UNIFORM CONVERGENCE OF A SEQUENCE OF QUASICONFORMAL MAPPINGS

By
Mr.Phruetthiphong Lohasuwan

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Graduate School, Silpakorn University
Academic Year 2015

การคู่เข้าเอกรูปของลำดับของการส่งกึ่งคงแบบ

โดย

นายพฤฒิพงศ์ โลหะสุวรรณ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์

ภาควิชาคณิตศาสตร์

บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ปีการศึกษา 2558

UNIFORM CONVERGENCE OF A SEQUENCE OF QUASICONFORMAL MAPPINGS

By
Mr.Phruetthiphong Lohasuwan

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Graduate School, Silpakorn University
Academic Year 2015

The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Uniform Convergence of a Sequence of Quasiconformal Mappings” submitted by MR.Phruetthiphong Lohasuwan as a partial fulfillment of the requirements for the degree of Master of Science in MATHEMATICS

....
(Associate Professor Panjai Tantatsanawong,Ph.D.)
Dean of Graduate School
........./.................../..........

The Thesis Advisor

1. Assistant Professor Somjate Chaiya, Ph.D.
2. Professor Raimo Näkki, Ph.D.

The Thesis Examination Committee

.... Chairman
(Jittisak Rakbud, Ph.D.)
............/......................./..............

.... Member
(Assistant Professor Thiradet Jiarasuksakun, Ph.D.)
............/......................./..............

.... Member
(Assistant Professor Somjate Chaiya, Ph.D.)
............/......................./..............

.... Member
(Professor Raimo Näkki, Ph.D.)
............/......................./..............

54305203 : สาขาวิชาคณิตศาสตร์

คำสำคัญ : การส่งกึ่งคงแบบ/ภาคขยายของขอบ/ความเชื่อมโยงเฉพาะที่

พฤฒิพงศ์ โลหะสุวรรณ์ : การลู่เข้าเอกรูปของลำดับของการส่งกึ่งคงแบบ. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : ผศ.ดร.สมเจตน์ ชัยยะ และ Professor Dr. RAIMO NÄKKI. 34 หน้า.

ทฤษฎีบทอันเป็นที่รู้จักทฤษฎีบทหนึ่งซึ่งนำเสนอโดย POMMERENKE ได้กล่าวไว้ว่า การส่งคงแบบของจานเปิดหนึ่งหน่วย B ไปทั่วถึงโดเมนที่มีขอบเขต D สามารถขยายความต่อเนื่องไปยังส่วนปิดคลุมของ B ได้ ก็ต่อเมื่อ ส่วนเติมเต็มของ D เป็นเซตเชื่อมโยงเฉพาะที่ ในวิทยานิพนธ์นี้ เราได้แสดงให้เห็นว่าทฤษฎีบทนี้ยังคงเป็นจริงสำหรับการส่งกึ่งคงแบบด้วย

นอกจากนี้ เราได้ศึกษาเกี่ยวกับการลู่เข้าเอกรูปของลำดับของการส่งกึ่งคงแบบ เราได้แสดงว่า ถ้าลำดับ (f_n) ของการส่งกึ่งคงแบบของจานเปิดหนึ่งหน่วย B ไปยังโดเมนที่มีขอบเขตลู่เข้าสู่สาทิสสัณฐาน f และแต่ละการส่ง f_n สามารถขยายความต่อเนื่องไปยังส่วนปิดคลุมของ B ได้ แล้ว ลำดับ (f_n) ลู่เข้าเอกรูปสู่ f เมื่อวงศ์ของส่วนเติมเต็มของภาพของ B ภายใต้การส่งกึ่งคงแบบ f_n เป็นเซตเชื่อมโยงเฉพาะที่เอกรูป

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ลายมือชื่อนักศึกษา .... ปีการศึกษา 2558

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ 1 .... 2 ....

54305203 : MAJOR : MATHEMATICS

KEY WORD : QUASICONFORMAL MAPPING/BOUNDARY EXTENSION

PHRUETTHIPHONG LOHASUWAN : UNIFORM CONVERGENCE OF A SEQUENCE OF QUASICONFORMAL MAPPINGS. THESIS ADVISOR : ASST.PROF.SOMJATE CHAIYA, Ph.D., AND PROF.RAIMO NÄKKI, Ph.D. 34 pp.

A well-known theorem, due to Pommerenke, states that a conformal mapping of the open unit disk B onto a bounded domain D can be extended continuously to the closure of B if and only if the complement of D is locally connected. In this thesis we show this result also holds for quasiconformal mappings.

We also investigate the uniform convergence of a sequence of quasiconformal mappings. For instance, we show that if (f_n) is a sequence of K-quasiconformal mappings of the open unit disk B into bounded domains converging to a homeomorphism f and if each f_n admits an extension to a continuous mapping of the closure of B, then the convergence is uniform B precisely when the complements of the image domains are uniformly locally connected.

Department of Mathematics Graduate School, Silpakorn University

Student's signature .... Academic Year 2015

Thesis Advisor's signature 1.. 2..

Acknowledgements

The thesis has been completed by the involvement of many people.

I would like to express my gratitude and sincere appreciation to Asst. Prof. Dr. Somjate Chaiya and Prof. Dr. Raimo Näkki, my advisors, for their valuable suggestions and throughout the study.

I would also like to thank Dr. Jittisak Rakbud and Asst. Prof. Dr. Thiradet Jiarasuksakun, Members of the Thesis Committee, for their valuable comments and suggestions.

Finally, I would like to express my gratitude to my family and friends for their understanding, encouragement and moral support during the study.

Table of Contents

Chapter 1

Introduction

The celebrated Riemann Mapping Theorem states that the open unit disk B in the complex plane $\mathbb{C}$ can be mapped conformally onto any simply connected domain D in $\mathbb{C}$ with non-degenerate boundary. A comprehensive treatment of the boundary behavior of conformal mappings is presented in Pommerenke's monograph [11], which also contains an extensive list of references.

In [10] Pommerenke characterized the domains D onto which the Riemann mapping of B admits a continuous boundary extension as follows: a conformal mapping of the open unit disk B onto D can be extended continuously to $\overline{B}$ if and only if the complement of D is locally connected. In this thesis we show that this result also holds for quasiconfomal mappings. Additional characterizations, valid not only for quasiconformal mappings in the plane but also for higher dimensional quasiconformal mappings as well, can be found in [7] and [13]. The main tools employed in establishing the quasiconformal analogue of Pommerenke's result are the modulus of a path family introduced by Ahlfors and Beurling in their landmark paper [2] and a quasiconformal version of Wolff's lemma which is introduced by Becker, see [3], in companion with some topological considerations.

We also investigate the uniform convergence of a sequence of K-quasiconformal mappings. Let (f_n) is a sequence of K-quasiconformal mappings of the open unit disk B into bounded domains which converges in B to a homeomorphism f. We show that the following statements hold:

(1) If the collection $\mathscr{E} = \{E_n = \hat{\mathbb{C}} \setminus f_n(B)\}$ is uniformly locally connected, then the mappings f_n can be extended to continuous mappings $\overline{f}_n$ of $\overline{B}$ and the sequence $(\overline{f}_n)$ converges uniformly in $\overline{B}$.

(2) If the convergence $f_n \to f$ is uniform and if each f_n extends continuously to $\overline{B}$, then $\mathscr{E}$ is uniformly locally connected.

Chapter 2

Modulus of a Path Family

In this chapter we study the main geometric tools used in studying quasiconformal mappings. We associate a number, called the modulus, to each path family to describe, albeit in an abstract way, how long or short or how plentiful the paths in the family are. The idea of doing so is due to Beurling and was first published in the seminal paper [2] by Ahlfors and Beurling in 1950. Most of the results follows Väisälä [13].

2.1 The modulus

Before defining the modulus, we need to introduce some pertinent terminology.

Let I be an interval in $\mathbb{R}$. A mapping γ from I into $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ is called a **path** if it is continuous. We call a restriction of the path γ to a subinterval of I a **subpath** of γ. The path γ is called **open** or **closed** if the interval I is open or closed, respectively.

To define the length of a path, let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed path. We call a subset $P = \{t_0, t_1, \ldots, t_n\}$ of the interval $[a, b]$ a **partition** of $[a, b]$ if

$$a = t_0 < t_1 < \ldots < t_n = b.$$

Define the **length** of γ by the supremum of the sums

$$\sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})|$$

over all partitions of $[a, b]$ and denote by $\ell(\gamma)$. If γ is a nonconstant path in $\overline{\mathbb{R}^2}$ that contains ∞, then we define $\ell(\gamma) = \infty$. For the constant path $\gamma(t) \equiv \infty$, we define $\ell(\gamma) = 0$.

Clearly, $0 \leq \ell(\gamma) \leq \infty$ for any path γ. A path γ is called **rectifiable** if it has a finite length, otherwise **non-rectifiable**. A path γ is **locally rectifiable** if every closed subpath of γ is rectifiable. Hence any path γ that contains ∞ is non-rectifiable, except only when it is a constant path.

Next, we will present a definition of a line integral. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a rectifiable path. The path $\gamma^0 : [0, \ell(\gamma)] \to \mathbb{R}^2$ is called the **normal representation** of γ if it satisfies the following properties:

1. there exists an increasing continuous mapping h from $[a, b]$ onto $[0, \ell(\gamma)]$ such that $\gamma = \gamma^0 \circ h$.
2. $\ell(\gamma^0|_{[0,t]}) = t$ for all $0 \leq t \leq \ell(\gamma)$.

The function γ^0 is unique, see [13, Theorem 2.4, p. 5].

Let ρ be a Borel function from a Borel set A in $\mathbb{R}^2$ into the interval $[0, \infty]$. For any rectifiable path $\gamma : [a, b] \to A$, we define the **line integral** of ρ over γ by

$$\int_\gamma \rho ds = \int_0^{\ell(\gamma)} \rho\left(\gamma^0(t)\right) dt.$$

We sometimes use the notation $\int_\gamma \rho(z)|dz|$ instead of $\int_\gamma \rho ds$. For an open or half-open locally rectifiable path γ the line integral of ρ is defined as the supremum of the line integrals of ρ over all closed subpaths of γ.

Now, we are ready to give the definition of the modulus of a path family. Let Γ be a family of paths in $\overline{\mathbb{R}^2}$. A Borel function $\rho : \mathbb{R}^2 \to [0, \infty]$ is called **admissible** for Γ if

$$\int_\gamma \rho ds \geq 1$$

for every locally rectifiable path γ in Γ. Let $\mathscr{F}(\Gamma)$ denote the collection of all admissible functions for Γ. Define the the **modulus** of Γ as

$$M(\Gamma) = \inf_{\rho \in \mathscr{F}(\Gamma)} \int_{\mathbb{R}^2} \rho^2 dm,$$

where m stands for the Lebesgue measure in $\mathbb{R}^2$. If $\mathscr{F}(\Gamma) = \emptyset$, we define $M(\Gamma) = \infty$. Obviously, $0 \leq M(\Gamma) \leq \infty$.

The following theorem establishes some basic properties about the modulus of path families.

Theorem 2.1.1. [13, Theorem 6.2, p. 16] *M is an outer measure in the collection of all path families:*

(1) $M(\emptyset) = 0$,

(2) $\Gamma_1 \subset \Gamma_2$ *implies* $M(\Gamma_1) \leq M(\Gamma_2)$,

(3) $M\left(\bigcup_{k=1}^{\infty} \Gamma_k\right) \leq \sum_{k=1}^{\infty} M(\Gamma_k)$.

Let Γ_1 and Γ_2 be two families of paths. We say that Γ_1 **minorizes** Γ_2, denoted by $\Gamma_1 < \Gamma_2$, if every path $\gamma \in \Gamma_2$ has a subpath in Γ_1.

