# CONSTRUCTIONS AND BOUNDS ON LINEAR CODES WITH PRESCRIBED HULL DIMENSION OVER FINITE FIELDS

By

Mr. Todsapol MANKEAN

A Thesis Proposal Submitted in Partial Fulfillment of the Requirements

for Doctor of Philosophy (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

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# การสร้างและค่าขอบเขตของรหัสเชิงเส้นบนฟีลด์จำกัดซึ่งกำหนดมิติของเปลือกหุ้ม

โดย
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| By | Todsapol MANKEAN |
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TODSAPOL MANKEAN : CONSTRUCTIONS AND BOUNDS ON LINEAR CODES WITH PRESCRIBED HULL DIMENSION OVER FINITE FIELDS. THESIS ADVISOR : ASSOCIATE PROFESSOR SOMPHONG JINMAN, Ph.D.

Due to their wide applications, hulls of linear codes have been of interest and extensively studied. In this thesis, bounds on the parameters and constructions of optimal linear codes with hull dimension one over small finite fields have been focused on. For all positive integers $n \geq 3$ and $q \in \{2,3\}$, optimal linear $[n, 2, d]_q$ codes with Euclidean hull dimension one are constructed. Moreover, for $q = 2$, the enumeration of such optimal codes is given up to equivalence. For all lengths $n \geq 3$ such that $n \equiv 1, 2, 4 \pmod 5$, optimal quaternary linear $[n, 2]_4$ codes with Hermitial hull dimension one are constructed. For positive integers $n \equiv 0, 3 \pmod 5$, good lower and upper bounds on the minimum weight of quaternary $[n, 2, d]_4$ codes with Hermitian hull dimension one are given as well as some illustrative examples.

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Todsapol MANKEAN

## Table of Contents

# Chapter 1

# Introduction

Coding theory introduced in 1948 by Claude Shannon is the study of the properties of codes and deals with the design of error-correcting codes for the reliable transmission of information across noisy channels. Linear codes, subspaces of the vector space $\mathbb{F}_q^n$, are interesting extensively studied due to their nice algebraic structures and wide applications. A linear code contained in its dual is called a *self-orthogonal code* and a linear code meets its dual trivialy is called a *linear complementary dual (LCD) code*. Self-orthogonal and linear complementary dual codes form important classes of linear codes due to their algebraic structures, practical applications in communications systems, and linked with other mathematical objects as shown in [2, 3, 5, 6, 8, 10, 11, 12, 15, 20, 21, 24] and references therein. Self-orthogonal codes have been applied in constructions of quantum codes in [11] and [20]. In [7], LCD codes have been shown to be asymptotically good and meet the asymptotic Gilbert-Varshamov bound.

The concepts of self-orthogonal codes and linear complementary dual codes have been generalized in the notion of hulls. This concept has been first introduced and applied in the characterization of finite projective planes in [1]. Precisely, the hull of a linear code is the intersection of the code and its dual. It is easily seen that self-orthogonal codes are linear codes with maximal hull and LCD codes are linear codes with minimal (trivial) hull. Later, it has been shown that the complexity of some algorithms in coding theory has been determined by the hull dimension of codes [13, 14, 25, 26]. Precisely, most of the algorithms work if the hull of codes is small. Since then, hulls of codes have become of interest and extensively studied. In [23], the number of linear codes with common hull dimension and the average hull dimension of inear codes have been studied. Recently, a rigorous treatment of hulls of linear codes have been given and applied in constructions of good quantum error correcting codes in [8, 17, 28]. Optimal and good LCD codes have been studied in [3, 7, 10, 19]. Bounds and the optimality of self-orthogonal codes have been established in [12, 20, 22, 27]. Therefore, it is of interest to study constructions and optimality of linear codes with prescribed hull dimension and their applications.

In this thesis, we focus on constructions and optimality of linear codes over small finite fields with hull dimension one with respect to the Euclidean and Hermitian inner products. Some basic properties of linear codes, hull of linear codes, and bounds on the parameters of codes are discussed in Chapter 2. Constructions of optimal binary $[n, 2]_2$ linear codes with Euclidean hull dimension one are given for all lengths $n \geq 3$ in Chapter 3 together with the enumeration of such codes up to equivalence. In Chapter 4, constructions of optimal ternary $[n, 2]_3$ linear codes with Euclidean hull dimension one are presented for all lengths $n \geq 3$. Optimal quaternary $[n, 2]_4$ codes with Hermitian hull dimension one are established in Chapter 5 for all lengths $n \geq 3$ such that $n \equiv 1, 2, 4 \pmod 5$. Subsequently, good lower and upper bounds on the minimum weight of quaternary $[n, 2]_4$ linear codes with Hermitian hull dimension one are given for positive integers $n \equiv 0, 3 \pmod 5$.

# Chapter 2

# Preliminaries

In this section, some definitions and properties of linear codes and hulls of linear codes are recalled as well as the proofs of preliminary results required in this study.

## 2.1 Linear Codes over Finite Fields

Let $\mathbb{F}_q$ denote the finite field of order $q$. For a positive integer $n$, a *linear code of length* $n$ over $\mathbb{F}_q$ is defined to be a subspace of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$. For an element $\boldsymbol{w}$ in $\mathbb{F}_q^n$, let $\mathrm{wt}(\boldsymbol{w})$ denote the *Hamming weight* of $\boldsymbol{w}$. Precisely, $\mathrm{wt}(\boldsymbol{w}) = |\{i \in \{1, 2, \ldots, n\} \mid w_i \neq 0\}|$ for all $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{F}_q^n$. A linear code $C$ of length $n$ over $\mathbb{F}_q$ is called an $[n, k, d]_q$ code if the $\mathbb{F}_q$-dimension is $k$ and the *minimum Hamming weight* of a linear code $C$ is

$$d = \mathrm{wt}(C) = \min\{\mathrm{wt}(\boldsymbol{c}) \mid \mathrm{wt}(\boldsymbol{c}) \in C \setminus \{0\}\},$$

or an $[n, k]_q$ code if the minimum Hamming weight of $C$ is not specified. The minimum Hamming weight is key to determine the error-correcting capability of a code. Let $t$ be a positive integer. A code $C$ is said to be *$t$-error-correcting* if the minimum distance decoding (see [16, Definition 2.5.1]) is able to correct $t$ or fewer errors. In [16, Theorem 2.5.10], it is proved that a code $C$ is $t$-error-correcting if and only if $\mathrm{wt}(C) \geq 2t+1$. Hence, $\mathrm{wt}(C)$ determines the efficiency of the code $C$. A $k \times n$ matrix

$G$ over $\mathbb{F}_q$ is called a *generator matrix* for an $[n, k, d]_q$ code $C$ if the rows of $G$ form a basis of $C$. A generator matrix of the form $\begin{bmatrix} I_k | X \end{bmatrix}$ is said to be in *standard form*.

**Example 2.1.1.** Let $C$ be linear code of length 5 over $\mathbb{F}_2$ with a generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $C = \{0000, 10101, 01111, 11010\}$. Since $\mathrm{wt}(10101) = 3$, $\mathrm{wt}(01111) = 4$, and $\mathrm{wt}(11010) = 3$, we have $\mathrm{wt}(C) = 3$. This implies that $C$ is a $[5, 2, 3]_2$ linear code.

Linear $[n, k]_q$ codes $C_1$ and $C_2$ are said to be *(permutation) equivalent* if there exists a permutation $\delta$ on $\{1, 2, \ldots, n\}$ such that $C_2 = \{(c_{\delta(1)}, c_{\delta(2)}, , \ldots, c_{\delta(n)}) \mid (c_1, c_2, \ldots, c_n) \in C_1\}$. For [16, Theorem 4.6.3] every linear code is equivalent to a linear code that has a generator matrix in standard form.

**Example 2.1.2.** Let $q = 2$ and $n = 4$. Let $\rho = (1\,4)(3\,4)$ be a permutation on $\{1, 2, 3, 4\}$. Then

$$C_1 = \{0000, 0101, 0010, 0111\}$$

is equivalent to the code

$$C_2 = \{0000, 1100, 0001, 1110\}$$

permuted by $\rho$.

For positive integers $m, n$, denote by $M_{m,n}(\mathbb{F}_q)$ the set of $m \times n$ matrices whose entries are in $\mathbb{F}_q$. For $A = [a_{ij}] \in M_{m,n}(\mathbb{F}_q)$, let $A^T$ denote the transpose matrix of A. In addition, if $q = r^2$ is square, let $A^\dagger = [a_{ji}^r]$.

## 2.2 Hulls of Linear Codes

For $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{F}_q^n$, two inner products between $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined as follows :

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathrm{E}} := \sum_{i=1}^{n} u_i v_i$ is called the *Euclidean inner product* between $\boldsymbol{u}$ and $\boldsymbol{v}$.

2. For $q = r^2$, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathrm{H}} := \sum_{i=1}^{n} u_i \overline{v_i} = \langle \boldsymbol{u}, \overline{\boldsymbol{v}} \rangle_{\mathrm{E}}$ is called the *Hermitian inner product* between $\boldsymbol{u}$ and $\boldsymbol{v}$, where $\overline{a} = a^r$ for all $a \in \mathbb{F}_q$

For a linear code $C$ of length $n$ over $\mathbb{F}_q$, the *Euclidean dual* and (resp., *Hermitian dual*) of a linear code $C$ is defined to be the set

$$C^{\perp_{\mathrm{E}}} := \{\boldsymbol{u} \in \mathbb{F}_q^n \mid \langle \boldsymbol{u}, \boldsymbol{c} \rangle_{\mathrm{E}} = 0 \text{ for all } \boldsymbol{c} \in C\}$$

$$(\text{resp., } C^{\perp_{\mathrm{H}}} := \{\boldsymbol{u} \in \mathbb{F}_q^n \mid \langle \boldsymbol{u}, \boldsymbol{c} \rangle_{\mathrm{H}} = 0 \text{ for all } \boldsymbol{c} \in C\})$$

and the *Euclidean* (resp., *Hermitian*) *hull* of $C$ is defined to be $\mathrm{Hull}(C) = C \cap C^{\perp_{\mathrm{E}}}$ (resp., $\mathrm{Hull}_{\mathrm{H}}(C) = C \cap C^{\perp_{\mathrm{H}}}$).

An $(n-k) \times n$ matrix $H$ over $\mathbb{F}_q$ is called a parity-check matrix of an $[n, k, d]_q$ code $C$ if $H$ is a generator matrix of $C^{\perp_E}$. A parity-check matrix of the form $\left[Y | I_{n-k}\right]$ is said to be in *standard form* (see [16, Definition 4.5.3]).

**Theorem 2.2.1** ([16, Theorem 4.5.9]). *If $G = [I_k | A]$ is the standard form generator matrix of an $[n, k]_q$-code $C$, then the standard form generator matrix of $C^{\perp_E}$ is $H = [-A^T | I_{n-k}]$.*

**Example 2.2.2.** Let $C$ be a linear code of length 5 over $\mathbb{F}_2$ with a generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

Then $C = \{00000, 10101, 01111, 11010\}$ has parameters $[5, 2, 3]_2$. By Theorem 2.2.1, we have that

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

is a parity-check matrix for $C$ and hence,

$$C^{\perp_{\mathrm{E}}} = \{00000, 10011, 11001, 01010, 01111, 11100, 10110, 00101\}.$$

By the definition of the hull, we have $\mathrm{Hull}(C) = \{00000, 01111\}$.

**Example 2.2.3.** Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2 = \omega + 1\}$ and let $C$ be a linear code of length 5 over $\mathbb{F}_4$ with a generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & \omega & \omega^2 \\ 0 & 1 & 0 & 1 & \omega \end{bmatrix}.$$

Then $C$ has parameters $[5, 2, 4]_4$. By Theorem 2.2.1, $C^{\perp_H}$ has a generator matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ \omega & 1 & 0 & 1 & 0 \\ \omega^2 & \omega & 0 & 0 & 1 \end{bmatrix}.$$

Using a direct calculation, we have $\mathrm{Hull_H}(C) = \{00000, 101\omega\omega^2, \omega0\omega\omega^21, \omega^20\omega^21\omega\}$.

For $n \in \{1, 2\}$, it is easy to see that there are no $[n, 2]_2$ codes with hull dimension one. Throughout, we assume that the length $n$ of the codes is greater than 2.

The hull dimension of a linear code can be determined using its generator matrix in [8] as follows.

**Proposition 2.2.4** ([8, Propositions 3.1])**.** *Let $C$ be a linear $[n, k, d]_q$ code with generator matrix $G$. Then rank($GG^T$) is independent of $G$ and*

$$\mathrm{rank}(GG^T) = k - \dim(\mathrm{Hull}(C)) = k - \dim(\mathrm{Hull}(C^{\perp_E})).$$

*In addition, if $q$ is square, then*

$$\mathrm{rank}(GG^\dagger) = k - \dim(\mathrm{Hull_H}(C)) = k - \dim(\mathrm{Hull_H}(C^{\perp_H})).$$

**Example 2.2.5.** From Example 2.2.2, we have $k = 2$, $G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ and

$$GG^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This implies that $\mathrm{rank}(GG^T) = 1$. By Proposition 2.2.4, it follows that

$$\begin{aligned} \dim(\mathrm{Hull}(C)) &= k - \mathrm{rank}(GG^T) \\ &= 2 - 1 = 1. \end{aligned}$$

Hence, $\dim(\mathrm{Hull}(C)) = 1$.

**Example 2.2.6.** From Example 2.2.3, we have $k = 2$, $G = \begin{bmatrix} 1 & 0 & 1 & \omega & \omega^2 \\ 0 & 1 & 0 & 1 & \omega \end{bmatrix}$ and

$$GG^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This implies that $\operatorname{rank}(GG^\dagger) = 1$. By Proposition 2.2.4, it can be deduced that

$$\begin{aligned} \dim(\mathrm{Hull}_{\mathrm{H}}(C)) &= k - \operatorname{rank}(GG^\dagger) \\ &= 2 - 1 = 1. \end{aligned}$$

Therefore, $\dim(\mathrm{Hull}_{\mathrm{H}}(C)) = 1$.

