DISTRIBUTIVE LATTICE–BASED ALGEBRAS

By

Aveya Charoenpol

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Doctor of Philosophy Program in Mathematics

International Program

Graduate School, Silpakorn University

Academic Year 2015

พีชคณิตฐานแลตทิชแจกแจง

โดย
นางสาวอาวีญา เจริญผล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาปรัชญาดุษฎีบัณฑิต
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ปีการศึกษา 2558

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The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Distributive Lattice−based Algebras” submitted by Miss Aveya Charoenpol as a partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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54305801 : สาขาวิชาคณิตศาสตร์
คำสำคัญ : แลตทิชแจกแจง / พีชคณิตฐานแลตทิช/ พีชคณิตเอกนามเชื่อมโยง / พีชคณิตลดทอนไม่ได้เชิงผลคูณย่อย

อาวีญา เจริญผล : พีชคณิตฐานแลตทิชแจกแจง. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : ศ.ดร.ฉวีวรรณ รัตนประเสริฐ. 54 หน้า.

พีชคณิตฐานแลตทิซแจกแจงหรือคำย่อว่าพีชคณิต-BDL เป็นพีชคณิต $\langle A; F\rangle$ ซึ่ง $\langle A;\vee,\wedge,0\rangle$ เป็นแลตทิซแจกแจงมีขอบเขตสำหรับบางสับเซต $\{\vee,\wedge,0,1\}$ ของF พีชคณิตชนิดนี้ถูกศึกษาอย่างกว้างขวางโดยนักพีชคณิตหลายๆท่าน โดยเฉพาะอย่างยิ่งพีชคณิต-BDL ซึ่ง $F\setminus\{\vee,\wedge,0,1\}$ เป็นเซตโทนของฟังก์ชันซึ่งเป็นฟังก์ชันสาทิสสัณฐานคู่กันบนแลตทิซฐานของพีชคณิต-BDL นั้น ตัวอย่างเช่น พีชคณิตออกควัม พีชคณิตบูลีน พีชคณิตเดอมอร์แกน พีชคณิตสโตน และพีชคณิตคลีเน

ในวิทยานิพนธ์นี้ เรานิยามคลาสของพีชคณิต BDL ตัวใหม่ ซึ่ง $F\setminus\{\vee,\wedge,0,1\}$ เป็นเซตโทนของฟังก์ชันสาทิสสัณฐานบนแลตทิซฐาน $\langle A;\vee,\wedge,0\rangle$ และเป็นฟังก์ชันเชื่อมโยง เราเรียกพีชคณิตชนิดนี้ว่า พีชคณิต-BDLC เนื่องจากผลคูณตรงไม่จำกัดครั้งของพีชคณิตเอกนามเชื่อมโยง อาจจะไม่เป็นพีชคณิตเอกนามเชื่อมโยง ดังนั้นคลาสของพีชคณิต BDLC ไม่เป็นวาไรตี เราศึกษาสมบัติเชิงโครงสร้างของพีชคณิต BDLC โดยศึกษาสมบัติทั่วไปของพีชคณิต พีชคณิตย่อย ผลคูณของพีชคณิต อิมเมจของฟังก์ชันสาทิสสัณฐานของพีชคณิต พีชคณิตเล็กสุดเฉพาะกลุ่ม และประยุกต์ผลเหล่านั้นเพื่อแสดงว่าคลาสย่อย $\mathcal{M}_n$ ของ $\mathcal{M}$ ซึ่ง $\lambda(f)$ น้อยกว่าหรือเท่ากับ n เป็นวาไรตี สำหรับทุกๆ จำนวนเต็มบวก n ยิ่งไปกว่านั้น มันสามารถถูกบรรยายโดยเอกลักษณ์ นอกจากนี้เราจำแนกพีชคณิตลดทอนไม่ได้เชิงผลคูณย่อยทั้งหมดใน $\mathcal{M}_n$ และแสดงว่ามันถูกก่อกำเนิดโดยพีชคณิตลดทอนไม่ได้เชิงผลคูณย่อยเพียงตัวเดียว ในส่วนสุดท้ายเราได้บรรยายแลตทิซของวาไรตีย่อยทั้งหมดของ $\mathcal{M}_n$

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A distributive lattice-based algebra; or shortly BDL-algebra, is an algebra $\langle A;F\rangle$ whose $\langle A;\vee,\wedge,0,1\rangle$ is a bounded distributive lattice for some $\{\vee,\wedge\, 0,1\}\subseteq F$. It is extensively studied by several algebraists. Especially, BDL-algebras whose $F\setminus\{\vee,\wedge,0,1\}$ is a singleton set of a dual endomorphism on its lattice-based; for instance, Ockham algebras, Boolean algebras, De Morgan algebras, Stone algebras and Kleene algebras.