Theorem 2.1.2. [13, Theorem 6.4, p. 17] *If* $\Gamma_1 < \Gamma_2$, *then* $M(\Gamma_1) \geq M(\Gamma_2)$.

Proof. Suppose first that $\Gamma_1 < \Gamma_2$. We will show that $\mathscr{F}(\Gamma_1) \subset \mathscr{F}(\Gamma_2)$. Let $\rho \in \mathscr{F}(\Gamma_1)$ and let γ be an arbitrary locally rectifiable path in Γ_2. Then γ has a subpath $\gamma' \in \Gamma_1$ and then

$$\int_{\gamma'} \rho |dz| \geq 1.$$

Hence

$$\int_{\gamma} \rho |dz| \geq \int_{\gamma'} \rho |dz| \geq 1$$

and hence $\rho \in \mathscr{F}(\Gamma_2)$. Since

$$\int_{\mathbb{R}^2} \rho^2 dm \geq M(\Gamma_2),$$

we obtain by taking the infimum over all such $\rho \in \mathscr{F}(\Gamma_1)$ that

$$M(\Gamma_1) = \inf_{\rho \in \mathscr{F}(\Gamma_1)} \int_{\mathbb{R}^2} \rho^2 dm.$$

Therefore $M(\Gamma_1) \geq M(\Gamma_2)$. □

The result of this theorem is called the minorizing principle. It is one of the most useful tools for finding an upper bound or a lower bound for the modulus of path families. We will use this principle several times latter.

Next, we introduce the notation and terminology that will be used for the rest of this thesis. All sets considered here are assumed to lie in the plane $\mathbb{R}^2$. We use $B(z, r)$ to denote the open disk of radius r centered at z, we let $S(z, r)$ denote the boundary of $B(z, r)$. For convenience, we abbreviate $B(r) = B(0, r)$, $S(r) = S(0, r)$, $B = B(0, 1)$ and $S = S(0, 1)$. The topology used for sets in $\mathbb{R}^2$ is the relative topology induced by the Euclidean metric. For a given path family Γ, it is very difficult (in general impossible) to compute $M(\Gamma)$. There are only very few such families Γ for which the modulus $M(\Gamma)$ can be computed precisely. We next compute the modulus $M(\Gamma)$ for some specific families Γ.

Example 2.1.3. *Let* A *be a spherical annulus* $B(b) \setminus \overline{B(a)}$ *where* $0 < a < b$. *Let* Γ *be the family of all radial segments in* A. *Then*

$$M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}.$$

Proof. We will first show that $M(\Gamma) \geq \frac{2\pi}{\log \frac{b}{a}}$. For each $\theta \in [0, 2\pi]$, let $\gamma_\theta : [a, b] \to A$ be the line segment in Γ defined by

$$\gamma_\theta(r) = re^{i\theta}.$$

Let $\rho \in \mathscr{F}(\Gamma)$. Then

$$\int_{\gamma_\theta} \rho|dz| = \int_a^b \rho(re^{i\theta})dr \geq 1.$$

By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 1 &\leq \left(\int_a^b \rho dr\right)^2 \\ &= \left(\int_a^b \rho r^{1/2} \cdot r^{-1/2} dr\right)^2 \\ &\leq \int_a^b (\rho r^{1/2})^2 dr \int_a^b (r^{-1/2})^2 dr \\ &= \int_a^b \rho^2 r dr \int_a^b \frac{1}{r} dr \\ &= \log \frac{b}{a} \int_a^b \rho^2 r dr. \end{aligned}$$

Thus we have

$$\frac{1}{\log \frac{b}{a}} \leq \int_a^b \rho^2 r dr.$$

Integrating over all $\theta \in [0, 2\pi]$, we get

$$\int_0^{2\pi} \frac{1}{\log \frac{b}{a}} d\theta \leq \int_0^{2\pi} \left(\int_a^b \rho^2 r dr\right) d\theta = \int_A \rho^2 dm \leq \int_{\mathbb{R}^2} \rho^2 dm.$$

It shows that

$$\frac{2\pi}{\log \frac{b}{a}} \leq \int_{\mathbb{R}^2} \rho^2 dm$$

for all $\rho \in \mathscr{F}(\Gamma)$, and hence

$$M(\Gamma) = \inf_{\rho \in \mathscr{F}(\Gamma)} \int_{\mathbb{R}^2} \rho^2 dm \geq \frac{2\pi}{\log \frac{b}{a}}.$$

Next, we will show that $M(\Gamma) \leq \frac{2\pi}{\log \frac{b}{a}}$. Let

$$\rho_0(z) = \begin{cases} \frac{1}{|z| \log \frac{b}{a}} & \text{if } z \in A, \\ 0 & \text{if } z \notin A. \end{cases}$$

For $\gamma \in \Gamma$, we have

$$\int_\gamma \rho_0 |dz| = \int_\gamma \frac{1}{|z| \log \frac{b}{a}} |dz| = \int_a^b \frac{1}{r \log \frac{b}{a}} dr = 1,$$

that is, ρ is admissible for Γ. Hence

$$\begin{aligned} M(\Gamma_0) &\leq \int_{\mathbb{R}^2} \rho_0^2 dm \\ &= \int_A \rho_0^2 dm \\ &= \int_A \frac{1}{|z|^2(\log \frac{b}{a})^2} dm \\ &= \int_0^{2\pi} \int_a^b \frac{1}{r^2(\log \frac{b}{a})^2} r dr d\theta \\ &= \frac{2\pi}{\log \frac{b}{a}}. \end{aligned}$$

It follows that $M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}$ as desired. □

Example 2.1.4. *Let A be a spherical annulus $B(b) \setminus \overline{B(a)}$ where $0 < a < b$, and let Γ be the family of all paths in A joining the boundary components of A. Then*

$$M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}.$$

Proof. We will show first that $M(\Gamma) \geq \frac{2\pi}{\log \frac{b}{a}}$. Let Γ_0 be the family of all radial segments in A. Clearly, $\Gamma_0 \subset \Gamma$. By Theorem 2.1.1 and Example 2.1.3, we obtain

$$M(\Gamma) \geq \frac{2\pi}{\log \frac{b}{a}}.$$

Next, we will show that $M(\Gamma) \leq \frac{2\pi}{\log \frac{b}{a}}$. Let

$$\rho_0(z) = \begin{cases} \frac{1}{|z| \log \frac{b}{a}} & \text{if } z \in A, \\ 0 & \text{if } z \notin A. \end{cases}$$

Given a locally rectifiable path γ in Γ, we have

$$\int_\gamma \rho_0 |dz| \geq \int_a^b \rho_0 dr = 1.$$

So ρ_0 is admissible for Γ. Hence, by Example 2.1.3 we obtain,

$$M(\Gamma) \leq \int_A \rho_0^2 dm = \frac{2\pi}{\log \frac{b}{a}}.$$

It follows that $M(\Gamma) = \frac{2\pi}{\log \frac{b}{a}}$ as desired. □

By using the similar argument, we can modify the previous computation to get the modulus for some subfamilies of the path family Γ in Example 2.1.4. More precisely, let A_α be a sector of central angle α and let Γ_α be the family of all paths in A_α joining the boundary components of A. Then we can show that

$$M(\Gamma_\alpha) = \frac{\alpha}{\log \frac{b}{a}}.$$

Example 2.1.5. *Let A be the spherical annulus $B(b) \smallsetminus \{0\}$ where $b > 0$, and let Γ be the family of all paths in A joining the boundary components of A. Then*

$$M(\Gamma) = 0.$$

Proof. Fix a number a with $0 < a < b$. Let Γ' be a family of all paths in the spherical annulus $B(b) \smallsetminus \overline{B(a)}$ joining the boundary components of the annulus. Clearly, $\Gamma' < \Gamma$. By Theorem 2.1.2, we obtain

$$M(\Gamma') \geq M(\Gamma).$$

Note that $M(\Gamma') = \frac{2\pi}{\log \frac{b}{a}}$ by Example 2.1.4. Thus

$$M(\Gamma) \leq \frac{2\pi}{\log \frac{b}{a}}.$$

Since $\frac{2\pi}{\log \frac{b}{a}} \to 0$ as $a \to 0$, we obtain that $M(\Gamma) = 0$ as desired. □

We next give 2 more examples whose results will be used several times later.

Example 2.1.6. *Let Δ be the family of all circles that separate the boundary components of the spherical annulus $A = B(b) \smallsetminus \overline{B(a)}$ where $0 < a < b$. Then*

$$M(\Delta) = \frac{1}{2\pi} \log \frac{b}{a}.$$

Proof. We will show first that $M(\Delta) \geq \frac{1}{2\pi} \log \frac{b}{a}$. For each $r \in (a, b)$, let $\gamma_r : [0, 2\pi] \to A$ be a circle in Δ defined by

$$\gamma_r(\theta) = re^{i\theta}.$$

Let $\rho \in \mathscr{F}(\Delta)$. Then

$$\int_{\gamma_r} \rho |dz| = \int_0^{2\pi} \rho(re^{i\theta}) r d\theta \geq 1.$$

By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} 1 &\leq \left(\int_0^{2\pi} \rho r d\theta \right)^2 \\ &= \left(\int_0^{2\pi} \rho r^{1/2} \cdot r^{1/2} d\theta \right)^2 \\ &\leq \left(\int_0^{2\pi} \rho^2 r d\theta \right) \left(\int_0^{2\pi} r d\theta \right) \\ &= 2\pi r \left(\int_0^{2\pi} \rho^2 r d\theta \right). \end{aligned}$$

That is

$$\frac{1}{2\pi r} \leq \int_0^{2\pi} \rho^2 r d\theta.$$

Integrating over all $r \in (a, b)$, we obtain

$$\frac{1}{2\pi} \log \frac{b}{a} = \int_a^b \frac{1}{2\pi r} dr \leq \int_a^b \left(\int_0^{2\pi} \rho^2 r d\theta \right) dr = \int_A \rho^2 dm \leq \int_{\mathbb{R}^2} \rho^2 dm,$$

for all $\rho \in \mathscr{F}(\Delta)$. Hence

$$M(\Delta) = \inf_{\rho \in \mathscr{F}(\Delta)} \int_{\mathbb{R}^2} \rho^2 dm \geq \frac{1}{2\pi} \log \frac{b}{a}.$$

Next we will show that $M(\Delta_0) \leq \frac{1}{2\pi} \log \frac{b}{a}$. Let

$$\rho_0(z) = \begin{cases} \frac{1}{2\pi|z|} & \text{if } z \in A, \\ 0 & \text{if } z \notin A. \end{cases}$$

For $\gamma \in \Delta$, we get

$$\int_\gamma \rho_0 |dz| = \int_\gamma \frac{1}{2\pi|z|} |z| d\theta = \int_0^{2\pi} \frac{1}{2\pi} d\theta = 1.$$

Hence ρ_0 is admissible for Δ. It follows that

$$\begin{aligned} M(\Delta_0) &\leq \int_{\mathbb{R}^2} \rho_0^2 dm \\ &= \int_A \rho_0^2 dm \\ &= \int_A \frac{1}{|z|^2 (2\pi)^2} dm \\ &= \int_0^{2\pi} \int_a^b \frac{1}{(2\pi)^2 r^2} r dr d\theta \\ &= \frac{1}{(2\pi)^2} \left(\int_0^{2\pi} d\theta \right) \left(\int_a^b \frac{1}{r} dr \right) \\ &= \frac{1}{2\pi} \log \frac{b}{a}. \end{aligned}$$