## 2.3 Constructions and Bounds

Using the analysis on a generator matrix of linear codes in [19], we have the following results. For an $[n, 2]_q$ code with generator matrix

$$G = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix},$$

we write

$$G = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \tag{2.1}$$

where $\alpha_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$.

Alternatively, by setting $\beta_l = \begin{bmatrix} a_{1l} \\ a_{2l} \end{bmatrix}$ for $1 \leq l \leq n$, $G$ can be viewed as

$$G = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}. \tag{2.2}$$

For $i, j \in \mathbb{F}_q$, let

$$S_{ij} := |\{l \in \{1, 2, \ldots, n\} \mid \beta_l = \begin{bmatrix} i \\ j \end{bmatrix}\}|. \tag{2.3}$$

**Example 2.3.1.** Let $C$ and $C'$ be ternary linear codes of length 7 generated by

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$G' = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

We see that $G$ and $G'$ are determined by $S_{10} = 2, S_{01} = 1, S_{11} = 2, S_{21} = 2$, and $S_{ij} = 0$ otherwise. It is easily seen that $C$ and $C'$ are equivalent by the permutation $(1\,4)(2\,5)$.

**Remark 2.3.2.** The generator matries determined by

$$S_{ij} := |\{l \in \{1, 2, \ldots, n\} \mid \beta_l = \begin{bmatrix} i \\ j \end{bmatrix}\}|$$

generate equivalent linear codes. The equivalences are induced by a permutation on $\{1, 2, \ldots, n\}$.

The Griesmer Bound in [9] is applied in this work.

**Theorem 2.3.3** (Griesmer Bound, [9, Theorem 2.7.4])**.** *Let $q$ be a prime power and let $n$, $k$ and $d$ be positive integers. If there exists an $[n, k, d]_q$ code, then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil.$$

The minimum Hamming weight of linear codes is important to determine the error-correcting capability of codes. In this thesis, we focus on the maximum of the minimum Hamming weight of linear codes. For a prime power $q$, positive integers $n, k$ and non-negative integer $\ell$, let

$$D_q(n, k, \ell) := \max\{d \mid \exists [n, k, d]_q \text{ code with } \dim(\mathrm{Hull}(C)) = \ell\}$$

and

$$D_q^{\mathrm{H}}(n, k, \ell) := \max\{d \mid \exists [n, k, d]_q \text{ code with } \dim(\mathrm{Hull}_{\mathrm{H}}(C)) = \ell\}.$$

Based on the Griesmer Bound, the following upper bounds on $D_q(n, k, \ell)$ and $D_q^{\mathrm{H}}(n, k, \ell)$ can be derived in the next lemma.

**Lemma 2.3.4.** *Let $q$ be a prime power and let $n, k$ and $\ell$ be inegers such that $1 \leq k \leq n$ and $0 \leq \ell$. Then*

$$D_q(n, k, \ell) \leq \left\lfloor \frac{(q-1)q^{k-1}n}{(q^k-1)} \right\rfloor$$

*and*

$$D_q^{\mathrm{H}}(n, k, \ell) \leq \left\lfloor \frac{(q-1)q^{k-1}n}{(q^k-1)} \right\rfloor \text{ if } q \text{ is square.}$$

*Proof.* From the Griesmer Bound in Theorem 2.3.3, for any linear $[n, k, d]_q$ code, we have

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \geq d \sum_{i=0}^{k-1} \frac{1}{q^i} = \frac{d(q^k-1)}{(q-1)q^{k-1}}.$$

It follows that

$$d \leq \left\lfloor \frac{(q-1)q^{k-1}n}{(q^k-1)} \right\rfloor.$$

Hence, $D_q(n, k, \ell) \leq \left\lfloor \frac{(q-1)q^{k-1}n}{(q^k-1)} \right\rfloor$ as desired.

Since the above proof is independent of the inner product, $D_q^{\mathrm{H}}(n, k, \ell) \leq \left\lfloor \frac{(q-1)q^{k-1}n}{(q^k-1)} \right\rfloor$ follows similarly. □

In this thesis, we focus on $D_q(n, k, 1)$ and $D_q^{\mathrm{H}}(n, k, 1)$. The details are discussed in the following chapters.

# Chapter 3

# Optimal Binary Linear Codes with Euclidean Hull Dimension One

In this chapter, we focus on construction of binary linear codes with hull dimension one. For each integer $n \geq 3$, an optimal $[n, 2]_2$ code with hull dimension one is constructed. Moreover, the enumeration of such optimal codes is given up to equivalence.

## 3.1 Basic Concepts

From the setup in Subsection 2.3 (c.f. [19]), a linear $[n, 2]_2$ code $C$ with generator matrix

$$G = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{3.1}$$

with $\alpha_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$ can be viewed of the form $C = \{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. From (2.3), we have $\alpha_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$ and it is not difficult to see that

$$\mathrm{wt}(\alpha_1) = S_{10} + S_{11}, \mathrm{wt}(\alpha_2) = S_{01} + S_{11}, \mathrm{wt}(\alpha_1 + \alpha_2) = S_{10} + S_{01}.$$

Hence,

$$\begin{aligned}\mathrm{wt}(C) &= \min\{\mathrm{wt}(\alpha_1), \mathrm{wt}(\alpha_2), \mathrm{wt}(\alpha_1+\alpha_2)\} \\ &= \min\{S_{10}+S_{11}, S_{01}+S_{11}, S_{10}+S_{01}\}. \qquad (3.2)\end{aligned}$$

From (3.1), we have

$$GG^T = \begin{bmatrix} S_{10}+S_{11} & S_{11} \\ S_{11} & S_{01}+S_{11} \end{bmatrix} \pmod 2. \qquad (3.3)$$

**Example 3.1.1.** *Let $C$ be the binary linear code of length 5 with generator matrix*

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

*Then $S_{00} = 0, S_{10} = 2, S_{01} = 2$ and $S_{11} = 1$ which implies that $S_{10} + S_{11} = 3$ and $S_{01} + S_{11} = 3$ By (3.3), it follows that*

$$GG^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

*which implies that $\mathrm{rank}(GG^T) = 1$ and $\dim(\mathrm{Hull}(C)) = 2 - \mathrm{rank}(GG^T) = 1$ by Proposition 2.2.4.*

**Remark 3.1.2.** *The binary codes determined by*

$$(S_{01}, S_{10}, S_{11}) \in \{(a,b,c),(b,a,c),(c,b,a),(a,c,b),(b,c,a),(c,a,b)\}$$

*are equivalent. This follows from the fact that*

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix}, \begin{bmatrix} \alpha_1+\alpha_2 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \alpha_1+\alpha_2 \end{bmatrix}, \begin{bmatrix} \alpha_1+\alpha_2 \\ \alpha_1 \end{bmatrix}, \text{ and } \begin{bmatrix} \alpha_2 \\ \alpha_1+\alpha_2 \end{bmatrix}$$

*generate the same code.*

**Example 3.1.3.** By Remark 3.1.2, let $a = 2, b = 0$ and $c = 3$. Then

$$(S_{01}, S_{10}, S_{11}) \in \{(2,0,3),(0,2,3),(3,0,2),(2,3,0),(0,3,2),(3,2,0)\}.$$

For $(S_{01}, S_{10}, S_{11}) = (2, 0, 3)$, we have that $\alpha_1 = 00111$ and $\alpha_2 = 11111$. Let $C$ be linear code with generator matrix

$$G = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $C = \{00000, 00111, 11111, 11000\}$. By Remark 3.1.2, we have that

$$\begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{determined by } (S_{01}, S_{10}, S_{11}) = (0, 2, 3)$$

$$\begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \text{determined by } (S_{01}, S_{10}, S_{11}) = (3, 0, 2)$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{determined by } (S_{01}, S_{10}, S_{11}) = (2, 3, 0)$$

$$\begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \text{determined by } (S_{01}, S_{10}, S_{11}) = (0, 3, 2), \text{and}$$

$$\begin{bmatrix} \alpha_2 \\ \alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{determined by } (S_{01}, S_{10}, S_{11}) = (3, 2, 0)$$

generate linear code C.

For binary linear codes with dimension two and hull dimension one, the upper bound in Lemma 2.3.4 can be simplified as

$$D_2(n, 2, 1) \leq \left\lfloor \frac{2n}{3} \right\rfloor. \tag{3.4}$$

Using the notations above, the exact value of $D_2(n, 2, 1)$ can be derived together with the enumeration of optimal $[n, 2]_2$ codes as follows.

## 3.2 $n \equiv 1 \pmod 6$

In the following theorem, we give constructions of optimal $[n, 2, \lfloor \frac{2n}{3} \rfloor]_2$ codes of such lengths with hull dimension one for all $n \equiv 1 \pmod 6$.

**Theorem 3.2.1.** *Let $n \geq 3$ be an integer. If $n \equiv 1 \pmod 6$, then*

$$D_2(n,2,1) = \left\lfloor \frac{2n}{3} \right\rfloor$$

*and there exists a unique (up to equivalence) optimal $\left[n, 2, \left\lfloor \frac{2n}{3} \right\rfloor\right]_2$ code with hull dimension one.*

*Proof.* Assume that $n \equiv 1 \pmod 6$. Then $n = 6t+1$ for some positive integer $t$ and $\left\lfloor \frac{2n}{3} \right\rfloor = \left\lfloor \frac{2(6t+1)}{3} \right\rfloor = 4t$.

For the existence, by (3.4), it suffices to show that there exists an $[n,2,4t]_2$ code with hull dimension one. Let $C$ be a linear code with generator matrix $G$ defined in (3.1) such that $(S_{01}, S_{10}, S_{11}) = (2t, 2t, 2t+1)$. Then $C$ is an $[n,2,d]_2$ code, where

$$\begin{aligned} d &= \min\{S_{01} + S_{11}, S_{10} + S_{11}, S_{10} + S_{01}\} \\ &= \min\{4t+1, 4t+1, 4t\} \\ &= 4t \quad \text{by (3.2)}. \end{aligned}$$

From (3.3), we have

$$GG^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which implies that $\operatorname{rank}(GG^T) = 1$. We therefore have $\dim(\operatorname{Hull}(C)) = 1$ by Proposition 2.2.4.

For the uniqueness, we first show that $S_{01} + S_{10} + S_{11} = n$. Suppose that there exists an $[n,2,4t]_2$ code with hull dimension one and $S_{00} > 0$. Then

$$S_{01} + S_{10} + S_{11} \leq n - 1 = 6t.$$

Since $\min\{S_{10} + S_{11}, S_{01} + S_{11}, S_{10} + S_{01}\} = 4t$ is required, the only possible choice of $(S_{01}, S_{10}, S_{11})$ is $(2t, 2t, 2t)$. In this case, we have

$$GG^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

by (3.3). By Proposition 2.2.4, we therefore have $\dim(\mathrm{Hull}(C)) = 2 \neq 1$, a contradiction. Hence, $n = S_{01} + S_{10} + S_{11}$.

It remains to focus on $(S_{01}, S_{10}, S_{11})$ satisfying $S_{01} + S_{10} + S_{11} = n = 6t + 1$ and $\min\{S_{10} + S_{11}, S_{01} + S_{11}, S_{10} + S_{01}\} = 4t$. It follows that

$$\begin{aligned}(S_{01}, S_{10}, S_{11}) \in &\{(2t, 2t, 2t+1), (2t, 2t+1, 2t), (2t+1, 2t, 2t),\\ &(2t-1, 2t+1, 2t+1), (2t+1, 2t-1, 2t+1),\\ &(2t+1, 2t+1, 2t-1)\}.\end{aligned}$$

We consider the following 2 cases.

**Case 1**. $(S_{01}, S_{10}, S_{11}) \in \{(2t, 2t, 2t+1), (2t, 2t+1, 2t), (2t+1, 2t, 2t)\}$. By (3.3), the corresponding matrices $GG^T$ are of the forms

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

respectively. In these cases, we have $\mathrm{rank}(GG^T) = 1$ which implies that there hull dimensions are one by Proposition 2.2.4. From Remark 3.1.2, these codes are equivalent.

**Case 2**. $(S_{01}, S_{10}, S_{11}) \in \{(2t-1, 2t+1, 2t+1), (2t+1, 2t-1, 2t+1), (2t+1, 2t+1, 2t-1)\}$. From (3.3), it is easily seen that

$$GG^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, $\mathrm{rank}(GG^T) = 2$ which implies that the codes are LCD by Proposition 2.2.4.

□

**Example 3.2.2.** Let $t = 2$. By Theorem 3.2.1, we have that $(S_{01}, S_{10}, S_{11}) = (2t, 2t, 2t+1) = (4, 4, 5)$. Let $C$ be the binary linear code of length 13 determined by the triple $(4, 4, 5)$ with generator matrix

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By (3.3), it follows that

$$GG^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$ and $\dim(\text{Hull}(C)) = 2 - \text{rank}(GG^T) = 1$ by Proposition 2.2.4. Then $C$ is an optimal $[13, 2, 8]_2$ code with hull dimension one.

## 3.3 $n \equiv 3 \pmod 6$

Next, we show that there does not exist $[n, 2, \lfloor \frac{2n}{3} \rfloor]_2$ codes with hull dimension one for all $n \equiv 3 \pmod 6$. Later, we give constructions of optimal $[n, 2, \lfloor \frac{2n}{3} \rfloor - 1]_2$ codes of such lengths with hull dimension one.

