$. Then, for every $\alpha \in \overline{\gamma}$, $\alpha \in \mathcal{R}(\overline{\gamma})$ if and only if $\alpha|_{W_j} \in \mathcal{R}\left(T(W_j, Y_{i_j})\right)$ for all $j \in E$.

Proof. Let $\alpha \in \overline{\gamma}$. Suppose that $\alpha \in \mathcal{R}(\overline{\gamma})$. Then there is a $\beta \in \overline{\gamma}$ such that $\alpha\beta\alpha = \alpha$. For each $j \in E$, let $\alpha_j = \alpha|_{W_j}$ and $\beta_j = \beta|_{W_j}$. Since $\chi^{(\alpha)} = \chi^{(\beta)} = \gamma$, we have that both α_j and β_j belong to $T(W_j, Y_{i_j})$ for all $j \in E$. And since $\alpha\beta\alpha = \alpha$, it follows for each $j \in E$ that $\alpha_j\beta_j\alpha_j = \alpha_j$. Therefore $\alpha_j \in \mathcal{R}\left(T(W_j, Y_{i_j})\right)$ for all $j \in E$. Conversely, suppose that $\alpha_j := \alpha|_{W_j} \in \mathcal{R}\left(T(W_j, Y_{i_j})\right)$ for all $j \in E$. Then for each $j \in E$, there is a $\beta_j \in T(W_j, Y_{i_j})$ such that $\alpha_j\beta_j\alpha_j = \alpha_j$. Since $\{W_j : j \in E\}$ is a partition of X, the following function $\beta : X \to X$ is well-defined:

$$\beta = \begin{pmatrix} W_j \\ \beta_j \end{pmatrix}.$$

It is obvious that $\beta \in \overline{\gamma}$, and that $\alpha\beta\alpha = \alpha$, which yields that α is a regular element of $\overline{\gamma}$. The proof is complete. □

Corollary 3.2.3. *Let $\gamma \in T(I)$ be an idempotent, and let $\{I_j : j \in E\}$ be the partition of I with the element i_j of I_j for all $j \in E$ such that*

$$\gamma = \begin{pmatrix} I_j \\ i_j \end{pmatrix}.$$

Let $W_j = \bigcup_{i \in I_j} Y_i$ for all $j \in E$, and let $E_0 = \{j \in E : |Y_{i_j}| = 1\}$. If $E_0 \neq \emptyset$, let $W = \bigcup_{j \in E_0} W_j$, $I_0 = \bigcup_{j \in E_0} I_j$, and define $\alpha : W \to W$ as follows:

$$\alpha = \begin{pmatrix} W_j \\ y_j \end{pmatrix},$$

where y_j is the only element of Y_{i_j} for each $j \in E_0$. Then the following are equivalent:

(1) *$\overline{\gamma}$ is regular;*

(2) *$T(W_j, Y_{i_j})$ is regular for all $j \in E$;*

(3) *for every $j \in E$, if $j \notin E_0$, then $W_j = Y_{i_j}$;*

(4) *γ is the identity function on I, or $\overline{\gamma} = \{\alpha\}$, or $\overline{\gamma}$ is exactly the set of all $\beta \in T_{\mathscr{F}}(X)$ such that $\beta|_W = \alpha$ and $\chi^{(\beta)}|_{I\setminus I_0}$ is the identity function on $I \setminus I_0$.*

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow immediately from Theorem 3.2.2 and Theorem 2 respectively. We now prove (3) $\Leftrightarrow$ (4).

(3) $\Rightarrow$ (4). Suppose that for every $j \in E$, if $j \notin E_0$, then $W_j = Y_{i_j}$. There are three cases to be considered.

Case 1 $E_0 = \emptyset$: We have in this case that $W_j = Y_{i_j}$ for all $j \in E$, which yields that $I_j = \{i_j\}$ for all $j \in E$. Thus $\{I_j : j \in E\} = \Lambda_I$, and hence γ is the identity function on I.

Case 2 $E_0 = E$: In this case, it is clear that $\overline{\gamma} = \{\alpha\}$.

Case 3 $\emptyset \neq E_0 \neq E$: In this case, we have that $\emptyset \neq W \neq X$ and $\emptyset \neq E \setminus E_0 \neq E$. Similarly to Case 1, we have $\{I_j : j \in E \setminus E_0\} = \Lambda_{I\setminus I_0}$. This yields that γ is the identity function on $I \setminus I_0$. Thus, for any $\beta \in \overline{\gamma}$, we get that $\chi^{(\beta)}|_{I\setminus I_0}$ is the identity function on $I \setminus I_0$. Since $W \neq \emptyset$, the function α can be considered in this case. For each $\beta \in \overline{\gamma}$, since $\chi^{(\beta)}|_{I_0} = \gamma|_{I_0}$, similarly to Case 2, we obtain that $\beta|_{W_j}$ is exactly $\alpha|_{W_j}$ for all $j \in E_0$. Accordingly, $\beta|_W = \alpha$ for all $\beta \in \overline{\gamma}$.