Therefore, $M(\Delta) = \frac{1}{2\pi} \log \frac{b}{a}$. □

Example 2.1.7. *Let Δ be the collection of all closed paths that separate the boundary components of the spherical annulus $A = B(b) \setminus \overline{B(a)}$ where $0 < a < b$. Then*

$$M(\Delta) = \frac{1}{2\pi} \log \frac{b}{a}.$$

Proof. We will show first that $M(\Delta) \geq \frac{1}{2\pi}\log\frac{b}{a}$. Let Δ_0 be the collection of all circles centered at the origin that separate the boundary components of A. Clearly, $\Delta_0 \subset \Delta$. By Theorem 2.1.1 and Example 2.1.6, we have

$$M(\Delta) \geq M(\Delta_0) = \frac{1}{2\pi}\log\frac{b}{a}.$$

Next we will show that $M(\Delta_0) \leq \frac{1}{2\pi}\log\frac{b}{a}$. Let

$$\rho_0(z) = \begin{cases} \frac{1}{2\pi|z|} & \text{if } z \in A, \\ 0 & \text{if } z \notin A. \end{cases}$$

Given a locally rectifiable path γ in Δ, we obtain

$$\int_\gamma \rho_0|dz| = \int_\gamma \frac{1}{2\pi|z|}|z|d\theta \geq \int_0^{2\pi}\frac{1}{2\pi}d\theta = 1.$$

So ρ_0 is admissible for Δ. Hence

$$\begin{aligned} M(\Delta) &\leq \int_{\mathbb{R}^2}\rho_0^2 dm \\ &= \int_A \rho_0^2 dm \\ &= \frac{1}{2\pi}\log\frac{b}{a}. \end{aligned}$$

Therefore, $M(\Delta) = \frac{1}{2\pi}\log\frac{b}{a}$. □

We can modify the argument in Example 2.1.7 to derive the modulus for some subfamilies of the path family Δ. More precisely, let A_α be a sector of central angle α and let Δ_α be the family of the subpaths of all paths in Δ that lie in A_α. Then

$$M(\Delta_\alpha) = \frac{1}{\alpha}\log\frac{b}{a}.$$

2.2 Rings

Given sets $E, F, G \subset \overline{\mathbb{R}^2}$. We denote $\Delta(E, F : G)$ the family of all closed paths with initial points in E, terminal points in F and otherwise lying in G.

Next, we will introduce the modulus of path families in a domain, namely a ring. A domain A in $\overline{\mathbb{R}^2}$ is called a **ring** if its complement consists of exactly 2 components, say C_0 and C_1. We denote A by $R(C_0, C_1)$. Then the boundary ∂A of A has also two components, namely $B_0 = C_0 \cap \overline{A}$ and $B_1 = C_1 \cap \overline{A}$, where $\overline{A}$ is the closure of A. For each ring $A = R(C_0, C_1)$, let $\Gamma_A = \Delta(B_0, B_1 : A)$ be the family of all closed paths that join B_0 and B_1 in A.

Observe that Γ_A has the same modulus as $\Gamma_1 = \Delta(B_0, B_1 : \overline{\mathbb{R}^2})$, $\Gamma_2 = \Delta(C_0, C_1 : A)$ and $\Gamma_3 = \Delta(C_0, C_1 : \overline{\mathbb{R}^2})$. For instance, $M(\Gamma_A) = M(\Gamma_1)$ because $\Gamma_A \subset \Gamma_1$ and Γ_A minorizes Γ_1.

Theorem 2.2.1. [13, Theorem 11.4, p. 34] *If* $A = R(C_0, C_1)$ *and* $A' = R(C_0', C_1')$ *are rings such that* $C_i \subset C_i'$*, then* $M(\Gamma_A) \le M(\Gamma_{A'})$.

Theorem 2.2.2. [13, Theorem 11.5, p. 34] *If* A *is a ring, then* $M(\Gamma_A)$ *is finite.*

Given a ring A, to obtain a lower bound for $M(\Gamma_A)$, we introduce the κ-function as follows. Given $r > 0$, let $\Phi(r)$ be the set of all rings $A = R(C_0, C_1)$ in $\overline{\mathbb{R}^2}$ with the following properties: (1) C_0 contains the origin and a point a such that $|a| = 1$, (2) C_1 contains ∞ and a point b such that $|b| = r$. Define the κ**-function** $\kappa : (0, \infty) \to \mathbb{R}$ by

$$\kappa(r) = \inf_{A \in \Phi(r)} M(\Gamma_A).$$

Theorem 2.2.3. [13, Theorem 11.7, p. 34] *The function* $\kappa : (0, \infty) \to \mathbb{R}$ *has the following properties:*

(1) κ *is decreasing.*

(2) $\lim\limits_{r \to \infty} \kappa(r) = 0$.

(3) $\lim\limits_{r \to 0} \kappa(r) = \infty$.

(4) $\kappa(r) > 0$ *for every* $r > 0$.

The generalization of the modulus of the path family in Example 2.1.6 is the following theorem.

Theorem 2.2.4. [13, Theorem 11.10, p. 36] *Suppose that* $A = R(C_0, C_1)$ *is a ring. Then* $M(\Gamma_A) = 0$ *if and only if* C_0 *or* C_1 *consists of a single point.*

2.3 Modulus estimates in the spherical metric

Here we consider the plane $\mathbb{R}^2$ as a complex plane and denote the extended complex plane $\mathbb{C} \cup \{\infty\}$ by $\hat{\mathbb{C}}$, that is $\hat{\mathbb{C}} = \overline{\mathbb{R}^2}$. The **chordal metric** or the **spherical metric** q in $\hat{\mathbb{C}}$ is defined by

$$\begin{aligned} q(z_1, z_2) &= \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}, \text{ if } z_1, z_2 \in \mathbb{C} \\ q(z_1, \infty) &= \frac{1}{\sqrt{1 + |z_1|^2}}, \text{ if } z_1 \in \mathbb{C}, \text{ and} \\ q(\infty, \infty) &= 0. \end{aligned}$$

Clearly, $q(z_1, z_2) \le 1$ for all $z_1, z_2 \in \hat{\mathbb{C}}$. Hence $\hat{\mathbb{C}}$ is indeed a compact space under the metric q.
Let E be a non-empty subset of $\overline{\mathbb{R}^2}$. Define the **diameter** of E by

$$\text{dia}(E) = q(E) = \sup\{q(a, b) | a, b \in E\}.$$

Clearly, $q(E) \le 1$ for any subset E of $\overline{\mathbb{R}^2}$.
If E and F are non-empty subsets of $\overline{\mathbb{R}^2}$, we define the distance from E to F by

$$q(E,F) = \inf \{q(a,b) | a \in E, b \in F\}.$$

Next we present some basic results of modulus estimates in the spherical metric. Given $0 < r \leq 1$, let $\Psi(r)$ be the collection of all rings $A = R(C_0, C_1)$ in $\overline{\mathbb{R}^2}$ such that $q(C_0) \geq r$ and $q(C_1) \geq r$. We denote

$$\lambda(r) = \inf_{A \in \Psi(r)} M(\Gamma_A).$$

For $0 < t \leq 1$, let $\Psi(r,t) = \{R(C_0, C_1) \in \Psi(r) : q(C_0, C_1) \leq t\}$. We denote

$$\lambda(r,t) = \inf_{A \in \Psi(r,t)} M(\Gamma_A).$$

Observe that the number $\lambda(r,1)$ is equal to the number $\lambda(r)$.

Theorem 2.3.1. [13, Theorem 12.5, p. 38] *The function $\lambda : (0,1] \to \mathbb{R}$ has the following properties:*

(1) *λ is increasing.*

(2) $\lim_{r \to 0} \lambda(r) = 0$.

(3) *$\lambda(r) > 0$ for every $0 < r \leq 1$.*

Theorem 2.3.2. [13, Theorem 12.7, p. 39] *The function $\lambda : (0,1] \times (0,1] \to \mathbb{R}$ has the following properties:*

(1) *$\lambda(r,t)$ is increasing in r.*

(2) *$\lambda(r,t)$ is decreasing in t.*

(3) *$\lambda(r,t) \geq \lambda(r) > 0$ for every r and t.*

(4) *$\lim_{t \to 0} \lambda(r,t) = \infty$ for every r.*

Chapter 3

Quasiconformal Mappings

In this chapter we present the definition of quasiconformal mappings and important results on equicontinuity and convergence for quasiconformal mappings.

Let D and D' be domains in $\overline{\mathbb{R}^2}$ and let $1 \le K < \infty$. A homeomorphism f from D onto D' is said to be K**-quasiconformal** if

$$\frac{1}{K}M(\Gamma) \le M(f(\Gamma)) \le KM(\Gamma)$$

for all path families Γ in D. The smallest K for which this double inequality holds is called the **dilatation** of f. The mapping f is said to be **quasiconformal** if it is K-quasiconformal for some K.

It is not difficult to show that a conformal mapping is 1-quasiconformal. The converse is also true: a 1-quasiconformal mapping is always conformal. However, to prove this is a somewhat more challenging assignment. One may think of the number K in the definition of quasiconformality to measure how much the mapping differs from being conformal.

Standard texts on quasiconformal mappings are Ahlfors [1] and Lehto and Virtanen [6] in two dimensions and Väisälä [13] in higher dimensions.

3.1 Equicontinuity of Quasiconformal Mappings

We consider sets in $\overline{\mathbb{R}^2}$. As a metric in $\overline{\mathbb{R}^2}$ we use the chordal metric q. We next recall the definition of equicontinuity.

A family $\mathscr{F}$ of mappings of a set E into $\overline{\mathbb{R}^2}$ is said to be **equicontinuous** at a point $x \in E$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$q(f(x), f(y)) < \varepsilon$$

whenever $f \in \mathscr{F}$ and $y \in E$ with $q(x, y) < \delta$. If $\mathscr{F}$ is equicontinuous at each point in E, we say that $\mathscr{F}$ is equicontinuous in E. Clearly, all mappings in an equicontinuous family are continuous and any finite family of continuous mappings is equicontinuous. To define the uniform equicontinuity the same way as continuity is altered to uniform continuity, we can define as follows:

The family $\mathscr{F}$ is **uniformly equicontinuous** if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$q(f(x), f(y)) < \varepsilon$$

whenever $f \in \mathscr{F}$ and $x, y \in E$ with $q(x, y) < \delta$.

In the topological space version for equicontinuity, we say that the family $\mathscr{F}$ is equicontinuous at a point $x \in E$ if for each $\varepsilon > 0$ there is a neighborhood U of x such that

$$q(f(x), f(y)) < \varepsilon$$

whenever $y \in U$ and $f \in \mathscr{F}$. The following theorem is the main result on equicontinuity:

Theorem 3.1.1. [13, Theorem 19.2, p. 65] *Let $\mathscr{F}$ be a family of K-quasiconformal mappings of a domain D into $\overline{\mathbb{R}}^2$. If each $f \in \mathscr{F}$ omits 2 values a_f and b_f with chordal distance*

$$q(a_f, b_f) \geq r,$$

where $r > 0$ is fixed, then $\mathscr{F}$ is equicontinuous.

Proof. Let $x_0 \in D$ and $0 < \varepsilon < r$. We can choose neighborhoods U and V of x_0, for example disks, so that $\overline{U} \subset V \subset D$, that $A = V \setminus \overline{U}$ is a ring domain, and so that $KM(\Gamma_A) < \lambda(\varepsilon)$, where λ is the function introduced in Theorem 2.3.1. Then $f(A) = R(C_0, C_1)$, where $C_0 = f(\overline{U})$ and $C_1 = (f(V))^c$, is a ring. Since C_1 contains a_f and b_f, we get

$$q(C_1) \geq q(a_f, b_f) \geq r.$$

For each $x \in U$,

$$q\left(f(x), f(x_0)\right) \leq q(C_0).$$

From the definition of $\lambda(t)$ and K-quasiconformality of f, we obtain

$$KM(\Gamma_A) \geq M(\Gamma_{f(A)}) \geq \lambda(t).$$

where $t = min\left(r, q\left(f(x), f(x_0)\right)\right)$. Thus $\lambda(t) < \lambda(\varepsilon)$. Since λ is increasing, we have $t < \varepsilon < r$. Hence

$$q\left(f(x), f(x_0)\right) < \varepsilon.$$

for all $x \in U$ and $f \in \mathscr{F}$. Therefore $\mathscr{F}$ is equicontinuous at x_0. □

This theorem leads us to get two corollaries that give the conditions for a family of K-quasiconformal mappings to be equicontinuous.