**Theorem 3.3.1.** *Let $n \geq 3$ be an integer. If $n \equiv 3 (\text{mod } 6)$, then*

$$D_2(n, 2, 1) = \left\lfloor \frac{2n}{3} \right\rfloor - 1.$$

*In this case, there exists a unique (up to equivalence) optimal $[3, 2, 1]_2$ code with hull dimension one and there are two (up to equivalence) optimal $\left[n, 2, \lfloor \frac{2n}{3} \rfloor - 1\right]_2$ codes with hull dimension one for all $n > 3$.*

*Proof.* Assume that $n \equiv 3 (\text{mod } 6)$. Then $n = 6t + 3$ for some integer $t \geq 0$. By (3.4), we have

$$D_2(n, 2, 1) \leq \left\lfloor \frac{2n}{3} \right\rfloor = \left\lfloor \frac{2(6t + 3)}{3} \right\rfloor = 4t + 2.$$

Suppose there exists an $\left[n, 2, \lfloor \frac{2n}{3} \rfloor = 4t + 2\right]_2$ code with hull dimension one. Then, by (3.2), we have $\text{wt}(C) = \min\{S_{01} + S_{11}, S_{10} + S_{11}, S_{10} + S_{01}\} = 4t + 2$ and $S_{01} + S_{10} + S_{11} \leq n = 6t + 3$. Hence, the only possible choice of $(S_{01}, S_{10}, S_{11})$ is $(2t + 1, 2t + 1, 2t + 1)$. By (3.3), we have

$$GG^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By Proposition 2.2.4, it follows that $\dim(\text{Hull}(C)) = 0 \neq 1$, a contradiction. Hence, there does not exist an $\left[n, 2, \lfloor \frac{2n}{3} \rfloor\right]_2$ code.

For the existence of an $[n, 2, 4t+1]_2$ code with hull dimension one, let $C$ be a linear code with generator matrix $G$ viewed in the form of (3.1) such that $(S_{01}, S_{10}, S_{11}) = (2t, 2t+2, 2t+1)$. Then $C$ is an $[n, 2, d]_2$ code, where

$$\begin{aligned} d &= \min\{S_{01} + S_{11}, S_{10} + S_{11}, S_{10} + S_{01}\} \\ &= \min\{4t+1, 4t+3, 4t+2\} \\ &= 4t+1 \text{ by (3.2).} \end{aligned}$$

From (3.3), we have

$$GG^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$. As desired, we have $\dim(\text{Hull}(C)) = 1$ by Proposition 2.2.4.

To determine the number of $[n, 2, 4t+1]_2$ codes with hull dimension one, we claim that $S_{01} + S_{10} + S_{11} = n$. Suppose that $S_{00} > 0$. Then

$$S_{01} + S_{10} + S_{11} \leq n - 1 = 6t + 2.$$

Since $\min\{S_{10}+S_{11}, S_{01}+S_{11}, S_{10}+S_{01}\} = 4t+1$, the possible choices of $(S_{01}, S_{10}, S_{11})$ are $(2t, 2t+1, 2t+1)$, $(2t+1, 2t+1, 2t)$, and $(2t+1, 2t, 2t+1)$. By (3.3), the matrix $GG^T$ is of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively. Hence, $\text{rank}(GG^T) = 2$ which implies that $\dim(\text{Hull}(C)) = 0 \neq 1$ by Proposition 2.2.4.

Now, it remains to focus on the triples $(S_{01}, S_{10}, S_{11})$ satisfying $S_{01} + S_{10} + S_{11} = n = 6t+3$ and $\min\{S_{10} + S_{11}, S_{01} + S_{11}, S_{10} + S_{01}\} = 4t+1$. By inspection, we have $(S_{01}, S_{10}, S_{11}) \in T_1 \cup T_2 \cup T_3$, where

$$\begin{aligned} T_1 &= \{(2t, 2t+2, 2t+1), (2t+2, 2t, 2t+1), (2t+2, 2t+2, 2t-1)\}, \\ T_2 &= \{(2t-1, 2t+2, 2t+2), (2t+1, 2t, 2t+2), (2t+1, 2t+2, 2t)\}, \\ T_3 &= \{(2t, 2t+1, 2t+2), (2t+2, 2t-1, 2t+2), (2t+2, 2t+1, 2t)\}. \end{aligned}$$

By (3.3), the matrices $GG^T$ are of the forms

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

for $(S_{01}, S_{10}, S_{11})$ in $T_1$, $T_2$, and $T_3$, respectively. It follows that $\text{rank}(GG^T) = 1$ and the codes have hull dimension one by Proposition 2.2.4.

From Remark 3.1.2, the codes determined by

$$(S_{01}, S_{10}, S_{11}) \in \{(2t, 2t+2, 2t+1), (2t+2, 2t, 2t+1), (2t+2, 2t+1, 2t), (2t, 2t+1, 2t+2), (2t+1, 2t, 2t+2), (2t+1, 2t+2, 2t)\}$$

are equivalent, and the codes determined by

$$(S_{01}, S_{10}, S_{11}) \in \{(2t+2, 2t+2, 2t-1), (2t-1, 2t+2, 2t+2), (2t+2, 2t-1, 2t+2)\}$$

are equivalent. Since a code in the first family contains a codeword of weight $4t+3$ but the latter does not, two codes from different families are not equivalent. Hence, there are two (up to equivalence) optimal $\left[n, 2, \left\lfloor \frac{2n}{3} \right\rfloor - 1\right]_2$ codes with hull dimension one for all $n > 3$. For $n = 3$, we have $t = 0$ and hence, the second family does not exist which implies that there exists a unique (up to equivalence) optimal $[3, 2, 1]_2$ code with hull dimension one. □

**Example 3.3.2.** Let $t = 2$. By Theorem 3.3.1, we have that $(S_{01}, S_{10}, S_{11}) = (2t, 2t+2, 2t+1) = (4, 6, 5)$ and $(S_{01}, S_{10}, S_{11}) = (2t+2, 2t+2, 2t-1) = (6, 6, 3)$ indeed 2 inequivalent codes.

1. Let $C_1$ be the binary linear code of length 15 determined by the triple $(4, 6, 5)$ with generator matrix

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By (3.3), it follows that

$$G_1 G_1^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which implies that $\text{rank}(G_1{G_1}^T) = 1$ and $\dim(\text{Hull}(C)) = 2 - \text{rank}(G_1{G_1}^T) = 1$ by Proposition 2.2.4. Then $C_1$ is an optimal $[15, 2, 9]_2$ code with hull dimension one.

2. Let $C_2$ be the binary linear code of length 15 determined by the triple $(6, 6, 3)$ with generator matrix

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

By (3.3), it follows that

$$G_2G_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies that $\text{rank}(G_2{G_2}^T) = 1$ and $\dim(\text{Hull}(C)) = 2 - \text{rank}(G_2{G_2}^T) = 1$ by Proposition 2.2.4. Then $C_2$ is an optimal $[15, 2, 9]_2$ code with hull dimension one.

By Theorem 3.2.1, we have that $C_1$ and $C_2$ are not equivalent.

## 3.4 $n \equiv 5 \pmod 6$

In the following theorem, we give constructions of an optimal $[n, 2, \lfloor \frac{2n}{3} \rfloor]_2$ code with dimesion 2 and Euclidean hull dimension one is given for all lengths $n \equiv 5 \pmod 6$.

**Theorem 3.4.1.** *Let $n \geq 3$ be an integer. If $n \equiv 5 (\text{mod } 6)$, then*

$$D_2(n, 2, 1) = \left\lfloor \frac{2n}{3} \right\rfloor$$

*and there exists a unique (up to equivalence) optimal $\left[n, 2, \lfloor \frac{2n}{3} \rfloor\right]_2$ code with hull dimension one.*

*Proof.* Assume that $n \equiv 5 (\text{mod } 6)$. Then $n = 6t + 5$ for some positive integer $t$. By (3.4), we have

$$D_2(n, 2, 1) \leq \left\lfloor \frac{2n}{3} \right\rfloor = \left\lfloor \frac{2(6t + 5)}{3} \right\rfloor = 4t + 3.$$

For the existence of an $[n, 2, 4t+3]_2$ code with hull dimension one, let $C$ be a linear code with generator matrix $G$ viewed in the form of (3.1) such that $(S_{01}, S_{10}, S_{11}) = (2t+2, 2t+1, 2t+2)$. Then $C$ is an $[n, 2, d]_2$ code, where

$$\begin{aligned} d &= \min\{S_{01} + S_{11}, S_{10} + S_{11}, S_{10} + S_{01}\} \\ &= \min\{4t+3, 4t+3, 4t+3\} \\ &= 4t+3 \ \text{ by (3.2)}. \end{aligned}$$

From (3.3), we have

$$GG^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$. Hence, we have $\dim(\text{Hull}(C)) = 1$ by Proposition 2.2.4.

To determine the number of $[n, 2, 4t+3]_2$ codes with hull dimension one, we claim that $S_{01} + S_{10} + S_{11} = n$. Suppose that $S_{00} > 0$. Then

$$S_{01} + S_{10} + S_{11} \le n - 1 = 6t + 4.$$

Since $\min\{S_{10} + S_{11}, S_{01} + S_{11}, S_{10} + S_{01}\} = 4t + 3$, there are no possible choices of $(S_{01}, S_{10}, S_{11})$, a contradiction. Hence, $S_{01} + S_{10} + S_{11} = n$.

It is not difficult to see that the triples $(S_{01}, S_{10}, S_{11})$ satisfying $S_{01} + S_{10} + S_{11} = n = 6t + 5$ and $\min\{S_{10} + S_{11}, S_{01} + S_{11}, S_{10} + S_{01}\} = 4t + 3$ are in the set

$$\{(2t+1, 2t+2, 2t+2), (2t+2, 2t+1, 2t+2), (2t+2, 2t+2, 2t+1)\}.$$

By (3.3), the corresponding matrices $GG^T$ are of the forms

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

respectively. We have $\text{rank}(GG^T) = 1$ and their hull dimensions are one by Proposition 2.2.4. These codes are equivalent by Remark 3.1.2. □

**Example 3.4.2.** Let $t = 2$. By Theorem 3.4.1, we have that $(S_{01}, S_{10}, S_{11}) = (2t + 2, 2t + 2, 2t + 1) = (6, 6, 5)$. Let $C$ be the binary linear code of length 17 determined by the triple $(6, 6, 5)$ with generator matrix

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By (3.3), it follows that

$$GG^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$ and $\dim(\text{Hull}(C)) = 2 - \text{rank}(GG^T) = 1$ by Proposition 2.2.4. Then $C$ is an optimal $[13, 2, 11]_2$ code with hull dimension one.

## 3.5 $n \equiv 0, 2, 4 \pmod 6$

Next, we show that there does not exist $[n, 2, \lfloor \frac{2n}{3} \rfloor]_2$ codes with hull dimension one for all $n \equiv 0, 2, 4 \pmod 6$. Later, we give constructions of optimal $[n, 2, \lfloor \frac{2n}{3} \rfloor - 1]_2$ codes of such lengths with hull dimension one.

**Lemma 3.5.1.** *Let $n \geq 3$ be an integer. If $n \equiv 0, 2, 4 \pmod 6$, then there are no $[n, 2, \lfloor \frac{2n}{3} \rfloor]_2$ codes with hull dimension one.*

*Proof.* Assume that $n \equiv 0, 2, 4 \pmod 6$. Suppose that there exists an $[n, 2, \lfloor \frac{2n}{3} \rfloor]_2$ code $C$ with hull dimension one.

First, we prove that $S_{10} + S_{01} + S_{11} = n$. Suppose that $S_{00} > 0$.

**Case I**. $n \equiv 0, 2 \pmod 6$. If follows that $\left\lfloor \frac{2(n-1)}{3} \right\rfloor < \frac{2n}{3} = \lfloor \frac{2n}{3} \rfloor$ for all $n \equiv 0 \pmod 6$, and $\left\lfloor \frac{2(n-1)}{3} \right\rfloor < \frac{2n-1}{3} = \lfloor \frac{2n}{3} \rfloor$ for all $n \equiv 2 \pmod 6$. Since $S_{00} > 0$, there exists an $[n - 1, 2, \lfloor \frac{2n}{3} \rfloor]_2$ code $C'$ by puncturing at one of the zero columns. From (3.4), we have

$$\left\lfloor \frac{2n}{3} \right\rfloor = \text{wt}(C') \leq \left\lfloor \frac{2(n-1)}{3} \right\rfloor < \left\lfloor \frac{2n}{3} \right\rfloor,$$

a contradiction.

**Case II**. $n \equiv 4 \pmod 6$. Since $S_{00} > 0$, there exists an $[n-1, 2, \lfloor \frac{2n}{3} \rfloor]_2$ code $C'$ by puncturing at one of the zero columns. Since $n - 1 \equiv 3 \pmod 6$, we have

$$\left\lfloor \frac{2n}{3} \right\rfloor = \mathrm{wt}(C') \leq D_2(n-1, 2, 1) = \left\lfloor \frac{2(n-1)}{3} \right\rfloor - 1 = \left\lfloor \frac{2n}{3} \right\rfloor - 1 < \left\lfloor \frac{2n}{3} \right\rfloor$$

by Theorem 3.3.1. This is a contradiction.

From the discussion above, we have $S_{10} + S_{01} + S_{11} = n$.

Next, we consider the following 2 cases.

**Case 1**. $S_{11}$ is odd. Since $n$ is even, it follows that either $S_{10}$ or $S_{01}$ is odd. From (3.3), we have

$$GG^T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ or } GG^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

In both cases, $\mathrm{rank}(GG^T) = 2$. By Proposition 2.2.4, we have $\dim(\mathrm{Hull}(C)) = 2 - \mathrm{rank}(GG^T) = 0$.

**Case 2**. $S_{11}$ is even. Then $S_{10}$ and $S_{01}$ have the same parity. If $S_{10}$ and $S_{01}$ are even, then

$$GG^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

by (3.3). If $S_{10}$ and $S_{01}$ are odd, then

$$GG^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

by (3.3). In both cases, $\dim(\mathrm{Hull}(C)) = 2 - \mathrm{rank}(GG^T) \neq 1$ by Proposition 2.2.4.