In this thesis, we define a new class of BDL-algebras whose $F\setminus\{\vee,\wedge,0,1\}$ contains only a connected endomorphism on $\langle A;\vee,\wedge,0\rangle$. This algebra is called a BDLC-algebra. Since the infinite direct product of connected unary algebras does not need to be connected, the class $\mathcal{M}$ of all BDLC-algebras is not a variety. We study algebraic properties of BDLC-algebras by investigating the concept of general properties of algebras, subalgebras, product of algebras, homomorphic image of algebras, minimal algebras and apply those results to show that the subclass $\mathcal{M}_n$ of $\mathcal{M}$ whose the pre-period is less than or equal to n is a variety for all positive integers n; moreover, it can be described by identities. Besides, we characterize all subdirectly irreducible algebras in $\mathcal{M}_n$ and show that it is generated by a single subdirectly irreducible algebra. Finally, we describe the lattice of all subvarieties of $\mathcal{M}_n$.

Department of Mathematics Graduate School, Silpakorn University

Student's signature .... Academic Year 2015

Thesis Advisor's signature ....

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Chapter 1

Introduction and Literature Review

An algebra $\underline{A} = \langle A; F\rangle$ is a structure consisting of a nonempty set A which is called *the universe of* $\underline{A}$, and a set F of operations defined on A which is called *the set of fundamental operations of* $\underline{A}$. If A is finite and every fundamental operation is finitary, $\underline{A}$ is called a *finite algebra*. However, we may consider F as $\{f_j\}_{j\in J}$ for some index set J. For convenience, we may write $\underline{A} = \langle A; f_1, f_2, ..., f_t\rangle$ when $F = \{f_1, f_2, ..., f_t\}$ is finite for some positive integers t. A *type* $\tau = (n_j)_{j\in J}$ of algebra is a function which map each $j \in J$ to the arity of f_j. Groups, rings and fields are examples of well-known algebras of type $(2, 1, 0)$, $(2, 2, 1, 0)$ and $(2, 2, 1, 1, 0, 0)$, respectively. Most of algebraist study algebraic properties of algebras through the concept of subalgebras, product of algebras, homomorphic image of algebras, minimal algebras and subdirectly irreducible algebras.

A *variety* is a class of algebras of the same type which is closed under homomorphic images, subalgebras and direct products of families of algebras. In [3], G. Birkhoff proved that $\mathcal{K}$ is a variety if and only if every algebra in $\mathcal{K}$ satisfies a certain set of laws. For instance, we know that all groups $\langle G; \cdot, ^{-1}, e\rangle$ satisfy the following laws:

- associative law : $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- identity law : $a \cdot e = a = e \cdot a$,
- inverse law : $a \cdot a^{-1} = e = a^{-1} \cdot a$

for all $a, b, c \in A$. Therefore, the class of all groups is a variety.

In the resent year, some complicate questions about algebras are not able studied through those algebraic properties; especially, representing some classes of algebras. Several algebraist are studying new methods to solve these problems. In 1970, H.A. Priestley [17] represented bounded distributive lattices by ordered Stone spaces; it is a new branch to use a topology to study an algebra. And also, this concept was used to describe subdirectly irreducible Ockham algebras by A. Urguhart [22]. In 1983, Davey and Werner [11] developed the method to represent every algebra as an algebra of continuous functions; this concept is known as natural duality.