(4) $\Rightarrow$ (3). Suppose that (4) holds. We now have three cases to consider.

Case 1 γ is the identity function on I: In this case, we have that $I_j = \{i_j\}$, which yields that $W_j = Y_{i_j}$ for all $j \in E$. So, (3) clearly holds.

Case 2 $\overline{\gamma} = \{\alpha\}$: We have in this case that $E_0 = E$. Thus (3) is true.

Case 3 $\overline{\gamma}$ is exactly the set of all $\beta \in T_{\mathscr{F}}(X)$ such that $\beta|_W = \alpha$ and $\chi^{(\beta)}|_{I\setminus I_0}$ is the identity function on $I \setminus I_0$: Let $j \in E$ and assume that $j \notin E_0$. Then $i_j \in I \setminus I_0$. Since $\gamma|_{I\setminus I_0}$ is the identity on $I \setminus I_0$, we have that $I_j = \{i_j\}$, which yields that $W_j = Y_{i_j}$. □

From our notion of character, we obtain three more subsemigroups of the full transformation semigroup $T(X)$ nested in the semigroup $T_{\mathscr{F}}(X)$ as follows.

Let $I_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $B_{\mathscr{F}}(X)$ be the sets of elements in $T_{\mathscr{F}}(X)$ whose characters are injective, surjective and bijective respectively. Then $B_{\mathscr{F}}(X) = I_{\mathscr{F}}(X) \cap S_{\mathscr{F}}(X)$. And by Lemma 3.1.3, both $I_{\mathscr{F}}(X)$ and $S_{\mathscr{F}}(X)$ are submonoids of $T_{\mathscr{F}}(X)$, and $B_{\mathscr{F}}(X)$ is a submonoid of both $I_{\mathscr{F}}(X)$ and $S_{\mathscr{F}}(X)$. Notice that $I_{\Lambda_X}(X)$, $S_{\Lambda_X}(X)$ and $B_{\Lambda_X}(X)$ are exactly the sets of elements in $T(X)$ which are injective, surjective and bijective respectively. And that $I_{\Sigma_X}(X) = S_{\Sigma_X}(X) = B_{\Sigma_X}(X) = T_{\Sigma_X}(X) = T(X)$. We end this chapter by studying the regularity of the semigroups $B_{\mathscr{F}}(X)$, $S_{\mathscr{F}}(X)$ and $I_{\mathscr{F}}(X)$.

Theorem 3.2.4. (1) *The semigroup $B_{\mathscr{F}}(X)$ is regular.*

(2) *$\mathcal{R}(I_{\mathscr{F}}(X)) = \mathcal{R}(S_{\mathscr{F}}(X)) = B_{\mathscr{F}}(X)$.*

Proof. (1) Let $\alpha \in B_{\mathscr{F}}(X)$, and let $\gamma = \chi^{(\alpha)}$. For $i \in I$, let α_i be the restriction of α to Y_i. Then α_i is a function from Y_i into $Y_{i\gamma}$. We now let $i \in I$ be arbitrarily fixed, and for each $x \in Y_i\alpha_i$, let $a_x^{(i)}$ be a fixed element in $x\alpha_i^{-1}$, and let a_i be another point in Y_i which is also fixed. Let $\beta_i : Y_{i\gamma} \to Y_i$ be defined by

$$\beta_i = \begin{pmatrix} x & Y_{i\gamma} \setminus Y_i\alpha_i \\ a_x^{(i)} & a_i \end{pmatrix}.$$