Corollary 3.1.2. [13, Theorem 19.4, p. 66] *Let $\mathscr{F}$ be a family of K-quasiconformal mappings of a domain D. Then $\mathscr{F}$ is equicontinuous if one of the following conditions is satisfied:*

(1) *There are points* $x_1, x_2 \in D$ *and a number* $r > 0$ *such that each* $f \in \mathscr{F}$ *omits a point* a_f *and* $q\,(a_f, f(x_i)) \geq r$ *for* $i = 1, 2$.

(2) *There are points* $x_1, x_2, x_3 \in D$ *and a number* $r > 0$ *such that each* $f \in \mathscr{F}$ *satisfies the three inequalities* $q\,(f(x_i), f(x_j)) \geq r$, *for* $i \neq j$.

Corollary 3.1.3. [13, Corollary 19.5, p. 67] *If* $\mathscr{F}$ *is a family of* K*-quasiconformal mappings of a domain* D *such that each* $f \in \mathscr{F}$ *assume at three given points three fixed values, then* $\mathscr{F}$ *is equicontinuous.*

3.2 Normal Families

Before defining the normal families, we recall the convergence of a sequence of mappings from a topological space T into a metric space M with metric d.

Suppose that (f_n) is a sequence of mappings from T into M. The sequence (f_n) **converges** in T **pointwise** to a mapping f if for each $z \in T$

$$f_n(z) \to f(z) \text{ as } n \to \infty,$$

that is,

$$\lim_{n\to\infty} d(f_n(z), f(z)) = 0.$$

We say that the sequence (f_n) **converges** in T **uniformly** to a mapping f if

$$\sup_{z\in T} d(f_n(z), f(z)) \to 0 \text{ as } n \to \infty.$$

If (f_n) converges uniformly to a mapping f on every compact subset of T, then (f_n) is said to **converges** in T **c-uniformly** to the mapping f.

A family $\mathscr{F}$ of continuous mappings f from T into M is called a **normal family** if every sequence in $\mathscr{F}$ has a subsequence that converges c-uniformly in T. Obviously, if $\mathscr{F}$ is a normal family, then every sequence in $\mathscr{F}$ contains a subsequence that converges pointwise in all of T.

Theorem 3.2.1. [13, Theorem 20.3, p. 68] *Let* (f_n) *be a sequence of continuous mappings of a topological space* T *into a complete metric space* (M, d) *which converges pointwise in a dense set* E *in* T. *If* (f_n) *is equicontinuous, then* (f_n) *converges c-uniformly in* T.

Proof. Let F be a compact set in T, and let $\varepsilon > 0$. From the equicontinuity it follows that every $x \in F$ has a neighborhood $U(x)$ such that

$$d(f_k(x), f_k(y)) < \frac{\varepsilon}{5}$$

whenever $y \in U(x)$ and $k \in \mathbb{N}$. We choose a finite covering $\{U(x_1), ..., U(x_j)\}$ of F. Since E is dense, we can find points $a_i \in U(x_i) \cap E$, $1 \leq i \leq j$.
Since (f_n) is converges pointwise in E, there are integers n_i such that

$$d(f_m(a_i), f_n(a_i)) < \frac{\varepsilon}{5}$$

whenever $m \geq n_i$, $n \geq n_i$ for each $i \in \{1, ..., j\}$. Set $n_0 = \max\{n_1, ..., n_j\}$.
If $x \in F$, $m \geq n_0$ and $n \geq n_0$, then x belongs to some $U(x_i)$, we obtain

$$\begin{aligned} d(f_m(x), f_n(x)) &\leq d(f_m(x), f_m(x_i)) + d(f_m(x_i), f_m(a_i)) + d(f_m(a_i), f_n(a_i)) \\ &+ d(f_n(a_i), f_n(x_i)) + d(f_n(x_i), f_n(x)) \\ &< \varepsilon. \end{aligned}$$

Since M is complete, $(f_n(x))$ converges in M. Furthermore, (f_n) converges uniformly on F because the choice of n_0 does not depend on x. □

Theorem 3.2.2. [13, Theorem 20.4, p. 68] ***(Ascoli's theorem)*** *If T is a separable topological space and M is a compact metric space, then every equicontinuous family $\mathscr{F}$ of mappings $f : T \to M$ is a normal family.*

Proof. Let $J = (f_1, f_2, ...)$ be a sequence of $\mathscr{F}$. Since T is separable, it contains a countable dense subset $E = \{a_1, a_2, ...\}$. Consider the sequence $(f_n(a_1))$. Since M is a compact metric space, this sequence has a converging subsequence. Denote the corresponding sequence of mappings as follows

$$J_1 = (f_{11}, f_{12}, f_{13}, ...).$$

Consider the sequence $(f_{11}(a_2), f_{12}(a_2), f_{13}(a_2), ...)$. It has a converging subsequence. Denote the corresponding sequence of mappings as follows

$$J_2 = (f_{21}, f_{22}, f_{23}, ...).$$

Continuing this process inductively, we obtain a sequence $J_k = (f_{k1}, f_{k2}, ...)$ such that J_k is a subsequence of J_{k-1} and such that J_k converges at a_k. Draw a picture of mappings obtained

$$\begin{aligned} J_1 &= (f_{11}, f_{12}, f_{13}, ...) \\ J_2 &= (f_{21}, f_{22}, f_{23}, ...) \\ J_3 &= (f_{31}, f_{32}, f_{33}, ...) \\ &\vdots \end{aligned}$$

Then the diagonal sequence $J' = (f_{11}, f_{22}, ..., f_{kk}, ...)$ converges at every point of E, By Theorem 3.2.1, J' converges c-uniformly in T. Hence $\mathscr{F}$ is normal family. □

Corollary 3.2.3. *Let $\mathscr{F}$ be a family of quasiconformal mappings of a domain D into $\overline{\mathbb{R}^2}$. If $\mathscr{F}$ is equicontinuous, then $\mathscr{F}$ is normal.*

3.3 Convergence of quasiconformal mappings

In this section we present the possibilities of the limit mapping of a sequence of quasiconformal mappings. We first explore few examples.

Let $f_n(z) = z + n$. Then f_n is a 1-quasiconformal mapping and then f_n is defined in $\overline{\mathbb{R}^2}$ and the limit mapping is $f \equiv \infty$. In $\mathbb{R}^2$, the convergence is c-uniform in $\mathbb{R}^2$. In $\overline{\mathbb{R}^2}$ by taking a closed chordal ball centered at ∞. This ball is compact, but the convergence to ∞ is not uniform on this ball.

Let (g_n) be a sequence of mappings defined by

$$g_n(z) = \begin{cases} \infty & \text{if } z = \infty, \\ nz & \text{otherwise.} \end{cases}$$

Then $g_n : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ is a 1-quasiconformal mapping and $g_n \to g$, where

$$g(z) = \begin{cases} 0 & \text{if } z = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Hence g assume 2 values. Furthermore, the mappings $g_n|_{\overline{\mathbb{R}^2}\setminus\{0\}}$ converge c-uniformly to the constant mapping $g \equiv \infty$.

Let h be a fixed K-quasiconformal mapping, and let $h_n \equiv h$ for each $n \in \mathbb{N}$. Trivially (h_n) converges c-uniformly to the K-quasiconformal mapping h.

In summary, there are at least three different kinds of limit mappings. The following theorem shows that no other possibilities exist:

Theorem 3.3.1. *Suppose that $f_n : D \to D_n$ is K-quasiconformal and $f_n \to f$ pointwise in D. Then one of the following three possibilities must occur:*

(1) *f is a constant. The convergence may be c-uniform or not.*

(2) *f assumes exactly 2 values, one of which is assumed only at exactly one point, namely a_1. The convergence is c-uniform in $D \setminus \{a_1\}$, but not uniform on all compact subsets of D.*

(3) *f is a homeomorphism which is K-quasiconformal. The convergence is c-uniform in D.*

Proof. By the above examples, the possibilities $(1) - (3)$ can occur. We will show that no other possibilities exist. Suppose first that f assumes exactly 2 values, say $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Since $f_n \to f$ pointwise in D, we get

$$f_n(a_1) \to f(a_1) = b_1 \text{ and } f_n(a_2) \to f(a_2) = b_2.$$

We see that there exists an $r > 0$ such that

$$q(f_n(a_1), f_n(a_2)) \geq r$$

for all n. In $D\setminus\{a_1, a_2\}$, each f_n omits 2 values whose distance is at least r. Hence, by Theorem 3.1.1, the family $\{f_n : n \in \mathbb{N}\}$ is equicontinuous in $D \setminus \{a_1, a_2\}$. By Theorem 3.2.1, $f_n \to f$ c-uniformly in $D \setminus \{a_1, a_2\}$. We conclude that f must be continuous in $D \setminus \{a_1, a_2\}$. Since $f(D) = \{b_1, b_2\}$, the set $D \setminus \{a_1, a_2\}$ must be mapped to one of these points, say b_2, because $D \setminus \{a_1, a_2\}$ is a connected set and f is continuous in $D \setminus \{a_1, a_2\}$. Hence, $f(D \setminus \{a_1\}) = \{b_2\}$, and so b_1 is assumed only at one point a_1.

Next we will prove that $f_n \to f$ c-uniformly in $D\setminus\{a_1\}$. Fix a compact set F in $D\setminus\{a_1\}$. Choose $0 < s < \text{dist}(a_1, F)$, where the distance is the chordal distance between the point a_1 and the set F, so that the sphere $S = S(a_1, s) \subset D\setminus\{a_1, a_2\}$. By what was proved above, the convergence on S is uniform, that is, $f_n(S) \to b_2$.

By topology, the set $f_n(S)$ as a Jordan curve divides $\overline{\mathbb{R}^2}$ into 2 components. Since a_1 and F lie in different components of $\overline{\mathbb{R}^2}\setminus S$, it follows that b_1 and $f_n(F)$ lie in different components of $\overline{\mathbb{R}^2} \setminus f_n(S)$. Therefore $f_n(F) \to b_2$ because $f_n(S) \to b_2$. This means that $f_n \to f$ uniformly on F.

It remains to show that the convergence is not c-uniform in D. For instance, on the compact set $\overline{B}(a_1, s)$ the convergence is not uniform, because if it was, the limit mapping $f|_{\overline{B}(a_1,s)}$ would be continuous. But $f|_{\overline{B}(a_1,s)}$ assumes exactly 2 values, b_1 and b_2, and f would map a connected set $\overline{B}(a_1, s)$ onto a disconnected set $\{b_1, b_2\}$.

Finally, assume neither (1) nor (2) occurs. We must show that the situation (3) happens. So now f assumes at least 3 values, say $y_1 = f(x_1)$, $y_2 = f(x_2)$ and $y_3 = f(x_3)$. We show first that f is continuous. Since $f_n(x_j) \to y_j$, $j = 1, 2, 3$, there is an $r > 0$ such that

$$q(f_n(x_i), f(x_j)) \geq r, \; i \neq j$$

for all n. By Corollary 3.1.2, the mappings f_n are equicontinuous in D. By Theorem 3.2.1, $f_n \to f$ c-uniformly in D. This forces f to be continuous.