For each algebra $\underline{\mathrm{A}}$, let $\mathscr{A} := ISP(\underline{\mathrm{A}})$ be the category consisting of all isomorphic copies of subalgebras of direct powers of $\underline{\mathrm{A}}$ and let $\mathscr{X} := IS_cP^+(\underset{\sim}{\mathrm{A}})$ be the category consisting of all isomorphic copies of closed substructures of non-empty direct powers of $\underset{\sim}{\mathrm{A}} := (A; R, \mathscr{T})$ where $R \subseteq \bigcup_{n\in\mathbb{N}} S(\underline{\mathrm{A}}^n)$ and $\mathscr{T}$ is the discrete topology on A. The *dual* $D(\underline{\mathrm{B}}) \in \mathscr{X}$ of $\underline{\mathrm{B}} \in \mathscr{A}$ and $E(\underset{\sim}{\mathrm{X}}) \in \mathscr{A}$ of $\underset{\sim}{\mathrm{X}} \in \mathscr{X}$ are the set of all homomorphisms from $\underline{\mathrm{B}}$ to $\underline{\mathrm{A}}$ and the set of all morphisms from $\underset{\sim}{\mathrm{X}}$ to $\underset{\sim}{\mathrm{A}}$, respectively. We say that $\underset{\sim}{\mathrm{A}}$ (or R) *yields a (natural) duality* on $\mathscr{A}$ or $\underset{\sim}{\mathrm{A}}$ *dualise* $\underline{\mathrm{A}}$ if $\underline{\mathrm{B}} \cong ED(\underline{\mathrm{B}})$ for all $\underline{\mathrm{B}} \in \mathscr{A}$; and we say that $\underline{\mathrm{A}}$ is *dualisable* if there is a structure $\underset{\sim}{\mathrm{A}}$ which dualise $\underline{\mathrm{A}}$. These mean that every algebra in $\mathscr{A}$ can be represented as a concrete algebra of morphisms from the structure $D(\underline{\mathrm{B}})$ to the structure $\underset{\sim}{\mathrm{A}}$; for further details, see in [7] or [11]. One of the famous theorem in the natural duality, which is named NU-duality Theorem [7], implies that the structure $\underset{\sim}{\mathrm{A}} := \big(A; S(\underline{\mathrm{A}}^2), \mathscr{T}\big)$ yields a duality on $ISP(\underline{\mathrm{A}})$ whenever $\underline{\mathrm{A}}$ is an algebra admitting a majority term operation; that is, there is a term operation $m : A^3 \to A$ satisfying $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ for all $x, y \in A$.

A *lattice* is an algebra $\langle A; \vee, \wedge\rangle$ of type $(2, 2)$ satisfying the following laws:

- commutative law : $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$,
- associative law : $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
- idempotent law : $a \vee a = a$ and $a \wedge a = a$,
- absorption law : $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$

for all $a, b, c \in A$. By Birkhoff's theorem [3], the class of all lattices is a variety. A *bounded lattice* is a lattice which has elements 0 and 1 satisfying $0 \wedge x = 0$ and $1 \vee x = 1$ for all $x \in A$. The elements 0 and 1 are called *the least and the greatest element* in A, respectively. If $\underline{A}$ is a bounded lattice, we write $\underline{A} = \langle A; \vee, \wedge, 0, 1 \rangle$. It is well known that the medean function $m : A^3 \to A$ on a set A defined by $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ is a majority term operation on a lattice $\langle A; \vee, \wedge \rangle$. By NU-duality Theorem [7], every lattice is dualisable. A *distributive lattice* is a lattice satisfying $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$. The power set $\mathscr{P}(X)$ of a set X is an example of a well-known bounded distributive lattice whose $\vee$ is the union, $\wedge$ is the intersection, 0 is the empty set and 1 is the set X. However, there are some lattices which are not distributive; for example, the diamond lattice M_3 and the pentagon lattice N_5 whose diagrams are shown in Figure 1. In [2], G. Birkhoff gave a characterization of distributive lattices by M_3 and N_5 which is known as the $M_3 - N_5$ Theorem:

$\langle A; \vee, \wedge \rangle$ is a distributive lattice if and only if it has no M_3 and N_5 as sublattices.

Figure 1 The lattice N_5 and M_3.

An algebra $\langle A; F \rangle$ is a *reduct* of an algebra $\langle A; F^* \rangle$ if $F \subseteq F^*$. All distributive lattices are precisely sublattices of $\langle \mathscr{P}(B); \vee, \wedge, 0, 1 \rangle$ for some sets B. If we consider a complement of sets as a unary operation $' : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $A' = X \backslash A$ for all $A \subseteq X$, we will have a new algebra $\langle \mathscr{P}(X); \vee, \wedge, ', 0, 1 \rangle$ whose $\langle \mathscr{P}(X); \vee, \wedge, 0, 1 \rangle$ is its reduct.

An algebra $\langle A; F \rangle$ is said to be a *lattice-based algebra* if $\langle A; \vee, \wedge \rangle$ is its reduct. So, NU-duality Theorem [7] implies that every lattice-based algebra is dualisable. The dualities of lattice-based algebras have been studied by various authors (see [9],

[11], [18]). If a bounded distributive lattice is a reduct of an algebra $\langle A; F\rangle$, we call $\langle A; F\rangle$ that a *bounded distributive lattice-based algebra*; or shortly, BDL-algebra. For a set X, one can notice that $\langle \mathscr{P}(X); \vee, \wedge, ', 0, 1\rangle$ is a BDL-algebra such that the complement $'$ is a dual endomorphism on its lattice-based. Lattice-based algebras are extensively studied; especially, BDL-algebras whose $F \setminus \{\vee, \wedge, 0, 1\}$ contains only a dual endomorphism on its lattice-based; for instance, Boolean algebras, De Morgan algebras, Ockham algebras.