Next, we define $\beta \in T(X)$ by

$$\beta = \begin{pmatrix} Y_i \\ \beta_j \end{pmatrix},$$

where $j \in I$ such that $i = j\gamma$. We now want to show that β is well defined. Let $x, z \in X$. Then there are $\nu, \mu \in I$ such that $x \in Y_\nu$ and $z \in Y_\mu$. Since γ is surjective, there are $i, j \in I$ such that $\nu = i\gamma$ and $\mu = j\gamma$. Assume that $x = z$. Then $Y_\nu = Y_\mu$, which yields that $i\gamma = \nu = \mu = j\gamma$. Thus, by the injectivity of γ, we have $i = j$. So $x\beta = x\beta_i = x\beta_j = z\beta_j = z\beta$. Hence β is well-defined, that is, $\beta \in T(X)$. It is evident that $\beta \in T_{\mathscr{F}}(X)$, and that the character of β is exactly γ^{-1}. Therefore $\beta \in B_{\mathscr{F}}(X)$. Finally, we show that $\alpha\beta\alpha = \alpha$. To see this, let $x \in X$. Then $x \in Y_i$ for some $i \in I$, and hence $x\alpha \in Y_i\alpha_i$. Thus $((x\alpha)\beta)\alpha = ((x\alpha_i)\beta_i)\alpha = \left(a_{x\alpha_i}^{(i)}\right)\alpha = \left(a_{x\alpha_i}^{(i)}\right)\alpha_i$. Since $a_{x\alpha_i}^{(i)} \in x\alpha_i\alpha_i^{-1}$, it follows that $((x\alpha)\beta)\alpha = \left(a_{x\alpha_i}^{(i)}\right)\alpha_i = x\alpha_i = x\alpha$. Consequently, α is a regular element of the semigroup $B_{\mathscr{F}}(X)$.

(2) We obtain immediately from (1) that $B_{\mathscr{F}}(X) \subseteq \mathcal{R}(I_{\mathscr{F}}(X))$ and that $B_{\mathscr{F}}(X) \subseteq \mathcal{R}(S_{\mathscr{F}}(X))$. To see that $B_{\mathscr{F}}(X) = \mathcal{R}(I_{\mathscr{F}}(X))$, let α be a regular element of $I_{\mathscr{F}}(X)$. We want to show that $\alpha \in B_{\mathscr{F}}(X)$. Suppose to the contrary that α is not a member of $B_{\mathscr{F}}(X)$, which means $\chi^{(\alpha)}$ is not surjective. Since α is regular, there is a $\beta \in I_{\mathscr{F}}(X)$ such that $\alpha\beta\alpha = \alpha$, which yields from Lemma 2.3 that $\chi^{(\alpha)}\chi^{(\beta)}\chi^{(\alpha)} = \chi^{(\alpha)}$. Thus, by the injectivity of $\chi^{(\alpha)}$ and $\chi^{(\beta)}$, we have that $\chi^{(\beta)}|_{I\chi^{(\alpha)}} = \left(\chi^{(\alpha)}\right)^{-1}$. Since $\chi^{(\alpha)}$ is not surjective, it follows that $I\chi^{(\alpha)} \neq I$, which implies that $\chi^{(\beta)}$ is not injective. This is a contradiction. Hence $\alpha \in B_{\mathscr{F}}(X)$, and so $\mathcal{R}(I_{\mathscr{F}}(X)) = B_{\mathscr{F}}(X)$. We now turn our attention to showing that $B_{\mathscr{F}}(X) = \mathcal{R}(S_{\mathscr{F}}(X))$. Let $\alpha \in \mathcal{R}(S_{\mathscr{F}}(X))$. Then there is a $\beta \in S_{\mathscr{F}}(X)$ such that $\alpha\beta\alpha = \alpha$. And by the surjectivity of $\chi^{(\alpha)}$, we have for each $i \in I$ that $i\left(\chi^{(\alpha)}\right)^{-1} \neq \emptyset$. We claim that for each $i \in I$, there is a unique $j_i \in i\left(\chi^{(\alpha)}\right)^{-1}$ such that $Y_i\beta \subseteq Y_{j_i}$. Let $i \in I$, and fix $x \in Y_i\alpha^{-1}$. Then $x\alpha \in Y_i$. To get what we claim, we will show that there is a $j_i \in i\left(\chi^{(\alpha)}\right)^{-1}$ such that $x\alpha\beta \in Y_{j_i}$. If $x\alpha\beta$ were not in Y_j for all $j \in i\left(\chi^{(\alpha)}\right)^{-1}$, there would be a $k \in I \setminus \{i\}$ such that $x\alpha\beta \in Y_\mu$ for some $\mu \in k\left(\chi^{(\alpha)}\right)^{-1}$, which yields that $x\alpha = x\alpha\beta\alpha \in Y_\mu\alpha \subseteq Y_k$. Since $x\alpha \in Y_i$, it follows that $Y_i \cap Y_k \neq \emptyset$, which is a contradiction. Hence there is a $j_i \in i\left(\chi^{(\alpha)}\right)^{-1}$ such that $x\alpha\beta \in Y_{j_i}$, which implies by the definition of $T_{\mathscr{F}}(X)$ that $Y_i\beta \subseteq Y_{j_i}$. It is clear that j_i is unique. Therefore $i\chi^{(\beta)} = j_i$ for all $i \in I$. To see that $\alpha \in B_{\mathscr{F}}(X)$, suppose to the contrary that $\alpha \notin B_{\mathscr{F}}(X)$, that is, $\chi^{(\alpha)}$ is not injective. Then there is a $\nu \in I$ such that $\left|\nu\left(\chi^{(\alpha)}\right)^{-1}\right| > 1$, which yields that $\chi^{(\beta)}$ is not surjective. This is