By topology, it suffices to show that f is one-to-one. For this, we first show that each x in D has a neighborhood U such that f is either one-to-one or constant in U. Fix $x \in D$. Choose any neighborhood U of x with $q(f_n(U)) < \frac{1}{2}$ for all n and $\overline{U} \subset D$. This is possible because the mappings f_n are equicontinuous.

Assume, contrary to the assertion, that there exist 3 points u, v, w in U such that $f(u) \neq f(v) = f(w)$. Join u and v by an arc $J_0 \subset U$. Next join w to a point on ∂U by an arc J_1 so that $J_0 \cap J_1 = \emptyset$. Then the domain $U \setminus (J_0 \cup J_1)$ is a ring, say A. Denote its image under f_n by $A_n = R(C_0^n, C_1^n)$, where $C_0^n = f_n(J_0)$ and $C_1^n = f_n(U\setminus J)^c$. For each $n \in \mathbb{N}$, let $r_n = q(f_n(u), f_n(v))$ and $t_n = q(f_n(v), f_n(w))$. Notice that $r_n > 0$ and $t_n > 0$ for all n, and $q(C_0^n) \geq r_n$, $q(C_1^n) \geq q(f_n(U))^c = 1$, and $q(C_0^n, C_1^n) \leq t_n$. Hence $M(\Gamma_{A_n}) \leq \lambda(r_n, t_n)$, where $\lambda(r, t)$ is the function in Theorem 2.3.2.

Since $r_n \to q(f(u), f(v)) > 0$ and $t_n \to q(f(v), f(w)) = 0$, $M(\Gamma_{A_n}) \to \infty$ by a property of the function $\lambda(r, t)$. But $M(\Gamma_A) > 0$ is a fixed number. This contradicts the fact that the mappings f_n are K-quasiconformal, which implies $M(\Gamma_{A_n}) \leq KM(\Gamma_A)$ for all $n \in \mathbb{N}$.

Finally, we show that f is one-to-one in D. Suppose that f is not one-to-one in D. Then there exist $x, y \in D$ with $x \neq y$ such that $f(x) = f(y)$. We show first that x has a neighborhood in which f is constant. Let U be a neighborhood of x, given earlier, where f is either one-to-one or constant. We may assume that $y \notin U$. We show that f is constant in U. For this, we look for a point $z \in U$ such that $f(x) = f(z)$. Choose a sphere $S \subset U$ separating x from y. The point z will be picked from S. Since f_n is a homeomorphism, there is $z_n \in S$ such that

$$q(f_n(x), f_n(z_n)) \leq q(f_n(x), f_n(y)).$$

Passing to a subsequence, if necessary, we may assume that $z_n \to z \in S$. Now

$$\begin{aligned} q(f(x), f(z)) &\leq q(f(x), f_n(x)) + q(f_n(x), f_n(z_n)) + q(f_n(z_n), f(z)) \\ &\leq q(f(x), f_n(x)) + q(f_n(x), f_n(y)) + q(f_n(z_n), f(z)). \end{aligned}$$

Obviously $q(f(x), f_n(x)) \to 0$ and $q(f_n(x), f_n(y)) \to q(f(x), f(y)) = 0$ as $n \to \infty$, because $f_n \to f$ pointwise in D. Since $\{f_n : n \in \mathbb{N}\}$ is equicontinuous at z, $q(f_n(z_n), f(z)) = 0$ as $n \to \infty$. Hence, $q(f(x), f(z)) = 0$, that is $f(x) = f(z)$.
To complete the proof, let

$$\begin{aligned} D_1 &= \{x \in D : x \text{ has a neighborhood in which } f \text{ is one-to-one}\} \text{ and} \\ D_2 &= \{x \in D : x \text{ has a neighborhood in which } f \text{ is constant}\}. \end{aligned}$$

Clearly, D_1, D_2 are disjoint open sets and their union is D. Since D is connected, either $D_1 = \emptyset$ or $D_2 = \emptyset$. We just proved that $x \in D_2$, hence $D_1 = \emptyset$. This forces f to be constant in D, a contradiction. Therefore f is one-to-one, and hence a homeomorphism. Furthermore, one can show that f is, in fact, K-quasiconformal, see [13, Corollary 37.3, p. 125]. □

Note that Theorem 3.3.1 is a refinement of Theorem 21.1 in [13].

Corollary 3.3.2. [13, Corollary 21.3, p. 71] *If $f_n : D \to D_n$ is a sequence of K-quasiconformal that converges c-uniformly to a mapping f in D, then f is either a homeomorphism onto a domain D' or a constant.*

Next we present the theorem that if f is a homeomorphism, then the inverse mappings f_n^{-1} converge to f^{-1}.

Theorem 3.3.3. [13, Theorem 21.10, p. 74] *Suppose that $f_n : D \to D_n$ is a sequence of K-quasiconformal mappings that converges to a homeomorphism $f : D \to D'$. Then for every compact set $F \subset D'$ there is a integer n_0 such that $F \subset D_n$ for $n \geq n_0$. Moreover, the mappings f_n^{-1} converge uniformly to f^{-1} in F.*

Chapter 4

Boundary Behavior of Quasiconformal Mappings

4.1 Boundary Behavior

We say that a sequence (E_n) of sets in $\hat{\mathbb{C}}$ **converges** to a point $c \in \hat{\mathbb{C}}$ if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $E_n \subset B(c, \varepsilon)$ for all $n \geq N$.

Lemma 4.1.1. *Let (E_n) be a sequence of sets in $\hat{\mathbb{C}}$ that converges to a point c and let A be a compact set not containing c. Then*

$$M(\Delta(A, E_n : \hat{\mathbb{C}})) \to 0$$

as $n \to \infty$.

Proof. Assume first that $c \neq \infty$ and that $A \neq \{\infty\}$. Let $R = \mathrm{dist}(\{c\}, A)$ where distance is the Euclidean distance between $\{c\}$ and A. Fix $\varepsilon > 0$ with $0 < \varepsilon < R$. Since (E_n) converges to c, there exists $N \in \mathbb{N}$ such that

$$E_n \subset B(c, \varepsilon)$$

for all $n \geq N$. For each $n \geq N$, let $r_n = \sup\{|c - x| : x \in E_n\}$. Let Γ_n be the family of all paths in $B(c, R) \setminus \overline{B}(c, r_n)$ joining the boundary components of $B(c, R) \setminus \overline{B}(c, r_n)$. Then, by Example 2.1.4

$$M(\Gamma_n) = \frac{2\pi}{\log \frac{R}{r_n}}.$$

We see that every path in $\Delta(A, E_n : \hat{\mathbb{C}})$ has a subpath in Γ_n, that is, Γ_n minorizes $\Delta(A, E_n : \hat{\mathbb{C}})$. Hence

$$M(\Delta(A, E_n : \hat{\mathbb{C}})) \leq M(\Gamma_n) = \frac{2\pi}{\log \frac{R}{r_n}}.$$

Since $r_n \to 0$ as $n \to \infty$, we have $M(\Gamma_n) \to 0$ as $n \to \infty$. Therefore $M(\Delta(A, E_n : \hat{\mathbb{C}})) \to 0$ as $n \to \infty$.

If $c = \infty$, then $E_n \to \infty$. Hence $M(\Delta(A, E_n : \hat{\mathbb{C}})) \to 0$.
If $A = \{\infty\}$, let B be a compact set such that $A \subset B$ and $c \notin B$. By the precious case, we already hence $M(\Delta(B, E_n : \hat{\mathbb{C}})) \to 0$ as $n \to \infty$. By the minorizing principle,

$$M(\Delta(A, E_n : \hat{\mathbb{C}})) \leq M(\Delta(B, E_n : \hat{\mathbb{C}}))$$

which gives $M(\Delta(A, E_n : \hat{\mathbb{C}})) \to 0$ as $n \to \infty$. □

Lemma 4.1.2. *Let f be a quasiconformal mapping of a domain D into the open unit disk B and let γ be an arc in the domain D terminating at point b on the boundary of D. Then f has a limit at b along γ.*

Proof. Assume that f does not have a limit along γ at b. Then there are sequences (x_n) and (y_n) in γ such that

$$x_n \to b,\ y_n \to b,\ f(x_n) \to b' \text{ and } f(y_n) \to b'' \text{ where } b' \neq b''$$

Represent γ as a continuous mapping $\gamma : [0, 1] \to D \cup \{b\}$. Denote $E_n = \gamma\left(\left[1 - \frac{1}{n}, 1\right)\right)$, $A = \overline{B}(0, \frac{1}{2})$ and $\Gamma_n = \Delta(A, f(E_n) : B)$. Then

$$f^{-1}(\Gamma_n) = \Delta(f^{-1}(A), E_n : D).$$

Now by Lemma 4.1.1, $M(f^{-1}(\Gamma_n)) \to 0$ as $n \to \infty$. But $M(\Gamma_n) \geq \frac{1}{2\pi} \log 2$, a contradiction with the quasiconformality of f. Therefore f has a limit at b along γ. □

Let D be a simply connected domain in $\hat{\mathbb{C}}$. A **cross-cut** C of D is an open Jordan arc in D such that $\overline{C} \setminus C$ consists of one or two points on ∂D.

Corollary 4.1.3. *Let f be a quasiconformal mapping of the disk B onto a domain D. If C is a cross-cut in D, then $f^{-1}(C)$ is a cross-cut in B.*

Proof. Since $f : B \to D$ is quasiconformal, f^{-1} is a quasiconformal mapping of the domain D onto the unit disk B. Let C be a cross-cut in D and let a be an endpoint of C. Then by virtue of Lemma 4.1.2, f^{-1} has a limit at a along C. In addition, the limit of f^{-1} at a along C must be in ∂B because f is homeomorphic. Thus $f^{-1}(C)$ is a cross-cut in B. □

Theorem 4.1.4. *Suppose that $\mathscr{F}$ is a compact and equicontinuous family of K-quasiconformal mappings of the unit disk B into $\hat{\mathbb{C}}$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$dia\left[f^{-1}(C)\right] < \varepsilon$$

whenever $f \in \mathscr{F}$ and C is a cross-cut in $f(B)$ with $dia(C) < \delta$.

Proof. Clearly, by Corollary 4.1.3, $f^{-1}(C)$ is a cross-cut in B for each cross-cut C in $f(B)$ where $f \in \mathscr{F}$. Suppose the assertion is false. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there are $f \in \mathscr{F}$ and a cross-cut C in $f(B)$ such that

$$\text{dia}(C) < \delta \ \text{ and } \ \text{dia}[f^{-1}(C)] \geq \varepsilon.$$

We see that for each $n \in \mathbb{N}$ there are $f_n \in \mathscr{F}$ and a cross-cut C_n in $f_n(B)$ such that

$$\operatorname{dia}(C_n) < \frac{1}{n} \text{ and } \operatorname{dia}[f_n^{-1}(C_n)] \geq \varepsilon.$$

Obviously, $\operatorname{dia}(C_n) \to 0$ as $n \to \infty$. Since $\mathscr{F}$ is compact and equicontinuous, by Ascoli's theorem, $\mathscr{F}$ is a normal family. So every sequence in $\mathscr{F}$ has a subsequence which converges pointwise in all subsets of B. Passing to a subsequence, we may assume that the sequence (f_n) converges pointwise in all subsets of B. Since $\mathscr{F}$ is a compact family of K-quasiconformal mappings, the limit mapping f of (f_n) lies in $\mathscr{F}$. Hence f is a K-quasiconformal mapping, and hence $f_n \to f$ uniformly on every compact subset of B.