From the two cases, we have $\dim(\mathrm{Hull}(C)) \neq 1$, a contradiction. Hence, there are no $[n, 2, \lfloor \frac{2n}{3} \rfloor]_2$ codes with hull dimension one. □

**Theorem 3.5.2.** *Let $n \geq 3$ be an integer. If $n \equiv 0, 2, 4 \pmod 6$, then*

$$D_2(n, 2, 1) = \left\lfloor \frac{2n}{3} \right\rfloor - 1.$$

*Moreover, the following statements hold.*

1. *There exists a unique (up to equivalence) optimal $\left[n, 2, \lfloor \frac{2n}{3} \rfloor - 1\right]_2$ code with hull dimension one for all $n = 4$ or $n \equiv 0, 2 \pmod 6$.*

2. *There are two (up to equivalence) optimal* $\left[n, 2, \left\lfloor \frac{2n}{3} \right\rfloor - 1\right]_2$ *codes with hull dimension one for all* $n > 4$ *such that* $n \equiv 4 \pmod 6$.

*Proof.* Assume that $n \equiv 0, 2, 4 \pmod 6$. By Lemma 3.5.1, we have

$$D_2(n, 2, 1) \leq \left\lfloor \frac{2n}{3} \right\rfloor - 1.$$

First, we observe that there are no triples $(S_{01}, S_{10}, S_{11})$ satisfying $S_{01}+S_{10}+S_{11} \leq n-2$ and $\min\{S_{10}+S_{11}, S_{01}+S_{11}, S_{10}+S_{01}\} = \left\lfloor \frac{2n}{3} \right\rfloor - 1$. Hence, $S_{01}+S_{10}+S_{11} \geq n-1$. We consider the following two cases.

**Case 1**. $n = S_{01} + S_{10} + S_{11}$. We show that there are no $[n, 2, \left\lfloor \frac{2n}{3} \right\rfloor - 1]_2$ codes with hull dimension one as follows.

**Case 1.1**. $n \equiv 0 \pmod 6$. Then $n = 6t$ for some positive integer $t$. The triples $(S_{01}, S_{10}, S_{11})$ satisfying $S_{01} + S_{10} + S_{11} = n = 6t$ and $\min\{S_{10} + S_{11}, S_{01} + S_{11}, S_{10} + S_{01}\} = \left\lfloor \frac{2n}{3} \right\rfloor - 1 = 4t - 1$ are in the set

$$\{(2t, 2t+1, 2t-1), (2t, 2t-1, 2t+1), (2t+1, 2t, 2t-1), (2t+1, 2t+1, 2t-2), (2t+1, 2t-2, 2t+1), (2t-2, 2t+1, 2t+1), (2t+1, 2t-1, 2t), (2t-1, 2t, 2t+1), (2t-1, 2t+1, 2t)\}.$$

The corresponding generator matrices $G$ have the property that $\text{rank}(GG^T) = 2$ which implies that $\dim(\text{Hull}(C)) = 0 \neq 1$ by Proposition 2.2.4.

**Case 1.2**. $n \equiv 2 \pmod 6$. Then $n = 6t + 2$ for some positive integer $t$. The triples $(S_{01}, S_{10}, S_{11})$ satisfying $S_{01}+S_{10}+S_{11} = n = 6t+2$ and $\min\{S_{10}+S_{11}, S_{01}+S_{11}, S_{10}+S_{01}\} = \left\lfloor \frac{2n}{3} \right\rfloor - 1 = 4t$ are in the set

$$\{(2t, 2t, 2t+2), (2t+2, 2t, 2t), (2t, 2t+2, 2t), (2t-2, 2t+2, 2t+2), (2t+2, 2t-2, 2t+2), (2t+2, 2t+2, 2t-2), (2t+1, 2t-1, 2t+2), (2t+2, 2t+1, 2t-1), (2t+1, 2t+2, 2t-1), (2t-1, 2t+1, 2t+2), (2t+2, 2t-1, 2t+1), (2t-1, 2t+2, 2t+1)\}.$$

The corresponding generator matrix $G$ of $(S_{01}, S_{10}, S_{11})$ in

$$\{(2t, 2t, 2t+2), (2t+2, 2t, 2t), (2t, 2t+2, 2t), (2t-2, 2t+2, 2t+2),$$
$$(2t+2, 2t-2, 2t+2), (2t+2, 2t+2, 2t-2)\}$$

has the property that $\text{rank}(GG^T) = 0$ which implies that $\dim(\text{Hull}(C)) = 2 \neq 1$ by Proposition 2.2.4. The corresponding generator matrix $G$ of $(S_{01}, S_{10}, S_{11})$ in

$$\{(2t+1, 2t-1, 2t+2), (2t+2, 2t+1, 2t-1), (2t+1, 2t+2, 2t-1),$$
$$(2t-1, 2t+1, 2t+2), (2t+2, 2t-1, 2t+1), (2t-1, 2t+2, 2t+1)\}$$

has the property that $\text{rank}(GG^T) = 2$ which implies that $\dim(\text{Hull}(C)) = 0 \neq 1$ by Proposition 2.2.4.

**Case 1.3**. $n \equiv 4 (\text{mod } 6)$. Then $n = 6t + 4$ for some positive integer $t$. The triples $(S_{01}, S_{10}, S_{11})$ satisfying $S_{01}+S_{10}+S_{11} = n = 6t+4$ and $\min\{S_{10}+S_{11}, S_{01}+S_{11}, S_{10}+S_{01}\} = \lfloor \frac{2n}{3} \rfloor - 1 = 4t + 1$ are in the set

$$\{(2t, 2t+1, 2t+3), (2t-1, 2t+2, 2t+3), (2t+3, 2t-1, 2t+2),$$
$$(2t+3, 2t+2, 2t-1), (2t-1, 2t+3, 2t+2), (2t+2, 2t+3, 2t-1),$$
$$(2t+2, 2t-1, 2t+3), (2t-1, 2t+2, 2t+3), (2t+3, 2t-1, 2t+2),$$
$$(2t+3, 2t+2, 2t-1), (2t-1, 2t+3, 2t+2), (2t+2, 2t+3, 2t-1),$$
$$(2t+3, 2t+3, 2t-2), (2t-2, 2t+3, 2t+3), (2t-2, 2t+3, 2t+3)\}.$$

The corresponding generator matrices $G$ have the property that $\text{rank}(GG^T) = 2$ which implies that $\dim(\text{Hull}(C)) = 0 \neq 1$ by Proposition 2.2.4.

**Case 2**. $n - 1 = S_{01} + S_{10} + S_{11}$. The existence of an $[n, 2, \lfloor \frac{2n}{3} \rfloor - 1]_2$ code with hull dimension one is given as follows.

**Case 2.1**. $n \equiv 0 (\text{mod } 6)$. Then $n - 1 \equiv 5 (\text{mod } 6)$. By Theorem 3.4.1, there exists an $[n-1, 2, \left\lfloor \frac{2(n-1)}{3} \right\rfloor]_2$ code with generator matrix $G$ and hull dimension one. Let $G' = [\mathbf{0}\ G]$. It is not difficult to see that $G'$ generates an $[n, 2, \left\lfloor \frac{2(n-1)}{3} \right\rfloor]_2$ code $D$ with hull dimension one. Since $\left\lfloor \frac{2(n-1)}{3} \right\rfloor = \lfloor \frac{2n}{3} \rfloor - 1$, $D$ has parameters $[n, 2, \lfloor \frac{2n}{3} \rfloor - 1]_2$.

**Case 2.2**. $n \equiv 2 (\text{mod } 6)$. Then $n-1 \equiv 1 (\text{mod } 6)$. By Theorem 3.2.1, there exists an an $[n-1, 2, \left\lfloor \frac{2(n-1)}{3} \right\rfloor]_2$ code with generator matrix $G$ and hull dimension one. Let $G' = [\mathbf{0}\ G]$. It is not difficult to see that $G'$ generates an $[n, 2, \left\lfloor \frac{2(n-1)}{3} \right\rfloor]_2$ code $D$ with hull dimension one. Since $\left\lfloor \frac{2(n-1)}{3} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor - 1$, $D$ has parameters $[n, 2, \left\lfloor \frac{2n}{3} \right\rfloor - 1]_2$.

**Case 2.3**. $n \equiv 4 (\text{mod } 6)$. Then $n-1 \equiv 3 (\text{mod } 6)$. By Theorem 3.3.1, there exists an an $[n-1, 2, \left\lfloor \frac{2(n-1)}{3} \right\rfloor - 1]_2$ code with generator matrix $G$ and hull dimension one. Let $G' = [\mathbf{0}\ G]$. It is not difficult to see that $G'$ generates an $[n, 2, \left\lfloor \frac{2(n-1)}{3} \right\rfloor - 1]_2$ code $D$ with hull dimension one. Since $\left\lfloor \frac{2(n-1)}{3} \right\rfloor - 1 = \left\lfloor \frac{2n}{3} \right\rfloor - 1$, $D$ has parameters $[n, 2, \left\lfloor \frac{2n}{3} \right\rfloor - 1]_2$.

The enumeration for each case follows from Theorem 3.4.1, Theorem 3.2.1, and Theorem 3.3.1, respectively. □

**Example 3.5.3.** By Example 3.4.2, we have that $C$ is a linear code with parameters $[17, 2, 8]_2$ generated by

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $G' = [\mathbf{0}\ G]$. By Theorem 3.5.2, we have that $G'$ generates an optimal $[18, 2, 7]_2$ code with hull dimension one.

**Example 3.5.4.** By Example 3.2.2, we have that $C$ is a linear code with parameters $[13, 2, 8]_2$ generated by

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $G' = [\mathbf{0}\ G]$. By Theorem 3.5.2, we have that $G'$ generates an optimal $[14, 2, 7]_2$ code with hull dimension one.

**Example 3.5.5.** By Example 3.3.2, we have that $C_1$ is a linear code with parameters $[15, 2, 9]_2$ generated by

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $G_1' = [\mathbf{0}\ G_1]$. By Theorem 3.5.2, we have that $G_1'$ generates an optimal $[16, 2, 8]_2$ code with hull dimension one.

**Example 3.5.6.** By Example 3.3.2, we have that $C_2$ is a linear code with parameters $[15, 2, 9]_2$ generated by

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Let $G_2' = [\mathbf{0}\ G_2]$. By Theorem 3.5.2, we have that $G_2'$ generates an optimal $[16, 2, 8]_2$ code with hull dimension one. By Theorem 3.5.2, we have that $G_1'$ and $G_2'$ generate two inequivalent codes.

The above results can be summarized as follows.

**Theorem 3.5.7.** *Let $n \geq 3$ be an integer. Then*

$$D_2(n, 2, 1) = \begin{cases} \left\lfloor \frac{2n}{3} \right\rfloor & \text{if } n \equiv 1, 5 \pmod 6, \\ \left\lfloor \frac{2n}{3} \right\rfloor - 1 & \text{if } n \equiv 0, 2, 3, 4 \pmod 6. \end{cases}$$

# Chapter 4

# Optimal Ternary Linear Codes with Euclidean Hull Dimension One

In this chapter, we focus on constructions of $[n,2]_3$ codes with hull dimension one. For each integer $n \geq 3$, an optimal $[n,2]_3$ code with hull dimension one is constructed. Consequently, the exact value of $D_3(n,2,1)$ is presented for all integers $n \geq 3$.

## 4.1 Basic Concepts

From the setup in Subsection 2.3 or in [19], a linear $[n,2]_3$ code $C$ with generator matrix

$$G = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{4.1}$$

can be viewed of the form $C = \{0, \alpha_1, \alpha_2, 2\alpha_1, 2\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+2\alpha_2, 2\alpha_1+2\alpha_2\}$. From (2.3), we have $\alpha_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$

and it is not difficult to see that

$$\mathrm{wt}(\alpha_1) = \mathrm{wt}(2\alpha_1) = \sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij}, \tag{4.2}$$

$$\mathrm{wt}(\alpha_2) = \mathrm{wt}(2\alpha_2) = \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij}, \tag{4.3}$$

$$\mathrm{wt}(\alpha_1 + \alpha_2) = \mathrm{wt}(2\alpha_1 + 2\alpha_2) = S_{01} + S_{02} + S_{10} + S_{11} + S_{20} + S_{22}, \tag{4.4}$$

$$\mathrm{wt}(2\alpha_1 + \alpha_2) = \mathrm{wt}(\alpha_1 + 2\alpha_2) = S_{01} + S_{02} + S_{10} + S_{12} + S_{20} + S_{21}. \tag{4.5}$$

Hence,

$$\begin{aligned}
\mathrm{wt}_H(C) &= \min\{\mathrm{wt}(\alpha_1), \mathrm{wt}(2\alpha_1), \mathrm{wt}(\alpha_2), \mathrm{wt}(2\alpha_2), \mathrm{wt}(\alpha_1 + \alpha_2),\\
&\qquad \mathrm{wt}(2\alpha_1 + 2\alpha_2), \mathrm{wt}(2\alpha_1 + \alpha_2), \mathrm{wt}(\alpha_1 + 2\alpha_2)\}\\
&= \min\{\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij}, \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij}, S_{01} + S_{02} + S_{10} + S_{11} + S_{20} + S_{22},\\
&\qquad S_{01} + S_{02} + S_{10} + S_{12} + S_{20} + S_{21}\}. \qquad (4.6)
\end{aligned}$$

From (4.1), we have

$$GG^T = \begin{bmatrix} \sum\limits_{i=1}^{2}\sum\limits_{j=0}^{2} S_{ij} & \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} ijS_{ij} \\ \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} ijS_{ij} & \sum\limits_{i=0}^{2}\sum\limits_{j=1}^{2} S_{ij} \end{bmatrix} \pmod 3. \tag{4.7}$$

**Example 4.1.1.** Let $C$ be the ternary linear code of length 5 with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

Then $S_{10} = S_{01} = S_{11} = S_{02} = S_{21} = 1$ and $S_{00} = S_{20} = S_{12} = S_{22} = 0$. In $\mathbb{F}_3$, it follows that

$$\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij} = 0, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 0, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 0 \text{ and } \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij} = 1.$$

By (4.7), it follows that

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which implies that $\mathrm{rank}(GG^T) = 1$ and $\dim(\mathrm{Hull_H}(C)) = 2 - \mathrm{rank}(GG^T) = 1$ by Proposition 2.2.4.