a contradiction. Therefore, $\alpha \in B_{\mathscr{F}}(X)$, and thus we obtain $B_{\mathscr{F}}(X) = \mathcal{R}(S_{\mathscr{F}}(X))$ as asserted. $\square$

From Theorem 3.2.4, we obtain the following result on the regularity of $I_{\mathscr{F}}(X)$.

Corollary 3.2.5. *$I_{\mathscr{F}}(X)$ is regular if and only if I is finite.*

Proof. Suppose that $I_{\mathscr{F}}(X)$ is regular. Then by Theorem 3.2.4, $I_{\mathscr{F}}(X) = B_{\mathscr{F}}(X)$. Thus, by Theorem 3.1.4, we have that for every $\gamma \in T(I)$, if γ is injective, then γ is bijective. This occurs only when I is finite (by Theorem 2.1.14). Conversely, suppose that I is finite. Then $I_{\mathscr{F}}(X) = B_{\mathscr{F}}(X)$. Hence, by Theorem 3.2.4 again, $I_{\mathscr{F}}(X)$ is regular. $\square$

Similarly, the following result on the regularity of the semigroup $S_{\mathscr{F}}(X)$ is obtained.

Corollary 3.2.6. *$S_{\mathscr{F}}(X)$ is regular if and only if I is finite.*

Chapter 4

Summary

In this thesis, by a *partition* of a set X, we mean a family $\{Y_i : i \in I\}$ of nonempty subsets of X possessing the properties that for each $i, j \in I$ with $i \neq j, Y_i \cap Y_j = \emptyset$, and that $X = \bigcup\limits_{i \in I} Y_i$. Let X be a nonempty set, and let $\mathscr{F} = \{Y_i : i \in I\}$ be a partition of X. Let

$$T_{\mathscr{F}}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in I, Y_i\alpha \subseteq Y_j\}.$$

This setting is another approach of the one of Huisheng in [8] by the well-known fact that any partition on a set induces an equivalence relation on that set in a natural way. The set $T_{\mathscr{F}}(X)$ can be generalized by fixing, in addition to the nonempty set X and the partition $\mathscr{F}$ of X, a nonempty subset J of the index set I as follows. Let $J \subseteq I$ with $J \neq \emptyset$, and let

$$T_{\mathscr{F}}^{(J)}(X) = \{\alpha \in T(X) : \forall i \in I \exists j \in J, Y_i\alpha \subseteq Y_j\}.$$

Let $Y^{(J)} = \bigcup\limits_{i \in J} Y_i$. Then we can easily see that $T_{\mathscr{F}}^{(J)}(X)$ is a subsemigroup of $T(X, Y^{(J)})$.

For each $\alpha \in T_{\mathscr{F}}^{(J)}(X)$, the function $\chi^{(\alpha)} : I \to J$ defined by $i\chi^{(\alpha)} = j$ if and only if $Y_i\alpha \subseteq Y_j$ is called the *character* of α. By the definition of a partition of a set stated here, the function $\chi^{(\alpha)}$ is well-defined for all $\alpha \in T_{\mathscr{F}}^{(J)}(X)$. For any $\alpha, \beta \in T_{\mathscr{F}}^{(J)}(X)$, we have that $\chi^{(\alpha\beta)} = \chi^{(\alpha)}\chi^{(\beta)}$. From the notion of character provided above, we can reasonably define two relations χ and $\widetilde{\chi}$ on $T_{\mathscr{F}}^{(J)}(X)$ as follows:

$$(\alpha, \beta) \in \chi \Leftrightarrow \chi^{(\alpha)} = \chi^{(\beta)},$$

and

$$(\alpha, \beta) \in \widetilde{\chi} \Leftrightarrow \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J.$$

We have that the relations χ and $\widetilde{\chi}$ are congruence relations on $T_{\mathscr{F}}^{(J)}(X)$. Thus both $T_{\mathscr{F}}^{(J)}(X)/\chi$ and $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi}$ are semigroups. The main results of this thesis are divided into two parts.