Next, we show that $f_n^{-1}(C_n)$ tends to ∂B. Suppose that there exists $\delta > 0$ such that

$$E \cap f_n^{-1}(C_n) \neq \emptyset$$

for infinitely many n, where $E = \overline{B}(0, 1-\delta)$. Passing to a subsequence, we may assume that $E \cap f_n^{-1}(C_n) \neq$ for all n. Let $x_n \in E \cap f_n^{-1}(C_n)$. Then $(f(x_n))$ is a sequence in $f(E)$. Since $f(E)$ is compact, the sequence $(f(x_n))$ has a convergent subsequence. We denote this subsequence again by $(f(x_n))$. Let $f(x_n) \to y$. Fix $\varepsilon_0 > 0$. Since $f_n \to f$ uniformly on the compact E, there exists $N_0 \in \mathbb{N}$ such that

$$q(f_n(x), f(x)) < \frac{\varepsilon_0}{2},$$

for all $n \geq N_0$ and all $x \in E$. Since $f(x_n) \to y$, there exists $N_1 \in \mathbb{N}$ such that

$$q(f(x_n), y) < \frac{\varepsilon_0}{2},$$

for all $n \geq N_1$. Let $N_2 = \max\{N_0, N_1\}$. Then for all $n \geq N_2$,

$$\begin{aligned} q(f_n(x_n), y) &\leq q(f_n(x_n), f(x_n)) + q(f(x_n), y) \\ &< \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} \\ &= \varepsilon_0. \end{aligned}$$

Therefore $f_n(x_n) \in y$. Since $f_n(x_n) \in C_n$, $\operatorname{dia}(C_n) \to 0$ and $f_n(x_n) \to y$, for any neighborhood U of y, there is a positive integer N such that $C_n \subset U$, for all $n \geq N$. That is

$$U \cap \partial f_n(B) \neq \emptyset, \tag{4.1.1}$$

for all $n \geq N$. On the other hand, $y \in f(E) \subset f(B)$, which means that there is a neighborhood U' of y such that $U' \subset f_n(B)$ for sufficiently large n. This contradicts (4.1.1). Therefore $f_n^{-1}(C_n)$ lies close to ∂B.

Finally, let $A = \overline{B}(0, \frac{1}{2})$ and consider the path family

$$\Gamma_n = \Delta\left(A, f_n^{-1}(C_n) : B\right).$$

Since $f_n^{-1}(C_n)$ tends to ∂B and $\operatorname{dia}(f_n^{-1}(C_n)) \geq \varepsilon$, for all sufficiently large n, by the minorizing principle,

$$M(\Gamma_n) \geq M(\Gamma_{\frac{\varepsilon}{2}}) = \frac{\varepsilon}{2 \log 2},$$

where $\Gamma_{\frac{\varepsilon}{2}}$ is the path family defined in Example 2.1.4. Since f is a homeomorphism, $f(A)$ is a compact subset of the domain $f(B)$. Choose a neighborhood V of $f(A)$ such that $\overline{V} \subset f(B)$. Then

$$f_n(A) \subset V,$$

for all sufficiently large n. Since $\overline{V}$ is compact, by virtue of Theorem 3.3.3, $\overline{V} \subset f_n(B)$ for all large n. We know already that $\mathrm{dia}(C_n) \to 0$. For large n, C_n must lie outside $\overline{V}$. Passing to a subsequence if necessary, we may assume that (C_n) converges to a point P, where $P \notin f(B)$. Note that if $P \in f(B)$ then $P \in f_n(B)$ for all large n which is impossible. Since $f_n(A) \subset \overline{V}$ for all large n,

$$\Delta(\overline{V}, C_n : \hat{\mathbb{C}}) < \Delta(f_n(A), C_n : f_n(B))$$

for all large n. It follows by Lemma 4.1.1 and minorizing principle that

$$\lim_{n\to\infty} M(f_n(\Gamma_n)) = \lim_{n\to\infty} M(\Delta(f_n(A), C_n : f_n(B)) \leq \lim_{n\to\infty} M(\overline{V}, C_n : \hat{\mathbb{C}}) = 0.$$

This contradicts the fact that the quasiconformality of the mapping f_n and $M(\Gamma_n) \geq \frac{\varepsilon}{2\log 2}$. □

4.2 Local connectedness

Let E be a set in $\mathbb{C}$ and let z be a point in E. The set E is said to be **locally connected** at z if every neighborhood U of z in E contains a connected neighborhood of z in E. This condition is often expressed by saying that z has arbitrarily small connected neighborhoods in E. The set E is locally connected if it is locally connected at each point.

If we want an "epsilon-delta" type of connectedness property, we are likely to end up with the following local property which is closely related to local connectedness and called connected im kleinen (an odd mixture of English and German): a set E in $\mathbb{C}$ is **connected im kleinen** at a point $z \in E$ provided that for each $\varepsilon > 0$ there is $\delta > 0$ such that each point $w \in E$ with $|z - w| < \delta$ can be joined to z by a connected set in E of diameter less than ε.

Lemma 4.2.1. [5, Theorem 3.2, p. 106] *A set E is locally connected if and only if the components of all open sets of E are open.*

Proof. Suppose first that the components of all open sets of E are open. If z is any point in E and U is any of its neighborhoods, the z-component of U can be chosen for the connected neighborhood of z.

For the converse, let C be a component of an open set U in E. Each point z in C has a connected neighborhood V_z in E lying in U. Each such V_z must lie in C. Thus $C = \bigcup_{z\in C} V_z$. As a union of open sets in E, the component C is open in E. □

Lemma 4.2.2. [5, Theorem 3.11, p. 114] *A set E is locally connected if and only if it is connected im kleinen at each point.*

Proof. Suppose first that E is locally connected. Fix $z \in E$ and $\varepsilon > 0$. Then $E \cap B(z, \frac{\varepsilon}{3})$ contains a connected neighborhood V of z in E. Choose $\delta > 0$ so small that $E \cap B(z, \delta)$ is contained in V. Then every point w in $E \cap B(z, \delta)$ can be joined to z by the connected set $V \subset E$. Since $\text{dia}(V) < \varepsilon$, the set E is connected im kleinen at z.

To prove the converse, it suffices to show, in view of Lemma 4.2.1, that the components of all open sets in E are open. Let U be such a set and let C be a component of U. Given a point $z \in C$, there is an open set V_z in E containing z and lying in U such that each point w in V_z can be joined to z by a connected set V_{zw} in U. Since C is the z-component of E, each V_{zw} must lie in C. Thus V_z lies in C and $C = \bigcup_{z \in C} V_z$. As a union of open sets in E, the component C is open in E. □

Lemma 4.2.3. [8, Theorem 8.2, p. 89] *Under a continuous mapping the image of a compact and locally connected set is compact and locally connected.*

Proof. Let E be compact and locally connected and let $f : E \to \mathbb{C}$ be continuous. Compactness of $f(E)$ is well-known. We will show that $f(E)$ is locally connected. For this, it suffices, in view of Lemma 4.2.1, to verify that the components of all open sets in $f(E)$ are open.

Write $E' = f(E)$. Let U be an open subset of E' and let C be a component of U. Since f is continuous, $f^{-1}(U)$ is open in E and so are the components of $f^{-1}(U)$ by virtue of Lemma 4.2.1. The set $f^{-1}(C)$ is a union of some components of $f^{-1}(U)$, because if A is such a component, then A is connected and, by the continuity of f, also $f(A)$ is connected, which implies that either $f(A)$ is disjoint from C or lies entirely in C. Hence $f^{-1}(C)$ consists of entire components of $f^{-1}(U)$. Consequently, $f^{-1}(C)$ is open.

Now, $f(f^{-1}(C)) = C$. Since a closed set in a compact space is compact and since compactness is preserved under continuous mappings, wee see that f is also a closed mapping, i.e. f preserves closed sets. It follows that the image $E' \setminus C$ of the closed set $E \setminus f^{-1}(C)$ is closed. Thus C is open. □

We say that a set E in $\mathbb{C}$ is **uniformly connected im kleinen** provided that, given $\varepsilon > 0$, there is $\delta > 0$ such that any pair of points in E with distance less than δ can be joined by a connected set in E with diameter less than ε.

Lemma 4.2.4. [5, Theorem 3.13, p. 114] *A locally connected compact set is uniformly connected im kleinen.*

Proof. Let E be a locally connected compact set in $\mathbb{C}$. Fix $\varepsilon > 0$. Since E is locally connected, each $z \in E$ lies in a connected open subset V_z of E with $\text{dia}(V_z) < \varepsilon$. Since E is compact, the covering $\{V_z\}$ of E has a Lebesgue number $\delta > 0$. Now if $z, w \in E$ and $|z - w| < \delta$, then z and w lie in one of the sets V_z. This set is the desired connected set. □

A classical result in conformal mapping theory, often referred to as Wolff's lemma, will be proved next for quasiconformal mappings. See Becker [3]. For conformal mappings see, for example, the book [4] of Collingwood and Lohwater.

Lemma 4.2.5. ***(Wolff's Lemma)*** *Let f be a bounded K-quasiconformal mapping of the open unit disk B, let $z_0 \in \partial B$ and let $0 < h < 1$. Then there exists s with $h^2 < s < h$ such that the image of the circular arc $C_s = S(z_0, s) \cap B$ has finite length*

$$\ell(f(C_s)) < \left[\frac{K\pi \ \mathit{area}(f(A))}{\log \frac{1}{h}}\right]^{\frac{1}{2}},$$

where $A = B \cap B(z_0, h) \setminus \overline{B}(z_0, h^2)$. In particular, there are numbers $s_k \to 0$ such that $\ell(f(C_{s_k})) \to 0$.

Proof. Let Γ be the family of all open circular paths C_t in A with $h^2 < t < h$. Then

$$M(\Gamma) > \frac{1}{\pi} \log \frac{h}{h^2} = \frac{1}{\pi} \log \frac{1}{h}.$$

Here the minorizing principle and example 2.1.7 do not yield the strict inequality above, but an easy modification of the argument used in the proof of example 2.1.7 will do it. Since f is K-quasiconformal,

$$\frac{1}{\pi} \log \frac{1}{h} < M(\Gamma) \le KM(f(\Gamma)).$$

Then at least one path in the family $f(\Gamma)$ must have finite length, because otherwise $\mathrm{M}(f(\Gamma)) = 0$. Set

$$\alpha = \inf_{C_t \in \Gamma} \ell(f(C_t)).$$

If $\alpha = 0$, the lemma is proved. Suppose $\alpha > 0$. Define a Borel function $\rho : \mathbb{C} \to [0, \infty]$ by setting

$$\rho(z) = \begin{cases} \frac{1}{\alpha} & \text{if } z \in f(A), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_{f(C_t)} \rho |dz| = \int_{f(C_t)} \frac{1}{\alpha} |dz| = \frac{1}{\alpha} \ell(f(C_t)) \ge \frac{1}{\alpha} \cdot \alpha = 1$$

for each locally rectifiable path $f(C_t)$ in $f(\Gamma)$. Thus ρ is an admissible function for the path family $f(\Gamma)$,

$$\begin{aligned} M(f(\Gamma)) &\le \int_{\mathbb{R}^2} \rho^2 dm \\ &= \int_{f(A)} \frac{1}{\alpha^2} dm \\ &= \frac{1}{\alpha^2} \mathrm{area}(f(A)) \end{aligned}$$

and, therefore, since

$$\frac{1}{\pi} \log \frac{1}{h} < M(\Gamma) \le \frac{K}{\alpha^2} \mathrm{area}(f(A)),$$

we obtain

$$\alpha^2 < \frac{K\pi \text{ area}(f(A))}{\log \frac{1}{h}}.$$

Consequently,

$$\ell(f(C_s)) < \left[\frac{K\pi \text{ area}(f(A))}{\log \frac{1}{h}}\right]^{\frac{1}{2}}$$

for at least one C_s in Γ. Since the right hand side tends to zero as $h \to 0$, there exist numbers $s_k \to 0$ such that $\ell(f(C_{s_k})) \to 0$. □

Theorem 4.2.6. *Let f be a quasiconformal mapping of the open unit disk B onto a bounded domain D. Then the following conditions are equivalent:*

(1) *f can be extended to a continuous mapping $\overline{f} : \overline{B} \to \overline{D}$.*

(2) *∂D is locally connected (uniformly connected im kleinen).*

(3) *$\mathbb{C} \setminus D$ is locally connected (uniformly connected im kleinen).*

Proof. (1) $\Rightarrow$ (2) : Since the unit circle is compact and locally connected, the image $\partial D = \overline{f}(\partial B)$ under the continuous mapping $\overline{f}$ is also compact and locally connected by virtue of Lemma 4.2.3, and Lemma 4.2.4 ensures that ∂D is locally connected.