From Lemma 2.3.4, it can be concluded that

$$D_3(n,2,1) \le \left\lfloor \frac{3n}{4} \right\rfloor. \tag{4.8}$$

Using the notations above, the exact value for $D_3(n,2,1)$ can be derived in the following sections.

## 4.2 $n \equiv 1 \pmod 4$

In the following theorem, we give constructions of an optimal $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$ code with dimesion 2 and Euclidean hull dimension one is given for all lengths $n \equiv 1 \pmod 4$.

**Theorem 4.2.1.** *Let $n \geq 3$ be an integer. If $n \equiv 1 \pmod 4$, then*

$$D_3(n,2,1) = \left\lfloor \frac{3n}{4} \right\rfloor.$$

*Proof.* Assume that $n \equiv 1 \pmod 4$. Then $n = 4t+1$ for some positive integer $t$ and $\lfloor \frac{3n}{4} \rfloor = \left\lfloor \frac{3(4t+1)}{4} \right\rfloor = 3t$.

From (3.4), it suffices to show the existence of an $[n, 2, 3t]_3$ code with hull dimension one.

**Case 1**. $t$ is even. Let $C$ be linear code with generator matrix $G$ of the form (4.1) determined by

$$S_{00} = 0,\ S_{01} = S_{10} = S_{02} = S_{20} = S_{12} = S_{11} = S_{22} = \frac{t}{2},\ \text{and } S_{21} = \frac{t}{2} + 1.$$

Then $C$ is an $[n, 2, d]_3$ code, where

$$\begin{aligned} d &= \min\{\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij}, \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij}, S_{01}+S_{02}+S_{10}+S_{11}+S_{20}+S_{22}, \\ &\qquad S_{01}+S_{02}+S_{10}+S_{12}+S_{20}+S_{21}\} \\ &= \min\{3t+1, 3t+1, 3t, 3t+1\} \qquad \text{by (4.6)} \\ &= 3t. \end{aligned}$$

From (4.7), we have

$$GG^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$. As desired, we have $\dim(\text{Hull}(C)) = 1$ by Proposition 2.2.4.

**Case 2**. $t$ is odd. Let $C$ be linear code with generator matrix $G$ of the form (4.1) determined by

$$S_{00} = 0,\ S_{02} = S_{20} = S_{12} = S_{21} = S_{11} = \frac{t+1}{2} \ \text{ and } \ S_{01} = S_{10} = S_{22} = \frac{t-1}{2}.$$

Then $C$ is an $[n, 2, d]_3$ code, where

$$\begin{aligned} d &= \min\{\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij}, \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij}, S_{01} + S_{02} + S_{10} + S_{11} + S_{20} + S_{22}, \\ &\qquad S_{01} + S_{02} + S_{10} + S_{12} + S_{20} + S_{21}\} \\ &= \min\{3t+1, 3t+1, 3t, 3t+1\} \qquad \text{by (4.6)} \\ &= 3t. \end{aligned}$$

From (4.7), we have

$$GG^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$. By Proposition 2.2.4, we therefor have $\dim(\text{Hull}(C)) = 1$ as desired. □

**Example 4.2.2.** Let $t = 2$. By Theorem 4.2.1 and $t$ is even, we have that $S_{00} = 0, S_{01} = S_{10} = S_{02} = S_{20} = S_{12} = S_{11} = S_{22} = 1$ and $S_{21} = 2$. Let $C$ be the ternary linear code of length 9 with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 2 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

In $\mathbb{F}_3$, we have

$$\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij} = 1, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 2, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 2 \text{ and } \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij} = 1.$$

By (4.7), it follows that

$$G = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

which implies that $\mathrm{rank}(GG^T) = 1$ and $\dim(\mathrm{Hull}(C)) = 2 - \mathrm{rank}(GG^T) = 1$ by Proposition 2.2.4. Then $C$ is an optimal $[9, 2, 6]_3$ code with hull dimension one by (4.8).

## 4.3 $n \equiv 2 \pmod 4$

In the following theorem, we give constructions of optimal $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$ codes of such lengths with hull dimension one where $n \equiv 2 \pmod 4$.

**Theorem 4.3.1.** *Let $n \geq 3$ be an integer. If $n \equiv 2 \pmod 4$, then*

$$D_3(n, 2, 1) = \left\lfloor \frac{3n}{4} \right\rfloor.$$

*Proof.* Assume that $n \equiv 2 \pmod 4$. Then $n = 4t + 2$ for some positive integer $t$ and $\lfloor \frac{3n}{4} \rfloor = \left\lfloor \frac{3(4t+2)}{4} \right\rfloor = 3t + 1$.

From the bound in (4.8), it suffices to show the existence of an $[n, 2, 3t + 1]_3$ code with hull dimension one.

**Case 1**. $t$ is even. Let $C$ be linear code with generator matrix $G$ of the form (4.1) determined by

$$S_{00} = 0,\ S_{01} = S_{10} = S_{11} = \frac{t}{2} + 1,\ S_{02} = S_{20} = S_{21} = S_{22} = \frac{t}{2} \text{ and } S_{12} = \frac{t}{2} - 1.$$

Then $C$ is an $[n, 2, d]_3$ code, where

$$\begin{aligned} d &= \min\{\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij}, \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij}, S_{01} + S_{02} + S_{10} + S_{11} + S_{20} + S_{22}, \\ &\qquad S_{01} + S_{02} + S_{10} + S_{12} + S_{20} + S_{21}\} \\ &= \min\{3t + 1, 3t + 1, 3t + 3, 3t + 1\} \qquad \text{by (4.6)} \\ &= 3t + 1. \end{aligned}$$

From (4.7), we have

$$GG^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$. Hence, $\dim(\text{Hull}(C)) = 1$ by Proposition 2.2.4.

**Case 2**. $t$ is odd. Let $C$ be linear code with generator matrix $G$ of the form (4.1) determined by

$$S_{00} = 0,\ S_{01} = S_{02} = S_{10} = S_{20} = S_{11} = S_{22} = \frac{t+1}{2} \text{ and } \ S_{12} = S_{21} = \frac{t-1}{2}.$$

Then $C$ is an $[n, 2, d]_3$ code, where

$$\begin{aligned} d &= \min\{\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij}, \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij}, S_{01} + S_{02} + S_{10} + S_{11} + S_{20} + S_{22}, \\ &\qquad S_{01} + S_{02} + S_{10} + S_{12} + S_{20} + S_{21}\} \\ &= \min\{3t+1, 3t+1, 3t+3, 3t+1\} \ \text{ by (4.6)} \\ &= 3t+1. \end{aligned}$$

From (4.7), we have

$$GG^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$. Hence, $\dim(\text{Hull}(C)) = 1$ by Proposition 2.2.4.

□

**Example 4.3.2.** For $t = 2$. By Theorem 4.3.1 and $t$ is even, we have that $S_{00} = S_{12} = 0, S_{02} = S_{20} = S_{21} = S_{22} = 1$ and $S_{01} = S_{10} = S_{11} = 2$. Let $C$ be the ternary linear code of length 10 with generator matrix

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}.$$

In $\mathbb{F}_3$, we have

$$\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij} = 1, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 2, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 2 \text{ and } \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij} = 1.$$

By (4.7), it follows that

$$G = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

which implies that $\mathrm{rank}(GG^T) = 1$ and $\dim(\mathrm{Hull}(C)) = 2 - \mathrm{rank}(GG^T) = 1$ by Proposition 2.2.4. Then $C$ is an optimal $[10, 2, 7]_3$ code with hull dimension one.

## 4.4 $n \equiv 3 \pmod 4$

In the following theorem, we give constructions of an optimal $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$ code with dimesion 2 and Euclidean hull dimension one is given for all lengths $n \equiv 3 \pmod 4$.

**Theorem 4.4.1.** *Let $n \geq 3$ be an integer. If $n \equiv 3 \pmod 4$, then*

$$D_3(n, 2, 1) = \left\lfloor \frac{3n}{4} \right\rfloor.$$

*Proof.* Assume that $n \equiv 3 \pmod 4$. Then $n = 4t + 3$ for some positive integer $t$ and $\lfloor \frac{3n}{4} \rfloor = \left\lfloor \frac{3(4t+3)}{4} \right\rfloor = 3t + 2$.

From (4.8), it suffices to construct an $[n, 2, 3t + 2]_2$ code with hull dimension one. Let $C$ be linear code with generator matrix $G$ of the form (4.1) determined by

$$S_{00} = 0,\ S_{01} + S_{02} = S_{10} + S_{20} = S_{11} + S_{22} = t + 1 \text{ and } S_{12} + S_{21} = t.$$

Then $C$ is an $[n, 2, d]_3$ code, where

$$\begin{aligned} d &= \min\{\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij}, \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij}, S_{01} + S_{02} + S_{10} + S_{11} + S_{20} + S_{22}, \\ &\qquad S_{01} + S_{02} + S_{10} + S_{12} + S_{20} + S_{21}\} \\ &= \min\{3t + 2, 3t + 2, 3t + 3, 3t + 2\} \qquad \text{by (4.6)} \\ &= 3t + 2. \end{aligned}$$

From (4.7), we have

$$GG^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$. Therefore, $\dim(\text{Hull}(C)) = 1$ by Proposition 2.2.4. $\square$

**Example 4.4.2.** Let $t = 2$. By Theorem 4.4.1 and $t$ is even, we have that $S_{00} = 0, S_{10} = S_{01} = S_{11} = S_{12} = S_{21} = 1$ and $S_{20} = S_{02} = S_{22} = 2$. Let $C$ be the ternary linear code of length 11 with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

In $\mathbb{F}_3$, we have

$$\sum_{i=1}^{2}\sum_{j=0}^{2} S_{ij} = 2, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 1, \sum_{i=1}^{2}\sum_{j=1}^{2} ijS_{ij} = 1 \text{ and } \sum_{i=0}^{2}\sum_{j=1}^{2} S_{ij} = 2.$$

By (4.7), it follows that

$$G = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$ and $\dim(\text{Hull}(C)) = 2 - \text{rank}(GG^T) = 1$ by Proposition 2.2.4. Then $C$ is an optimal $[11, 2, 8]_3$ code with hull dimension one by (4.8).

## 4.5 $n \equiv 0 \pmod 4$

Next lemma, we show that there does not exist $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$ codes with hull dimension one for all $n \equiv 0 \pmod 4$. Later, a construction of an optimal $[n, 2, \lfloor \frac{3n}{4} \rfloor - 1]_3$ code with hull dimension one is given for all lengths $n \equiv 0 \pmod 4$.

**Lemma 4.5.1.** *Let $n \geq 3$ be an integer and let $C$ be an $[n, 2]_3$ code. If $n \equiv 0 \pmod 4$ and $\text{wt}(C) = \lfloor \frac{3n}{4} \rfloor$, then weight of every non-zero codeword is $\lfloor \frac{3n}{4} \rfloor$.*

*Proof.* Assume that $n \equiv 0 \pmod 4$ and $C$ has parameters $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$. Then $n = 4t$ for some positive integer $t$ and $\lfloor \frac{3n}{4} \rfloor = 3t$.

Suppose that $S_{00} > 0$. Then, by puncturing $C$ at one of the zero columns, an $[n-1, 2, \lfloor \frac{3n}{4} \rfloor]_3$ code $C'$ is obtained. By Theorem 2.3.3, we have

$$3t = \left\lfloor \frac{3n}{4} \right\rfloor = \mathrm{wt}(C') \le \left\lfloor \frac{3(n-1)}{4} \right\rfloor = 3t - 1,$$

a contradiction. Hence,

$$S_{00} = 0 \text{ and } n = S_{01} + S_{02} + S_{10} + S_{20} + S_{11} + S_{12} + S_{21} + S_{22}.$$

From (4.2), (4.3), (4.4) and (4.5), it is easily seen that

$$\sum_{\boldsymbol{c} \in C \setminus \{\boldsymbol{0}\}} \mathrm{wt}(\boldsymbol{c}) = 6(S_{01} + S_{02} + S_{10} + S_{20} + S_{11} + S_{12} + S_{21} + S_{22}) = 6n = 24t.$$

Since $C \setminus \{\boldsymbol{0}\}$ contains 8 codewords and $\mathrm{wt}(C) = 3t$, every codeword in $C \setminus \{\boldsymbol{0}\}$ has weight $3t = \lfloor \frac{3n}{4} \rfloor$. □

**Lemma 4.5.2.** *Let $n \ge 3$ be an integer. If $n \equiv 0 \pmod 4$, then there are no $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$ codes with hull dimension one.*

*Proof.* Assume that $n \equiv 0 \pmod 4$. Then

$$\left\lfloor \frac{3(n-1)}{4} \right\rfloor < \frac{3n}{4} = \left\lfloor \frac{3n}{4} \right\rfloor.$$