In the first part, the regularity of the quotient semigroups $T_{\mathscr{F}}^{(J)}(X)/\chi$ and $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi}$ and the semigroup $T_{\mathscr{F}}^{(J)}(X)$ and are studied. We obtain that the quotient semigroup $T_{\mathscr{F}}^{(J)}(X)/\chi$ is regular if and only if $J = I$ or $|J| = 1$. While, the quotient semigroup $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi}$ is always regular. To obtain the results on the regularity of the quotient semigroups $T_{\mathscr{F}}^{(J)}(X)/\chi$ and $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi}$, we establish the following results: $T_{\mathscr{F}}^{(J)}(X)/\chi \cong T(I,J)$ by the isomorphism $[\alpha] \mapsto \chi^{(\alpha)}$ and $T_{\mathscr{F}}^{(J)}(X)/\widetilde{\chi} \cong T(J)$ by the isomorphism $\widetilde{[\alpha]} \mapsto \chi^{(\alpha)}|_J$, where $[\alpha]$ is the equivalence class of α under χ and $\widetilde{[\alpha]}$ is the equivalence class of α under $\widetilde{\chi}$. For the semigroup $T_{\mathscr{F}}^{(J)}(X)$ we get that it is regular if and only if $\left|T_{\mathscr{F}}^{(J)}(X)\right| = 1$ or $T_{\mathscr{F}}^{(J)}(X) = T(X)$.

In the second part of our results, we study the regularity of some subsemigroups of the semigroup $T_{\mathscr{F}}(X)$. From the fact that $T_{\mathscr{F}}(X)/\chi \cong T(I)$ by the isomorphism $[\alpha] \mapsto \chi^{(\alpha)}$, we can identify the semigroup $T(I)$ with the semigroup $T_{\mathscr{F}}(X)/\chi$. For any $\gamma \in T(I)$, let $\overline{\gamma}$ denote the equivalence class of α whose character is γ. We see that for each $\gamma \in T(I)$, $\overline{\gamma}$ is a subsemigroup of $T_{\mathscr{F}}(X)$ if and only if γ is an idempotent. Let $\gamma \in T(I)$ be an idempotent. Then there is a partition $\{I_j : j \in E\}$ of I with the element i_j for I_j for all $j \in E$ such that

$$\gamma = \begin{pmatrix} I_j \\ i_j \end{pmatrix}.$$

Let $W_j = \displaystyle\bigcup_{i \in I_j} Y_i$ for all $j \in E$, and let $E_0 = \{j \in E : |Y_{i_j}| = 1\}$. If $E_0 \neq \emptyset$, let $W = \displaystyle\bigcup_{j \in E_0} W_j$, and let $I_0 = \displaystyle\bigcup_{j \in E_0} I_j$, and define $\alpha : W \to W$ as follows:

$$\alpha = \begin{pmatrix} W_j \\ y_j \end{pmatrix},$$

where y_j is the only element of Y_{i_j} for each $j \in E_0$. We obtain that $\overline{\gamma}$ is regular if and only if γ is the identity function on I, or $\overline{\gamma} = \{\alpha\}$, or $\overline{\gamma}$ is exactly the set of all $\beta \in T_{\mathscr{F}}(X)$ such that $\beta|_W = \alpha$ and $\chi^{(\beta)}|_{I \setminus I_0}$ is the identity function on $I \setminus I_0$. From our notion of character defined above, we can define some further subsemigroups of $T_{\mathscr{F}}(X)$ as follows. Let

$$I_{\mathscr{F}}(X) := \left\{\alpha \in T_{\mathscr{F}}(X) : \chi^{(\alpha)} \text{ is injective}\right\},$$

$$S_{\mathscr{F}}(X) := \left\{\alpha \in T_{\mathscr{F}}(X) : \chi^{(\alpha)} \text{ is surjective}\right\}$$

and

$$B_{\mathscr{F}}(X) := \left\{\alpha \in T_{\mathscr{F}}(X) : \chi^{(\alpha)} \text{ is bijective}\right\}.$$

We have that the semigroup $B_{\mathscr{F}}(X)$ is regular, and that $\mathcal{R}(I_{\mathscr{F}}(X)) = \mathcal{R}(S_{\mathscr{F}}(X)) = B_{\mathscr{F}}(X)$. From these, we can deduce that $I_{\mathscr{F}}(X)$ is regular if and only if I is finite. Also, we can deuce that $S_{\mathscr{F}}(X)$ is regular if and only if I is finite.

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