(2) $\Rightarrow$ (3) : Assume that ∂D is locally connected. Fix $\varepsilon > 0$. Since ∂D is compact, ∂D is uniformly connected im kleinen by Lemma 4.2.4. Choose δ, $0 < \delta < \frac{\varepsilon}{3}$, such that any pair of points in ∂D with distance less than δ can be joined in ∂D by a connected set with diameter less than $\frac{\varepsilon}{3}$. Now let a and b be points in $\mathbb{C} \setminus D$ with $|a-b| < \delta$. If $[a, b] \cap \partial D = \emptyset$, then the line segment $[a, b]$ is a connected set in $\mathbb{C} \setminus D$ of diameter less than ε. Suppose next that $[a, b] \cap \partial D \neq \emptyset$. Let a' be the first point and b' the last point in $[a, b] \cap \partial D$ when traversing from a toward b along $[a, b]$. The points a' and b' can be joined by a connected set A in ∂D with $\text{dia}(A) < \frac{\varepsilon}{3}$. The set $F = [a, a'] \cup A \cup [b', b]$ is connected, lies in $\mathbb{C} \setminus D$, contains a and b and satisfies

$$\begin{aligned} \text{dia}(F) &\leq \text{dia}([a, a']) + \text{dia}(A) + \text{dia}([b', b]) \\ &< \delta + \frac{\varepsilon}{3} + \delta \\ &< \varepsilon. \end{aligned}$$

Thus $\mathbb{C} \setminus D$ is uniformly connected im kleinen.

(3) $\Rightarrow$ (1) : Assume that $\mathbb{C} \setminus D$ is locally connected. To show that f has a continuous extension to $\overline{B}$, fix $z_0 \in \partial B$. It suffices to verify that f has a limit at z_0. For this, fix ε, $0 < \varepsilon < \text{dist}\,(f(0), \partial D)$. The existence of the limit at z_0 will follow if we can show that $\text{dia}\,[f(U \cap B)] < \varepsilon$ for some neighborhood U of z_0.

By Wolff's lemma, there is a nested sequence of circular arcs $C_k = S(z_0, s_k) \cap B$, with $0 < s_k < \frac{1}{2}$ and $s_k \to 0$ as $k \to \infty$, such that the length of $f(C_k)$ is finite and tends to 0 as $k \to \infty$. Thus

$$\text{dia}\,[f(C_k)] \to 0$$

as $k \to \infty$. Fix k large enough so that

$$\text{dia}\,[f(C_k)] < \frac{\varepsilon}{2}.$$

Let a_k and b_k denote the endpoints of the arc $f(C_k)$. Then $a_k, b_k \in \partial D \subset \mathbb{C} \setminus D$.

Suppose first that $a_k = b_k$. Then $f(C_k)$, together with the endpoints a_k and b_k, is a simple closed curve or a Jordan curve J, a set homeomorphic to the unit circle. The arc $f(C_k)$ divides D into two subdomains, D_0 and D_1. Let D_0 be the $f(0)$-component of $D \setminus f(C_k)$. Then f maps 0 into D_0 and $B \cap B(z_0, s_k)$ onto D_1. Since

$$\text{dia}(D_1) = \text{dia}(J) = \text{dia}(f(C_k)) < \varepsilon,$$

the disk $B(z_0, s_k)$ can be chosen for the neighborhood U sought for.

Suppose next that $a_k \neq b_k$. By passing to subsequences and relabeling, we may assume that the sequences (a_k) and (b_k) converge towards one and the same point, w_0. We can thereby choose $\delta > 0$ so that each pair of points in $B(w_0, \delta) \cap (\mathbb{C} \setminus D)$ can be joined by a connected set in $\mathbb{C} \setminus D$ of diameter less than $\frac{\varepsilon}{2}$. For large k, this can also be done for the points a_k and b_k. We may assume that our k above is such a k. Join a_k to b_k by a connected set A_k, with $\text{dia}(A_k) < \frac{\varepsilon}{2}$, in $\mathbb{C} \setminus D$. We divide the rest of the proof into two cases depending upon whether A_k is a Jordan arc with endpoints a_k and b_k or not. A Jordan arc means a set homeomorphic to the line segment $[0, 1]$.

Consider first the case where A_k is a Jordan arc with endpoints a_k and b_k. Then

$$J = f(C_k) \cup A_k$$

is a Jordan curve. By the Jordan Curve Theorem, J divides $\mathbb{C}$ into two domains and is their common boundary. One of these domains, say D_1, is bounded, the other is unbounded. The domain D_1 satisfies

$$\text{dia}(D_1) = \text{dia}(J) \leq \text{dia}(f(C_k)) + \text{dia}(A_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now 0 and $B \cap B(z_0, s_k)$ are separated in B by the crosscut C_k. Thus $f(0)$ and $f(B \cap B(z_0, s_k))$ are separated in D by the crosscut $f(C_k)$. Since $0 < \varepsilon < \text{dist}\,(f(0), \partial D)$ and $\text{dia}(D_1) < \varepsilon$, the domain D_1 cannot contain the $f(0)$-component of $D \setminus f(C_k)$. Hence D_1 must contain the other component, which is $f(B \cap B(z_0, s_k))$. It follows that

$$\text{dia}\,[f(B \cap B(z_0, s_k))] \leq \text{dia}(D_1) < \varepsilon,$$

and we can again choose the disk $B(z_0, s_k)$ for the desired neighborhood U of z_0.

Finally, consider the case where A_k is not a Jordan arc, just a connected set in $\mathbb{C} \setminus D$ joining a_k to b_k. We will use Janiszewski's separation theorem to complete the proof. However, present an argument that is perhaps more transparent.

Since $\mathbb{C} \setminus D$ is closed, the closure $\overline{A}_k$ of A_k taken with respect to $\mathbb{C}$ lies in $\mathbb{C} \setminus D$. It has the same diameter as A_k and it is connected, closed and bounded, hence compact. In other words, $\overline{A}_k$ is a continuum. Consider the set

$$F = \overline{A}_k \cup f(C_k).$$

It also is a continuum and has diameter less than ε. The complementary components of F are domains, in fact simply connected domains, except for the unbounded component. Since $0 < \varepsilon < \operatorname{dist}(f(0), \partial D)$, the $f(0)$-component of $D \setminus f(C_k)$ cannot lie in any of the bounded components of $\mathbb{C} \setminus F$. Thus $f(B \cap B(z_0, s_k))$ must be contained in one of the bounded components of $\mathbb{C} \setminus F$, call it G. Since ∂G lies in F, we obtain

$$\operatorname{dia}[f(B \cap B(z_0, s_k))] \leq \operatorname{dia}(G) = \operatorname{dia}(\partial G) \leq \operatorname{dia}(F) < \varepsilon.$$

Thus again we can choose the disk $B(z_0, s_k)$ for the desired neighborhood U of z_0. The proof is complete. □

Chapter 5

Uniform Convergence of Quasiconformal Mappings

In this chapter we prove the main result about the uniform convergence of a sequence of quasiconformal mappings.

5.1 Uniform Convergence of Quasiconformal Mappings

First we present the definition of uniformly local connectedness of compact sets as follows. A family $\mathscr{E}$ of compact sets in $\hat{\mathbb{C}}$ is called **uniformly locally connected** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathscr{E}$ if $z, w \in E$ and $|z - w| < \delta$, we can find a connected set in E of diameter less than ε joining z and w.

Lemma 5.1.1. *Let D be a domain in $\hat{\mathbb{C}}$ and let a, b be points in $\mathbb{C} \setminus D$. Then $[a, b] \cap D$ consists of a countable union of disjoint open line segments.*

Proof. Without loss of generality, by rotation and translation, we may assume that the line segment $[a, b]$ lies in the real line. Then $[a, b] \cap D$ is a union of disjoint open intervals. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each open interval there is a rational number that belongs to such open interval. Hence $[a, b] \cap D$ consists of a countable union of disjoint open interval because $\mathbb{Q}$ is countable. □

We are now prepared to prove the main theorem.

Theorem 5.1.2. *Let $\mathscr{F}$ be a compact family of K-quasiconformal mappings defined in the unit disk B into bounded domains. Then the following conditions are equivalent:*

(1) *$\mathscr{F}$ is uniformly equicontinuous in B*

(2) *Each $f \in \mathscr{F}$ can be extended to a continuous mapping $\overline{f}$ of $\overline{B}$ and the family $\{\overline{f} : f \in \mathscr{F}\}$ is uniformly equicontinuous in $\overline{B}$.*

(3) *The family* $\mathscr{E} = \{\hat{\mathbb{C}} \setminus f(B) : f \in \mathscr{F}\}$ *of the complements of the domains* $f(B)$ *is uniformly locally connected.*

Proof. (1) ⇒ (2) : Since each $f \in \mathscr{F}$ is uniformly continuous in B, such an f can be extended continuously to $\overline{B}$. The extended family $\{\overline{f} : f \in \mathscr{F}\}$ is easily seen to be equicontinuous at each point of ∂B, because $\mathscr{F}$ is uniformly equicontinuous in B. By compactness of $\overline{B}$, the extended family is uniformly equicontinuous in $\overline{B}$.

(2) ⇒ (3) : For each $f \in \mathscr{F}$, let $E(f)$ be the complement $\hat{\mathbb{C}} \setminus f(B)$ of $f(B)$. By virtue of Theorem 4.1.4, to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$\operatorname{dia}\left[f^{-1}(C)\right] < \varepsilon,$$

where $f \in \mathscr{F}$ and $\operatorname{dia}(C) < \delta$ for each cross-cut C of $f(B)$. We will show that $\mathscr{E} = \{E(f) : f \in \mathscr{F}\}$ is uniformly locally connected. Fix $\varepsilon > 0$. By hypothesis, there is $\eta > 0$ such that

$$\operatorname{dia}\left[f(L)\right] < \frac{\varepsilon}{3}, \tag{5.1.1}$$

whenever $f \in \mathscr{F}$ and L is a set in $\overline{B}$ with $\operatorname{dia}(L) < \eta$. Choose $0 < \delta < \frac{\varepsilon}{3}$ so that

$$\operatorname{dia}\left[f^{-1}(C)\right] < \eta, \tag{5.1.2}$$

whenever $f \in \mathscr{F}$ and C is a cross-cut of $f(B)$ with $\operatorname{dia}(C) < \delta$. Now, for an arbitrary $f \in \mathscr{F}$, fix points $a, b \in E(f)$ with $q(a, b) < \delta$. We may assume that $a, b \neq \infty$. If the line segment $[a, b]$ lies in $\hat{\mathbb{C}} \setminus f(B)$, we can choose $[a, b]$ as a connected set with

$$\operatorname{dia}([a, b]) = q(a, b) < \delta < \varepsilon.$$

So assume that $[a, b] \cap f(B) \neq \emptyset$. By Lemma 5.1.1, we have $[a, b] \cap f(B) = \bigcup_j C_j$, a countable union of disjoint open line segments C_j which are cross-cuts of $f(B)$. Then by virtue of (5.1.2), the preimages satisfy

$$\operatorname{dia}\left[f^{-1}(C_j)\right] < \eta,$$

and they are cross-cuts of B by virtue of Theorem 4.1.4. The endpoints of each cross-cut $f^{-1}(C_j)$ lie on the circle ∂B, they are distinct, and they determine a closed circular L_j on ∂B with

$$\operatorname{dia}(L_j) < \eta.$$

Hence $\operatorname{dia}\left[f(L_j)\right] < \frac{\varepsilon}{3}$ by virtue of (5.1.1). By replacing the line segment $\overline{C_j}$ by $f(L_j)$, we see that the set

$$F = ([a, b] \setminus f(B)) \cup \left(\bigcup_j f(L_j)\right)$$

is a connected set joining a and b in $E(f)$. We will show that $\operatorname{dia}(F) < \varepsilon$. Let $x, y \in F$. Clearly, if $x, y \in [a, b] \setminus f(B)$, then

$$q(x, y) \leq q(a, b) < \delta < \frac{\varepsilon}{3}.$$

So $\text{dia}(F) \le \frac{\varepsilon}{3} < \varepsilon$.