Suppose that there exists an $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$ code $C$ with generator matrix $G$. If $S_{00} > 0$, then there exists an $[n-1, 2, \lfloor \frac{3n}{4} \rfloor]_2$ code $C'$ by puncturing at one of the zero columns. By (4.8), we have $\lfloor \frac{3n}{4} \rfloor = \mathrm{wt}(C') \le \lfloor \frac{3(n-1)}{4} \rfloor < \lfloor \frac{3n}{4} \rfloor$, a contradiction. Hence, $S_{00} = 0$ which implies that

$$S_{01} + S_{02} + S_{10} + S_{20} + S_{11} + S_{12} + S_{21} + S_{22} = n.$$

By Lemma 4.5.1, the weight of every non-zero codeword in $C$ is $\lfloor \frac{3n}{4} \rfloor$. From (4.7), we have that

$$GG^T = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$$

where $x \in \{0, 1, 2\}$ which implies that $\mathrm{rank}(GG^T) = 0$ or 2. By Proposition 2.2.4, $\dim(\mathrm{Hull}(C)) = 0$ or 2 which means $\dim(\mathrm{Hull}(C)) \neq 1$. Hence, there are no $[n, 2, \lfloor \frac{3n}{4} \rfloor]_3$ codes with hull dimension one. □

**Theorem 4.5.3.** *Let $n \geq 3$ be an integer. If $n \equiv 0 \pmod 4$, then*

$$D_3(n,2,1) = \left\lfloor \frac{3n}{4} \right\rfloor - 1.$$

*Proof.* Assume that $n \equiv 0 \pmod 4$. Then $n = 4t$ for some positive integer $t$ and

$$\left\lfloor \frac{3n}{4} \right\rfloor - 1 = \left\lfloor \frac{3(4t)}{4} \right\rfloor - 1 = 3t - 1.$$

From Lemma 4.5.2, it is enough to give a construction an $[n, 2, 3t-1]_3$ code with hull dimension one. Since $n - 1 = 4t - 1 = 4(t-1) + 3$, there exists an $[n-1, 2, 3(t-1)+2 = 3t-1]_3$ code with generator matrix $G$ and hull dimension one by Theorem 4.4.1. Let $G' = [\mathbf{0}\ G]$. It is not difficult to see that $G'$ generates an $[n, 2, 3t-1]_3$ code with hull dimension one. □

**Example 4.5.4.** By Example 4.4.2, we have that $C$ is a linear code with parameters $[11, 2, 8]_3$ generated by

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 2 & 1 & 2 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 2 & 2 & 1 \end{bmatrix}.$$

Let $G' = [\mathbf{0}\ G]$. By Theorem 4.5.3, we have that $G'$ generates an optimal $[12, 2, 7]_3$ code with hull dimension one.

The above results on $D_3(n, 2, 1)$ can be summarized as follows.

**Theorem 4.5.5.** *Let $n \geq 3$ be an integer. Then*

$$D_3(n,2,1) = \begin{cases} \lfloor \frac{3n}{4} \rfloor - 1 & \text{if } n \equiv 0 \pmod 4, \\ \lfloor \frac{3n}{4} \rfloor & \text{if } n \equiv 1, 2, 3 \pmod 4. \end{cases}$$

# Chapter 5

# Quaternary Linear Codes with Hermitian Hull Dimension One

In this chapter, a brief discussion on Hermitian hull of linear codes is given as well as the construction ideas for linear codes with prescribed Hermitian hull dimension. From now on, we focus on quaternary linear codes of dimension two and Hermitian hull dimension one.

Using the analysis on a generator matrix of a linear code similar to that of [7], we derived the following results.

## 5.1 Basic Concepts

Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2 = \omega + 1\}$ and let $C$ be an $[n, 2]_4$ code over $\mathbb{F}_4$ with generator matrix

$$G = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}.$$

By setting $\beta_l = \begin{bmatrix} a_{1l} \\ a_{2l} \end{bmatrix}$ for all $1 \leq l \leq n$, the matrix $G$ can be viewed of the form

$$G = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \end{bmatrix}. \tag{5.1}$$

For $i, j \in \mathbb{F}_4$, let $S_{ij} := |\{l \in \{1, 2, \ldots, n\} \mid \beta_l = \begin{bmatrix} i \\ j \end{bmatrix}\}|$. It is not difficult to see that the generator matrix $G$ and the code $C$ are determined explicitly by the values $S_{ij}$ for all $i, j \in \mathbb{F}_4$. For constructions of quternary linear codes, it suffices to established the values $S_{ij}$.

Alternatively, let $\alpha_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$. Then $G$ and $C$ can be view as $G = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and

$$C = \{0, \alpha_1, \alpha_2, \omega\alpha_1, \omega^2\alpha_1, \omega\alpha_2, \omega^2\alpha_2, \alpha_1 + \alpha_2, \omega\alpha_1 + \alpha_2, \omega^2\alpha_1 + \alpha_2, \alpha_1 + \omega\alpha_2, \alpha_1 + \omega^2\alpha_2, \omega\alpha_1 + \omega\alpha_2, \omega^2\alpha_1 + \omega\alpha_2, \omega\alpha_1 + \omega^2\alpha_2, \omega^2\alpha_1 + \omega^2\alpha_2\},$$

respectively.

By inspection, it is easily seen that

$$\mathrm{wt}(\alpha_1) = \mathrm{wt}(\omega\alpha_1) = \mathrm{wt}(\omega^2\alpha_1) = \sum_{i \in \mathbb{F}_4^*} \sum_{j \in \mathbb{F}_4} S_{ij}, \tag{5.2}$$

$$\mathrm{wt}(\alpha_2) = \mathrm{wt}(\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_2) = \sum_{i \in \mathbb{F}_4} \sum_{j \in \mathbb{F}_4^*} S_{ij}, \tag{5.3}$$

$$\mathrm{wt}(\alpha_1 + \alpha_2) = \mathrm{wt}(\omega\alpha_1 + \omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_1 + \omega^2\alpha_2) = \sum_{i \in \mathbb{F}_4} \sum_{j \in \mathbb{F}_4, i \neq j} S_{ij}, \tag{5.4}$$

$$\mathrm{wt}(\alpha_1 + \omega\alpha_2) = \mathrm{wt}(\omega\alpha_1 + \omega^2\alpha_2) = \mathrm{wt}(\omega^2\alpha_1 + \alpha_2) = \sum_{i \in \mathbb{F}_4} \sum_{j \in \mathbb{F}_4, i \neq \omega j} S_{ij}, \tag{5.5}$$

and

$$\mathrm{wt}(\alpha_1 + \omega^2\alpha_2) = \mathrm{wt}(\omega\alpha_1 + \alpha_2) = \mathrm{wt}(\omega^2\alpha_1 + \omega\alpha_2) = \sum_{i \in \mathbb{F}_4} \sum_{j \in \mathbb{F}_4, i \neq \omega^2 j} S_{ij}. \tag{5.6}$$

Let $y_0 = S_{11} + S_{\omega\omega} + S_{\omega^2\omega^2},\ y_1 = S_{1\omega^2} + S_{\omega 1} + S_{\omega^2\omega}$, and $y_2 = S_{\omega^2 1} + S_{1\omega} + S_{\omega\omega^2}$. Then we have

$$GG^{\dagger} = \begin{bmatrix} \sum_{i\in\mathbb{F}_4^*}\sum_{j\in\mathbb{F}_4} S_{ij} & y_0 + \omega y_1 + \omega^2 y_2 \\ y_0 + \omega^2 y_1 + \omega y_2 & \sum_{i\in\mathbb{F}_4}\sum_{j\in\mathbb{F}_4^*} S_{ij} \end{bmatrix}, \tag{5.7}$$

where the calculation is done in $\mathbb{F}_4$.

The construction is illustrated in the following example.

**Example 5.1.1.** Let $C$ be a quaternary linear code of length 4 determined by $S_{10} = S_{01} = S_{11} = S_{1\omega} = 1$ and $S_{00} = S_{0\omega} = S_{0\omega^2} = S_{1\omega^2} = S_{\omega 0} = S_{\omega 1} = S_{\omega\omega} = S_{\omega\omega^2} = S_{\omega^2 0} = S_{\omega^2 1} = S_{\omega^2\omega} = S_{\omega^2\omega^2} = 0$. Then the generator matrix of $C$ is of the form

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \omega \end{bmatrix}.$$

Consequently, we have

$$y_0 = 1,\ y_1 = 0,\ \text{and}\ y_2 = 1.$$

In $\mathbb{F}_4$, we have

$$\sum_{i\in\mathbb{F}_4^*}\sum_{j\in\mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i\in\mathbb{F}_4}\sum_{j\in\mathbb{F}_4^*} S_{ij} = 1.$$

By (5.7), it follows that

$$GG^{\dagger} = \begin{bmatrix} 1 & 1+\omega^2 \\ 1+\omega & 1 \end{bmatrix} = \begin{bmatrix} 1 & \omega \\ \omega^2 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^{\dagger}) = 1$ and $\dim(\text{Hull}_{\text{H}}(C)) = 2 - \text{rank}(GG^{\dagger}) = 1$ by Proposition 2.2.4.

By setting $k = 2$ and $q = 4$ in Lemma 2.3.4, the next lemma follows.

**Lemma 5.1.2.** $D_4^{\text{H}}(n,2,1) \leq \left\lfloor \frac{4n}{5} \right\rfloor$ *for all integers* $n \geq 3$.

## 5.2 Optimal Quaternary Linear Codes with Hermitian Hull Dimension One with $n \equiv 1, 2, 4 \pmod 5$

In this section, we focus on constructions of optimal quaternary linear codes with dimension 2 and Hermitian hull dimension one, and establish the exact values of $D_4^{\mathrm{H}}(n, 2, 1)$ for arbitrary lengths $n \geq 3$ such that $n \equiv 1, 2, 4 \pmod 5$.

**Theorem 5.2.1.** *Let $n \geq 3$ be an integer. If $n \equiv 1, 2, 4 \pmod 5$, then*

$$D_4^{\mathrm{H}}(n, 2, 1) = \left\lfloor \frac{4n}{5} \right\rfloor.$$

*Proof.* Assume that $n \equiv 1, 2, 4 \pmod 5$. From Lemma 5.1.2, it follows that $D_4^{\mathrm{H}}(n, 2, 1) \leq \left\lfloor \frac{4n}{5} \right\rfloor$. It remains to show the existence of a $\left[n, 2, \left\lfloor \frac{4n}{5} \right\rfloor\right]_4$ code whose Hermitian hull dimension is one.

We consider the constructions in the following 3 cases.

**Case 1**. $n \equiv 1 \pmod 5$. Then $n = 5t + 1$ for some positive integer $t$. Based on the parity of $t$, we consider the following two subcases.

**Case 1.1**. $t$ is even. For $1 \leq r \leq n$, let $\beta_r$ be the $2 \times 1$ matrix over $\mathbb{F}_4$ defined by

$$\beta_r = \begin{cases} [0\ 1]^T & \text{if } 1 \leq r \leq \frac{t}{2}, \\ [0\ \omega]^T & \text{if } \frac{t}{2}+1 \leq r \leq \frac{2t}{2}, \\ [\omega\ 0]^T & \text{if } \frac{2t}{2}+1 \leq r \leq \frac{3t}{2}, \\ [1\ 1]^T & \text{if } \frac{3t}{2}+1 \leq r \leq \frac{4t}{2}, \\ [\omega^2\ \omega^2]^T & \text{if } \frac{4t}{2}+1 \leq r \leq \frac{5t}{2} \\ [1\ \omega^2]^T & \text{if } \frac{5t}{2}+1 \leq r \leq \frac{6t}{2}, \\ [\omega^2\ \omega]^T & \text{if } \frac{6t}{2}+1 \leq r \leq \frac{7t}{2}, \\ [\omega^2\ 1]^T & \text{if } \frac{7t}{2}+1 \leq r \leq \frac{8t}{2}, \\ [\omega\ \omega^2]^T & \text{if } \frac{8t}{2}+1 \leq r \leq \frac{9t}{2}, \\ [1\ 0]^T & \text{if } \frac{9t}{2}+1 \leq r \leq \frac{10t}{2}+1. \end{cases}.$$

Let $C$ be an $[n, 2]_4$ code generated by

$$G = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}.$$

Then $S_{01} = S_{0\omega} = S_{\omega 0} = S_{11} = S_{\omega^2\omega^2} = S_{1\omega^2} = S_{\omega^2\omega} = S_{\omega^2 1} = S_{\omega\omega^2} = \frac{t}{2}$, $S_{10} = \frac{t}{2} + 1$, and $S_{ij} = 0$ otherwise. It follows that

$$\begin{aligned} y_0 &= S_{11} + S_{\omega\omega} + S_{\omega^2\omega^2} = t, \\ y_1 &= S_{1\omega^2} + S_{\omega 1} + S_{\omega^2\omega} = t, \text{ and} \\ y_2 &= S_{\omega^2 1} + S_{1\omega} + S_{\omega\omega^2} = t. \end{aligned}$$

In $\mathbb{F}_4$, it can be deduced that

$$\sum_{i \in \mathbb{F}_4^*} \sum_{j \in \mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i \in \mathbb{F}_4} \sum_{j \in \mathbb{F}_4^*} S_{ij} = 0.$$

By (5.7), we have

$$GG^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies that $\mathrm{rank}(GG^\dagger) = 1$. By Proposition 2.2.4, $C$ has Hermitian hull dimension $2 - \mathrm{rank}(GG^\dagger) = 1$.

By the definition of $S_{ij}$ and (5.2)–(5.6), it can be deduced that

$$\begin{aligned} \mathrm{wt}(\alpha_1) = \mathrm{wt}(\omega\alpha_1) = \mathrm{wt}(\omega^2\alpha_1) &= 4t + 1, \\ \mathrm{wt}(\alpha_2) = \mathrm{wt}(\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_2) &= 4t, \\ \mathrm{wt}(\alpha_1 + \alpha_2) = \mathrm{wt}(\omega\alpha_1 + \omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_1 + \omega^2\alpha_2) &= 4t + 1, \\ \mathrm{wt}(\alpha_1 + \omega\alpha_2) = \mathrm{wt}(\omega\alpha_1 + \omega^2\alpha_2) = \mathrm{wt}(\omega^2\alpha_1 + \alpha_2) &= 4t + 1, \text{ and} \\ \mathrm{wt}(\alpha_1 + \omega^2\alpha_2) = \mathrm{wt}(\omega\alpha_1 + \alpha_2) = \mathrm{wt}(\omega^2\alpha_1 + \omega\alpha_2) &= 4t + 1. \end{aligned}$$

Hence, $C$ has minimum weight $4t = \left\lfloor \frac{4(5t+1)}{5} \right\rfloor = \left\lfloor \frac{4n}{5} \right\rfloor$. Therefore, $C$ is an optimal $[n, 2, \lfloor \frac{4n}{5} \rfloor]_4$ code with Hermitian hull dimension one.

**Case 1.2**. $t$ is odd. For $1 \leq r \leq n$, let $\beta_r$ be the $2 \times 1$ matrix over $\mathbb{F}_4$ defined by

$$\beta_r = \begin{cases} [0\ \omega]^T & \text{if } 1 \leq r \leq \frac{t+1}{2}, \\ [1\ 0]^T & \text{if } \frac{t+1}{2} + 1 \leq r \leq \frac{2(t+1)}{2}, \\ [\omega\ 0]^T & \text{if } \frac{2(t+1)}{2} + 1 \leq r \leq \frac{3(t+1)}{2}, \\ [\omega^2\ \omega^2]^T & \text{if } \frac{3(t+1)}{2} + 1 \leq r \leq \frac{4(t+1)}{2}, \\ [\omega^2\ \omega]^T & \text{if } \frac{4(t+1)}{2} + 1 \leq r \leq \frac{5(t+1)}{2}, \\ [\omega\ \omega^2]^T & \text{if } \frac{5(t+1)}{2} + 1 \leq r \leq \frac{6(t+1)}{2}, \\ [0\ 1]^T & \text{if } \frac{6(t+1)}{2} + 1 \leq r \leq \frac{7t+5}{2}, \\ [1\ 1]^T & \text{if } \frac{7t+5}{2} + 1 \leq r \leq \frac{8t+4}{2}, \\ [1\ \omega^2]^T & \text{if } \frac{8t+4}{2} + 1 \leq r \leq \frac{9t+3}{2}, \\ [\omega^2\ 1]^T & \text{if } \frac{9t+3}{2} + 1 \leq r \leq \frac{10t+2}{2}. \end{cases}$$

Let $C$ be an $[n, 2]_4$ code generated by

$$G = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}.$$

Then $S_{0\omega} = S_{10} = S_{\omega 0} = S_{\omega^2\omega^2} = S_{\omega^2\omega} = S_{\omega\omega^2} = \frac{t+1}{2}$, $S_{01} = S_{11} = S_{1\omega^2} = S_{\omega^2 1} = \frac{t-1}{2}$, and $S_{ij} = 0$ otherwise. It follows that

$$y_0 = S_{11} + S_{\omega\omega} + S_{\omega^2\omega^2} = t,$$

$$y_1 = S_{1\omega^2} + S_{\omega 1} + S_{\omega^2\omega} = t, \text{ and}$$

$$y_2 = S_{\omega^2 1} + S_{1\omega} + S_{\omega\omega^2} = t.$$

In $\mathbb{F}_4$, we have

$$\sum_{i \in \mathbb{F}_4^*} \sum_{j \in \mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i \in \mathbb{F}_4} \sum_{j \in \mathbb{F}_4^*} S_{ij} = 0.$$

From (5.7), it can be concluded that

$$GG^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

From Proposition 2.2.4, it can be concluded that $C$ has Hermitian hull dimension $2 - \text{rank}(GG^{\dagger}) = 2 - 1 = 1$.

Using the definition of $S_{ij}$ and (5.2)–(5.6), we have

$$\begin{aligned}
\text{wt}(\alpha_1) = \text{wt}(\omega\alpha_1) &= \text{wt}(\omega^2\alpha_1) = 4t+1,\\
\text{wt}(\alpha_2) = \text{wt}(\omega\alpha_2) &= \text{wt}(\omega^2\alpha_2) = 4t,\\
\text{wt}(\alpha_1+\alpha_2) = \text{wt}(\omega\alpha_1+\omega\alpha_2) &= \text{wt}(\omega^2\alpha_1+\omega^2\alpha_2) = 4t+1,\\
\text{wt}(\alpha_1+\omega\alpha_2) = \text{wt}(\omega\alpha_1+\omega^2\alpha_2) &= \text{wt}(\omega^2\alpha_1+\alpha_2) = 4t+1, \text{ and}\\
\text{wt}(\alpha_1+\omega^2\alpha_2) = \text{wt}(\omega\alpha_1+\alpha_2) &= \text{wt}(\omega^2\alpha_1+\omega\alpha_2) = 4t+1.
\end{aligned}$$

Hence, $C$ has minimum weight $4t = \left\lfloor \frac{4(5t+1)}{5} \right\rfloor = \lfloor \frac{4n}{5} \rfloor$. Consequently, $C$ is an optimal $[n, 2, \lfloor \frac{4n}{5} \rfloor]_4$ code with Hermitian hull dimension one.

From the two subcases, it follows that

$$D_4^{\text{H}}(n, 2, 1) = \left\lfloor \frac{4n}{5} \right\rfloor$$

for all $n \equiv 1 \pmod 5$.

**Case 2**. $n \equiv 2 \pmod 5$. Then $n = 5t + 2$ for some positive integer $t$. Since a quaternary linear code $C$ is determined by the values $S_{ij}$ for all $i, j \in \mathbb{F}_4$, in the remaining parts, we give constructions of linear codes in terms of $S_{ij}$. However, an explicit form of its generator matrix $G$ can be determined in the same way as Case 1. We consider the following two subcases.

**Case 2.1** $t$ is even. Let $S_{01} = S_{\omega 0} = S_{\omega^2\omega^2} = S_{1\omega^2} = S_{\omega^2\omega} = S_{\omega\omega^2} = \frac{t}{2}$, $S_{0\omega} = \frac{t}{2} - 1$, $S_{10} = S_{11} = S_{\omega^2 1} = \frac{t}{2} + 1$, and $S_{ij} = 0$ otherwise. It can be concluded that

$$\begin{aligned}
y_0 &= S_{11} + S_{\omega\omega} + S_{\omega^2\omega^2} = t + 1,\\
y_1 &= S_{1\omega^2} + S_{\omega 1} + S_{\omega^2\omega} = t, \text{ and}\\
y_2 &= S_{\omega^2 1} + S_{1\omega} + S_{\omega\omega^2} = t + 1.
\end{aligned}$$

In $\mathbb{F}_4$, it can be deduced that

$$\sum_{i\in\mathbb{F}_4^*}\sum_{j\in\mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i\in\mathbb{F}_4}\sum_{j\in\mathbb{F}_4^*} S_{ij} = 1.$$

By (5.7), we have

$$GG^{\dagger} = \begin{bmatrix} 1 & \omega \\ \omega^2 & 1 \end{bmatrix}$$

which implies that $\text{rank}(GG^T) = 1$ and $\dim(\text{Hull}_{\text{H}}(C)) = 1$ by Proposition 2.2.4.

Based on the definition of $S_{ij}$ and (5.2)–(5.6), it can be deduced that

$$\begin{aligned}
\text{wt}(\alpha_1) = \text{wt}(\omega\alpha_1) &= \text{wt}(\omega^2\alpha_1) = 4t+3, \\
\text{wt}(\alpha_2) = \text{wt}(\omega\alpha_2) &= \text{wt}(\omega^2\alpha_2) = 4t+1, \\
\text{wt}(\alpha_1+\alpha_2) = \text{wt}(\omega\alpha_1+\omega\alpha_2) &= \text{wt}(\omega^2\alpha_1+\omega^2\alpha_2) = 4t+1, \\
\text{wt}(\alpha_1+\omega\alpha_2) = \text{wt}(\omega\alpha_1+\omega^2\alpha_2) &= \text{wt}(\omega^2\alpha_1+\alpha_2) = 4t+2, \text{ and} \\
\text{wt}(\alpha_1+\omega^2\alpha_2) = \text{wt}(\omega\alpha_1+\alpha_2) &= \text{wt}(\omega^2\alpha_1+\omega\alpha_2) = 4t+1.
\end{aligned}$$

Hence, the minimum weight of $C$ is $4t+1 = \left\lfloor \frac{4(5t+2)}{5} \right\rfloor = \left\lfloor \frac{4n}{5} \right\rfloor$. As desired, $C$ is an optimal $[n, 2, \lfloor \frac{4n}{5} \rfloor]_4$ code with Hermitian hull dimension one.

**Case 2.2**. $t$ is odd. Let $S_{10} = S_{11} = S_{\omega^2\omega^2} = S_{1\omega^2} = S_{\omega^2\omega} = S_{\omega^2 1} = S_{\omega\omega^2} = \frac{t+1}{2}$, $S_{01} = S_{0\omega} = S_{\omega 0} = \frac{t-1}{2}$, and $S_{ij} = 0$ otherwise. Then

$$\begin{aligned}
y_0 &= S_{11} + S_{\omega\omega} + S_{\omega^2\omega^2} = t+1, \\
y_1 &= S_{1\omega^2} + S_{\omega 1} + S_{\omega^2\omega} = t+1, \text{ and} \\
y_2 &= S_{\omega^2 1} + S_{1\omega} + S_{\omega\omega^2} = t+1.
\end{aligned}$$

In $\mathbb{F}_4$, we have

$$\sum_{i\in\mathbb{F}_4^*}\sum_{j\in\mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i\in\mathbb{F}_4}\sum_{j\in\mathbb{F}_4^*} S_{ij} = 0.$$

By (5.7), it follows that

$$GG^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case, $\text{rank}(GG^T) = 1$ and $\dim(\text{Hull}_{\text{H}}(C)) = 2 - \text{rank}(GG^T) = 1$ by Proposition 2.2.4.

Using the definition of $S_{ij}$ and (5.2)–(5.6), we have

$$\begin{aligned}
\mathrm{wt}(\alpha_1) = \mathrm{wt}(\omega\alpha_1) = \mathrm{wt}(\omega^2\alpha_1) &= 4t+3,\\
\mathrm{wt}(\alpha_2) = \mathrm{wt}(\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_2) &= 4t+2,\\
\mathrm{wt}(\alpha_1+\alpha_2) = \mathrm{wt}(\omega\alpha_1+\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\omega^2\alpha_2) &= 4t+1,\\
\mathrm{wt}(\alpha_1+\omega\alpha_2) = \mathrm{wt}(\omega\alpha_1+\omega^2\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\alpha_2) &= 4t+1, \text{ and}\\
\mathrm{wt}(\alpha_1+\omega^2\alpha_2) = \mathrm{wt}(\omega\alpha_1+\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\omega\alpha_2) &= 4t+1
\end{aligned}$$

which implies that $C$ has minimum weight $4t+1 = \left\lfloor \frac{4(5t+2)}{5} \right\rfloor = \left\lfloor \frac{4n}{5} \right\rfloor$. In this case, $C$ is an optimal $[n, 2, \lfloor \frac{4n}{5} \rfloor]_4$ code with Hermitian hull dimension one.

From the two subcases, it follows that

$$D_4^{\mathrm{H}}(n,2,1) = \left\lfloor \frac{4n}{5} \right\rfloor$$

for all $n \equiv 2 \pmod 5$.

**Case 3**. $n \equiv 4 \pmod 5$. Then $n = 5t+4$ for some positive integer $t$. We consider the following two cases based on the parity of $t$.

**Case 3.1**. $t$ is even. Let $S_{0\omega} = S_{10} = S_{\omega 0} = S_{\omega^2\omega^2} = S_{\omega\omega^2} = S_{\omega^2\omega} = \frac{t}{2}$, $S_{01} = S_{11} = S_{1\omega^2} = S_{\omega^2 1} = \frac{t}{2}+1$, and $S_{ij} = 0$ otherwise. Then

$$\begin{aligned}
y_0 &= S_{11} + S_{\omega\omega} + S_{\omega^2\omega^2} = t+1,\\
y_1 &= S_{1\omega^2} + S_{\omega 1} + S_{\omega^2\omega} = t+1, \text{ and}\\
y_2 &= S_{\omega^2 1} + S_{1\omega} + S_{\omega\omega^2} = t+1.
\end{aligned}$$

In $\mathbb{F}_4$, we have

$$\sum_{i\in\mathbb{F}_4^*}\sum_{j\in\mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i\in\mathbb{F}_4}\sum_{j\in\mathbb{F}_4^*} S_{ij} = 0.$$

By (5.7), it can be concluded that

$$GG^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It means that $\mathrm{rank}(GG^T) = 1$ and $C$ has Hermitian hull dimension one by Proposition 2.2.4.

From the definition of $S_{ij}$ and (5.2)–(5.6), it can be deduced that

$$\begin{aligned}
\mathrm{wt}(\alpha_1) = \mathrm{wt}(\omega\alpha_1) = \mathrm{wt}(\omega^2\alpha_1) &= 4t+3,\\
\mathrm{wt}(\alpha_2) = \mathrm{wt}(\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_2) &= 4t+4,\\
\mathrm{wt}(\alpha_1+\alpha_2) = \mathrm{wt}(\omega\alpha_1+\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\omega^2\alpha_2) &= 4t+3,\\
\mathrm{wt}(\alpha_1+\omega\alpha_2) = \mathrm{wt}(\omega\alpha_1+\omega^2\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\alpha_2) &= 4t+3, \text{ and}\\
\mathrm{wt}(\alpha_1+\omega^2\alpha_2) = \mathrm{wt}(\omega\alpha_1+\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\omega\alpha_2) &= 4t+3.
\end{aligned}$$

Hence, $C$ has minimum weight $4t+3 = \left\lfloor \frac{4(5t+4)}{5} \right\rfloor \lfloor \frac{4n}{5} \rfloor$. Therefore, $C$ is an optimal quaternary linear $[n, 2, \lfloor \frac{4n}{5} \rfloor]_4$ code with Hermitian hull dimension one.