If $x \in [a,b] \setminus f(B)$ and $y \in \bigcup_j f(L_j)$, then there is an index j such that $y \in f(L_j)$. Let be y' the first point in $[a,b] \cap \partial f(L_j)$ when traversing from x toward $\partial f(L_j)$ along $[a,b]$. Then

$$\begin{aligned} q(x,y) &\le q(x,y') + q(y',y) \\ &\le q(a,b) + \text{dia}(f(L_j)) \\ &< \delta + \frac{\varepsilon}{3}. \end{aligned}$$

Hence $\text{dia}(F) \le \delta + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$.

If $x, y \in \bigcup_j f(L_j)$. Then there are indices i, j such that $x \in f(L_i)$ and $y \in f(L_j)$. Obviously, if $i = j$, then

$$q(x,y) \le \text{dia}\,[f(L_j)] < \frac{\varepsilon}{3}.$$

In the case $i \ne j$, let x' be the last point in $[a,b] \cap \partial f(L_i)$ and y' the first point in $[a,b] \cap \partial f(L_j)$ when traversing from a toward b along $[a,b]$. Then

$$\begin{aligned} q(x,y) &\le q(x,x') + q(x',y') + q(y',y) \\ &\le \text{dia}(f(L_i)) + q(a,b) + \text{dia}(f(L_j)) \\ &< \frac{\varepsilon}{3} + \delta + \frac{\varepsilon}{3} \\ &= \delta + \frac{2\varepsilon}{3}. \end{aligned}$$

Thus $\text{dia}(F) \le \delta + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$.

(3) $\Rightarrow$ (1) : By Theorem 4.2.6, the mappings in $\mathscr{F}$ can be extended continuously to $\overline{B}$. We retain the notation f for the extended mapping. We will show that the extended mappings are equicontinuous on the boundary of B. Fix $z_0 \in \partial B$ and $\varepsilon > 0$. It suffices to find $\eta > 0$ so that

$$q\left(f(z'), f(z'')\right) < \varepsilon,$$

whenever $z', z'' \in B \cap B(z_0, \eta)$ for all $f \in \mathscr{F}$. Since $f(0) \notin \partial f(B)$ and $\mathscr{F}$ is compact, there exists $d > 0$ such that

$$\text{dist}(\{f(0)\}, E(f)) \ge d$$

for all $f \in \mathscr{F}$. We may assume that $\varepsilon < 2d$. By hypothesis, there is δ with $0 < \delta < \frac{\varepsilon}{2}$ such that any pair of points $a, b \in E(f)$ with $q(a,b) < \delta$ can be joined in $E(f)$ by a connected set F such that $\text{dia}(F) < \frac{\varepsilon}{2}$. Fix h, $0 < h < 1$ so that

$$\left[\frac{K\pi\ \text{area}(f(R))}{\log \frac{1}{h}}\right]^{\frac{1}{2}} < \delta,$$

where $R = B \cap B(z_0, h) \setminus \overline{B}(z_0, h^2)$ for any $f \in \mathscr{F}$. Let $f \in \mathscr{F}$ be arbitrary. By Wolff's lemma, we can choose a circular cross-cut $Q_s = B \cap S(z_0, s)$ with $h^2 < s < h$ so that

$$\ell(f(Q_s)) < \left[\frac{K\pi \text{ area}(f(R))}{\log \frac{1}{h}}\right]^{\frac{1}{2}} < \delta < \frac{\varepsilon}{2}.$$

The endpoints a and b of $f(Q_s)$ lie in $E(f)$ and they satisfy $q(a, b) < \delta$. Hence they can be joined in $E(f)$ by a connected set F with $\text{dia}(F) < \frac{\varepsilon}{2}$. Therefore

$$F \cup f(Q_s) \subset B(a, \frac{\varepsilon}{2}).$$

Now, let z be an arbitrary point in B with $q(a, f(z)) \geq \frac{\varepsilon}{2}$. Then the points $f(z)$ and $f(0)$ are not separated by the set $E(f)$. Neither are they separated by the set $F \cup f(Q_s)$. Since $E(f) \cap (F \cup f(Q_s)) = f(Q_s)$ is connected, we conclude by Janiszewski's theorem that these points are not separated by the union

$$E(f) \cup (F \cup f(Q_s)) = E(f) \cup f(Q_s).$$

Hence they can be joined in $f(B)$ by a path which does not intersect the cross-cut $f(Q_s)$. Since $|0 - z_0| = 1 > h > s$, we have $|z - z_0| > s > h^2$. Consequently, if z is a point in B with $|z - z_0| \leq h^2$, we must have $q(a, f(z)) < \frac{\varepsilon}{2}$. Set $\eta = h^2$. Then by the triangle inequality

$$\begin{aligned} q\left(f(z'), f(z'')\right) &\leq q\left(a, f(z')\right) + q\left(a, f(z'')\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

whenever z' and z'' are points in $B \cap B(z_0, \eta)$. Since f is arbitrary, $\mathscr{F}$ is equicontinuous at z_0. It holds for any point z_0 in ∂B. Hence $\mathscr{F}$ is equicontinuous in ∂B. Finally, since $\mathscr{F}$ is a compact family of K-quasiconformal mappings in B, $\mathscr{F}$ is equicontinuous in B. Hence $\mathscr{F}$ is equicontinuous in $\overline{B}$. Since $\overline{B}$ is compact, $\mathscr{F}$ is uniformly equicontinuous in $\overline{B}$. This proves (1) as desired. □

We next give a condition for uniform convergence in terms of domains containing ∞ as follows:

Corollary 5.1.3. *Let (f_n) be a sequence of K-quasiconformal mappings of the unit disk B into bounded domains. Suppose that (f_n) converges to a homeomorphism f.*

(1) *If the collection $\mathscr{E} = \{E_n = \hat{\mathbb{C}} \setminus f_n(B)\}$ is uniformly locally connected, then the mappings f_n extend to continuous mappings $\overline{f}_n$ of $\overline{B}$ and the sequence $(\overline{f}_n)$ converges uniformly in $\overline{B}$.*

(2) *If the convergence $f_n \to f$ is uniform and if each f_n extends continuously to $\overline{B}$, then $\mathscr{E}$ is uniformly locally connected.*

Proof. (1) Since each E_n is locally connected, by Theorem 4.2.6 each mapping f_n can be extended to a continuous mapping $\overline{f}_n$ of $\overline{B}$. By using the assumption that (f_n) converges to a homeomorphism f, and that $\mathscr{E} = \{E_n = \hat{\mathbb{C}} \setminus f_n(B)\}$ is uniformly locally connected, and applying the similar arguments used in the proof of part (3) $\Rightarrow$ (1) of Theorem 5.1.2, we can prove that the family $\{f_n : n \in \mathbb{N}\}$ is uniformly equicontinous in $\overline{B}$.

Let $\varepsilon > 0$. Since $\{f_n : n \in \mathbb{N}\}$ is uniformly equicontinous in $\overline{B}$, there is a positive number δ such that

$$q(\overline{f}_n(x), \overline{f}_n(y)) < \frac{\varepsilon}{3}$$

for all $n \in \mathbb{N}$, whenever $x, y \in \overline{B}$ with $|x - y| < \delta$. Since $\overline{B}$ is compact, there are points, say $x_1, x_2, \ldots, x_M$, in B such that $\overline{B} \subset \bigcup_{k=1}^{M} B(x_k, \delta)$. Since (f_n) converges to a homeomorphism f, there is an integer N such that

$$q(f_n(x_k), f_m(x_k)) < \frac{\varepsilon}{3}$$

for all $k = 1, 2, \ldots, M$ whenever $m, n \geq N$. Let $y \in \overline{B}$. Then $y \in B(x_k, \delta)$ for some $k \in \{x_1, x_2, \ldots, x_M\}$. For $m, n \geq N$, we have

$$\begin{aligned} q(\overline{f}_n(y), \overline{f}_m(y)) &\leq q(\overline{f}_n(y), \overline{f}_n(x_k)) + q(\overline{f}_n(x_k), \overline{f}_m(x_k)) + q(\overline{f}_m(x_k), \overline{f}_m(y)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore $(\overline{f}_n)$ converges uniformly in $\overline{B}$, as desired.
(2) Suppose that the convergence $f_n \to f$ is uniform and if each f_n extends continuously to $\overline{B}$. It can be shown that f can be extended to a continuous mapping $\overline{f}$ of $\overline{B}$. By Theorem 3.3.1, f is K-quasiconformal. Therefore the family $\mathscr{F} = \{f_n : n \in \mathbb{N}\} \cup \{f\}$ is a compact family of K-quasiconformal mappings defined in the unit disk B into bounded domains. Indeed $(\overline{f}_n)$ converges uniformly in $\overline{B}$, which implies that $\{\overline{f} : f \in \mathscr{F}\}$ is uniformly equicontinuous in $\overline{B}$. Hence, by Theorem 5.1.2, $\mathscr{E}$ is uniformly locally connected. $\square$

Bibliography

[1] L.V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand Co., Princeton 1966.

[2] L.V. Ahlfors and A. Beurling, Conformal invariants and function-theoretic null-sets, *Acta. Math.*, **83**(1950), 101–129.

[3] V.J. Becker, Über gleichgradig stetig fortsetzbare quasikonforme Abbildungen, *Math. Nachr.*, **80**(1977), 289–294.

[4] E.F. Collingwood and A.J. Lohwater, *The theory of cluster sets*, Cambridge University Press, 1966.

[5] J.G. Hocking and G.S. Young, *Topology*, Addison-Wesley, 1961.

[6] O. Lehto and K.I. Virtanen, *Quasikonforme Abbildungen*, Springer-Verlag, Berlin 1965.

[7] R. Näkki, Continuous boundary extension of quasiconformal mappings, *Ann. Acad. Sci. Fenn.*, **AI 511**(1972), 1–10.

[8] M.H.A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1964.

[9] B.P. Palka, *An Introduction to Complex Function Theory.* Springer-Verlag, 1991.

[10] Ch. Pommerenke, *Univalent functions*, Vandenhoek & Ruprecht, Göttingen 1975.

[11] Ch. Pommerenke, *Boundary behavior of conformal maps*, Springer-Verlag, Berlin 1992.

[12] L.A. Steen and J.A. Seebach, *Counterexamples in topology*, Springer-Verlag, New York 1978.

[13] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. 229, Springer-Verlag, Berlin 1971.

Biography

NAME	Mister Phruetthiphong Lohasuwan
ADDRESS	80/9 Moo 4 Tambol Phayayen, Amphur Pak Chong, Nakhon Ratchasima, 30320
INSTITUTION ATTENDED	
2010	Bachelor of Science in Mathematics, Silpakorn University
2015	Master of Science in Mathematics, Silpakorn University