**Case 3.2**. $t$ is odd. Let $S_{01} = S_{0\omega} = S_{\omega 0} = S_{11} = S_{\omega^2\omega^2} = S_{1\omega^2} = S_{\omega^2\omega} = S_{\omega^2 1} = S_{\omega\omega^2} = \frac{t+1}{2}$, $S_{10} = \frac{t-1}{2}$, and $S_{ij} = 0$ otherwise. Then

$$\begin{aligned}
y_0 &= S_{11} + S_{\omega\omega} + S_{\omega^2\omega^2} = t+1,\\
y_1 &= S_{1\omega^2} + S_{\omega 1} + S_{\omega^2\omega} = t+1, \text{ and}\\
y_2 &= S_{\omega^2 1} + S_{1\omega} + S_{\omega\omega^2} = t+1.
\end{aligned}$$

In $\mathbb{F}_4$, it can be deduced that

$$\sum_{i\in\mathbb{F}_4^*}\sum_{j\in\mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i\in\mathbb{F}_4}\sum_{j\in\mathbb{F}_4^*} S_{ij} = 0.$$

By (5.7), we have

$$GG^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies that $\mathrm{rank}(GG^T) = 1$ and $\dim(\mathrm{Hull}_{\mathrm{H}}(C)) = 2 - \mathrm{rank}(GG^T) = 1$.

Applying the definition of $S_{ij}$ and (5.2)–(5.6), we have

$$\begin{aligned}
\mathrm{wt}(\alpha_1) = \mathrm{wt}(\omega\alpha_1) = \mathrm{wt}(\omega^2\alpha_1) &= 4t+3,\\
\mathrm{wt}(\alpha_2) = \mathrm{wt}(\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_2) &= 4t+4,\\
\mathrm{wt}(\alpha_1+\alpha_2) = \mathrm{wt}(\omega\alpha_1+\omega\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\omega^2\alpha_2) &= 4t+3,\\
\mathrm{wt}(\alpha_1+\omega\alpha_2) = \mathrm{wt}(\omega\alpha_1+\omega^2\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\alpha_2) &= 4t+3, \text{ and}\\
\mathrm{wt}(\alpha_1+\omega^2\alpha_2) = \mathrm{wt}(\omega\alpha_1+\alpha_2) = \mathrm{wt}(\omega^2\alpha_1+\omega\alpha_2) &= 4t+3
\end{aligned}$$

which implies that $\mathrm{wt}(C) = 4t+3 = \left\lfloor \frac{4(5t+4)}{5} \right\rfloor \left\lfloor \frac{4n}{5} \right\rfloor$. Consequently, $C$ is an optimal $[n, 2, \lfloor \frac{4n}{5} \rfloor]_4$ code with Hermitian hull dimension one.

From the two subcases, it can be concluded that

$$D_4^{\mathrm{H}}(n, 2, 1) = \left\lfloor \frac{4n}{5} \right\rfloor$$

for all $n \equiv 4 \pmod 5$. □

**Example 5.2.2.** Let $t = 2$. By Theorem 5.2.1 and $t$ is even, we have that $S_{01} = S_{0\omega} = S_{\omega 0} = S_{11} = S_{\omega^2\omega^2} = S_{1\omega^2} = S_{\omega^2\omega} = S_{\omega^2 1} = S_{\omega\omega^2} = 1, S_{10} = 2$ and $S_{ij} = 0$ otherwise. Let $C$ be a quaternary linear code of length 11 with generator matrix

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & \omega & 1 & \omega^2 & 1 & \omega^2 & \omega^2 & \omega \\ 0 & 0 & 1 & \omega & 0 & 1 & \omega^2 & \omega^2 & \omega & 1 & \omega^2 \end{bmatrix}.$$

Then By Theorem 5.2.1, it can be deduced that

$$y_0 = t, \; y_1 = t, \text{ and } y_2 = t.$$

In $\mathbb{F}_4$, we have

$$\sum_{i \in \mathbb{F}_4^*} \sum_{j \in \mathbb{F}_4} S_{ij} = 1 \text{ and } \sum_{i \in \mathbb{F}_4} \sum_{j \in \mathbb{F}_4^*} S_{ij} = 0.$$

By (5.7), it follows that

$$GG^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies that $\mathrm{rank}(GG^\dagger) = 1$ and $\dim(\mathrm{Hull_H}(C)) = 2 - \mathrm{rank}(GG^\dagger) = 1$ by Proposition 2.2.4. Then $C$ is an optimal $[11, 2, 8]_4$ code with hull dimension one.

## 5.3 Bounds on $D_4^{\mathrm{H}}(n, 2, 1)$ with $n \equiv 0, 3 \pmod 5$

In this section, we focus on the two remaining cases where $n \equiv 0, 3 \pmod 5$. For these cases, we provide good lower and upper bounds on the minimum weight of $[n, 2]_4$ codes with Hermitian hull dimension one. Precisely, upper and lower bounds on $D_4^{\mathrm{H}}(n, 2, 1)$ are given with the gap one.

**Theorem 5.3.1.** *Let $n$ be a positive integer. If $n \equiv 0, 3 \pmod 5$, then there exists an $[n, 2, \lfloor \frac{4n}{5} \rfloor - 1]_4$ code with Hermitian hull dimension one.*

*Proof.* Assume that $n \equiv 0, 3 \pmod 5$. We consider the following two cases.

**Case 1.** $n \equiv 0 \pmod 5$. Then $n-1 \equiv 4 \pmod 5$ and $n = 5t$ for some positive integer $t$. By Theorem 5.2.1, there exists an $[n-1, 2, \left\lfloor \frac{4(n-1)}{5} \right\rfloor]_4$ code $C$ with Hermitian hull dimension one. Let $G$ be a generator matrix for $C$ and let $C'$ be a quaternary linear code generated by $G' = [\mathbf{0}\ G]$. Since

$$\left\lfloor \frac{4(n-1)}{5} \right\rfloor = \left\lfloor \frac{4(5t-1)}{5} \right\rfloor = 4t - 1 = \left\lfloor \frac{4(5t)}{5} \right\rfloor - 1 = \left\lfloor \frac{4n}{5} \right\rfloor - 1,$$

it follows that $C'$ is an $[n, 2, \left\lfloor \frac{4(n-1)}{5} \right\rfloor]_2 = [n, 2, \lfloor \frac{4n}{5} \rfloor - 1]_2$ code with Hermitian hull dimension one.

**Case 2.** $n \equiv 3 \pmod 5$. Then $n-1 \equiv 2 \pmod 5$ and $n = 5t+3$ for some positive integer $t$. By Theorem 5.2.1, there exists an $[n-1, 2, \left\lfloor \frac{4(n-1)}{5} \right\rfloor]_4$ code $C$ with Hermitian hull dimension one. Let $G$ be a generator matrix for $C$ and let $C'$ be a quaternary linear code generated by $G' = [\mathbf{0}\ G]$. Since

$$\left\lfloor \frac{4(n-1)}{5} \right\rfloor = \left\lfloor \frac{4(5t+2)}{5} \right\rfloor = 4t + 1 = \left\lfloor \frac{4(5t+3)}{5} \right\rfloor - 1 = \left\lfloor \frac{4n}{5} \right\rfloor - 1,$$

it can be easily seen that $C'$ is an $[n, 2, \left\lfloor \frac{4(n-1)}{5} \right\rfloor]_2 = [n, 2, \lfloor \frac{4n}{5} \rfloor - 1]_2$ code with Hermitian hull dimension one. ☐

**Example 5.3.2.** By Example 5.2.2, we have that $C$ is a linear code with parameters $[11, 2, 8]_4$ generated by

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & \omega & 1 & \omega^2 & 1 & \omega^2 & \omega^2 & \omega \\ 0 & 0 & 1 & \omega & 0 & 1 & \omega^2 & \omega^2 & \omega & 1 & \omega^2 \end{bmatrix}.$$

Let $G' = [\mathbf{0}\ G]$. By Theorem 5.2.1, we have that $G'$ generates an optimal $[10, 2, 7]_3$ code with hull dimension one.

**Corollary 5.3.3.** *Let $n$ be a positive integer. If $n \equiv 0, 3 \pmod 5$ then*

$$\left\lfloor \frac{4n}{5} \right\rfloor - 1 \leq D_4^{\mathrm{H}}(n, 2, 1) \leq \left\lfloor \frac{4n}{5} \right\rfloor.$$

*Proof.* While the lower bound is given in Theorem 5.3.1, the upper bound is guaranteed by Lemma 5.1.2. □

Constructions and bounds on quaternary linear codes with dimension two and Hermitian hull dimension one have been studied. Optimal quaternay $[n,2]_4$ codes with Hermitian hull dimension one have been constructed for all lengths $n \geq 3$ such that $n \equiv 1,2,4 \pmod 5$. Good lower and upper bounds on the minimum weight of quaternay $[n,2]_4$ codes with Hermitian hull dimension one have been given for all lengths $n \equiv 0,3 \pmod 5$. The above results are summarized as follows.

**Theorem 5.3.4.** *Let $n \geq 3$ be an integer. Then*

$$D_4^{\mathrm{H}}(n,2,1) = \left\lfloor \frac{4n}{5} \right\rfloor \text{ for all } n \equiv 1,2,4 \pmod 5$$

*and*

$$\left\lfloor \frac{4n}{5} \right\rfloor - 1 \leq D_4^{\mathrm{H}}(n,2,1) \leq \left\lfloor \frac{4n}{5} \right\rfloor \text{ for all } n \equiv 0,3 \pmod 5.$$

Based on our inspection, we propose the following conjecture.

**Conjecture 5.3.5.** $D_4^{\mathrm{H}}(n,2,1) = \left\lfloor \frac{4n}{5} \right\rfloor - 1$ for all positive integers $n \equiv 0,3 \pmod 5$.

# Chapter 6

# Conclusion

In this chapter, we summarize the existence of binary and ternary linear codes with Euclidean hull dimension one and quaternary linear codes with Hermitian hull dimension one. For a linear $[n, k, d]_q$ code, the minimum weight $d$ measures the efficiency of the code. A linear code $C$ is optimal if $C$ has the highest minimum weight amoung all linear $[n, k]_q$ codes. Precisely, a linear code with Euclidean hull dimension one and Hermitian hull dimension one are optimal if $d = D_q(n, k, 1)$ and $d = D_q^H(n, k, 1)$, respectively. The results in Chapters 3-6 are summarized in the following tables.

Table 6.1: Existence of optimal $[n, 2, d]_2$ and $[n, 2, d]_3$ codes with dimension two Euclidean hull dimension one.

| $q$ | $n$ | $d$ | Optimal | Remark |
|---|---|---|---|---|
| 2 | $n \equiv 0 \pmod 6$ | $\lfloor \frac{2n}{3} \rfloor - 1$ | ✓ | Theorem 3.5.2 |
| | $n \equiv 1 \pmod 6$ | $\lfloor \frac{2n}{3} \rfloor$ | ✓ | Theorem 3.2.1 |
| | $n \equiv 2 \pmod 6$ | $\lfloor \frac{2n}{3} \rfloor - 1$ | ✓ | Theorem 3.5.2 |
| | $n \equiv 3 \pmod 6$ | $\lfloor \frac{2n}{3} \rfloor - 1$ | ✓ | Theorem 3.3.1 |
| | $n \equiv 4 \pmod 6$ | $\lfloor \frac{2n}{3} \rfloor - 1$ | ✓ | Theorem 3.5.2 |
| | $n \equiv 5 \pmod 6$ | $\lfloor \frac{2n}{3} \rfloor$ | ✓ | Theorem 3.4.1 |
| 3 | $n \equiv 0 \pmod 4$ | $\lfloor \frac{3n}{4} \rfloor - 1$ | ✓ | Theorem 4.5.3 |
| | $n \equiv 1 \pmod 4$ | $\lfloor \frac{3n}{4} \rfloor$ | ✓ | Theorem 4.2.1 |
| | $n \equiv 2 \pmod 4$ | $\lfloor \frac{3n}{4} \rfloor$ | ✓ | Theorem 4.3.1 |
| | $n \equiv 3 \pmod 4$ | $\lfloor \frac{3n}{4} \rfloor$ | ✓ | Theorem 4.4.1 |



Table 6.2: Existence of optimal $[n, 2, d]_4$ codes with dimension two and Hermitian hull dimension one.

| $q$ | $n$ | $d$ | Optimal | Remark |
|---|---|---|---|---|
| 4 | $n \equiv 0 \pmod 5$ | $\lfloor \frac{4n}{5} \rfloor - 1$ | ? | Theorem 5.3.1 |
| | $n \equiv 1 \pmod 5$ | $\lfloor \frac{4n}{5} \rfloor$ | ✓ | Theorem 5.2.1 |
| | $n \equiv 2 \pmod 5$ | $\lfloor \frac{4n}{5} \rfloor$ | ✓ | Theorem 5.2.1 |
| | $n \equiv 3 \pmod 5$ | $\lfloor \frac{4n}{5} \rfloor - 1$ | ? | Theorem 5.3.1 |
| | $n \equiv 4 \pmod 5$ | $\lfloor \frac{4n}{5} \rfloor$ | ✓ | Theorem 5.2.1 |

In Table 6.1 and 6.2, $n \geq 3$ is a positive integer, $d$ in the three column indicates the maximum minimum weight constructed in the corresponding theorem, and ✓ (resp., ?) certifies the optimality (resp., the open cases).

The study of $D_q(n,k,1)$ and $D_q^H(n,k,1)$ for all possible values $q,n,k$, and $l$ is an interesting problem.

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# Publications

## Publications

- Mankean, T., Jitman, S.: *Optimal binary and ternary linear codes with hull dimension one,* J.Appl. Math. Comput. **64**, 137–155 (2020).
- Mankean, T., Jitman, S.: (2021, Jan. 9). *Constructions and bounds on quaternary linear codes with Hermitian hull dimension one,* Arab. J. Math. **10**, 175–184 (2021).

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