A *Boolean algebra* was introduced by George Boole [6] to be a BDL-algebra $\langle A; \vee, \wedge, f, 0, 1\rangle$ whose f is a unary operation on A satisfying for each $x, y \in A$,

- $f(x \vee y) = f(x) \wedge f(y)$,
- $f(x \wedge y) = f(x) \vee f(y)$,
- $x \wedge f(x) = 0$,
- $x \vee f(x) = 1$

In [5], G. Birkhoff and M. Ward proved that every finite Boolean algebra is isomorphic to $\langle \mathscr{P}(B); \vee, \wedge, 0, 1\rangle$ for some finite sets B.

A *De Morgan algebra* was introduced by Moisil [16] to be a generalization of a Boolean algebra. It is a BDL-algebra $\langle A; \vee, \wedge, f, 0, 1\rangle$ whose its unary operation f satisfies for each $x, y \in A$,

- $f(x \vee y) = f(x) \wedge f(y)$,
- $f(x \wedge y) = f(x) \vee f(y)$,
- $f(0) = 1$,
- $f^2(x) = x$

An *Ockham algebra* is a BDL-algebra $\langle A; \vee, \wedge, f, 0, 1\rangle$ whose f is a dual endomorphism on its lattice-based. This algebra is a generalization of Boolean algebras and De Morgan algebras. It was first introduced by Berman [1]. Later, A. Urguhart [22] characterized congruences and subdirectly irreducible Ockham algebras. Besides, M.S. Goldberg [14] applied the concept of duality to characterize all finite subdirectly irreducible Ockham algebras.

BDL-algebras not only are popularly studied in mathematics but also can be applied in computer science; for instance, Boolean algebras have been fundamental in the development of computer science and digital logic.

We are interested in introducing a new kind of BDL-algebras by considering conditions on the unary operation f, especially when f is connected.

In the literature, if f is a unary operation on a set A then $\langle A; f\rangle$ is called *mono-unary algebra*. A unary operation f on a set A is *connected* if for each $a, b \in A$, there exist nonnegative integers n, m such that $f^n(a) = f^m(b)$. If f is connected, $\langle A; f\rangle$ is called a *connected mono-unary algebra*. In [24], M. Yoeli characterized all subdirectly irreducible connected mono-unary algebras. Later, G.H. Wenzel [23] extended this result to any mono-unary algebras. It is well-known fact that every mono-unary algebra is a disjoint union of connected mono-unary algebras. So, we study mono-unary algebras via connected mono-unary algebras. One direction of studying mono-unary algebra is the concept of *pre-period* which is the least nonnegative integer $\lambda(f)$ satisfying $\mathrm{Im} f^{\lambda(f)} = \mathrm{Im} f^{\lambda(f)+1}$ (see e.g.[25]). If $\lambda(f) = |A| - 1$ then f is called a *long-tailed function* [12]. C. Ratanaprasert and K. Denecke [20] characterized all congruence relations on $\langle A; f\rangle$ whose f is a long-tailed function; besides, C. Ratanaprasert , K. Denecke and S.L. Wismath [20],[13] proved that there exists $d \in A$ such that $A = \left\{d, f(d), ..., f^{\lambda(f)}(d) = f^{\lambda(f)+1}(d)\right\}$. The result from [20] and [13] implies that if f is a long-tailed function then f is connected. If f is a long-tailed function on a finite set A, one can define a totally order $\leq$ on A by $d > f(d) > ... > f^{\lambda(f)}(d) = f^{\lambda(f)+1}(d)$ which implies that f is an endomorphism on $\langle A; \vee, \wedge, 0\rangle$ where $f^{\lambda(f)}(d) = 0$.

In this thesis, we define a BDLC-algebra to be an algebra $\underline{\mathrm{A}} := \langle A; \vee, \wedge, f, 0, 1\rangle$ whose $\langle A; f\rangle$ is a connected unary algebra and f is an endomorphism on the bounded below distributive lattice $\langle A; \vee, \wedge, 0\rangle$. For example, $A = \left\{1, f(1), ..., f^{\lambda(f)}(1) = 0\right\}$ equipped with $\{\vee, \wedge, f, 0, 1\}$ and $1 > f(1) > ... > f^{\lambda(f)}(1)$ forms a BDLC whose f is a long-tailed function on A and we prove later that this algebra is contained in every BDLC algebra. Since the infinite direct product of connected unary algebras does not need to be connected, the class $\mathcal{M}$ of all BDLC-algebras is not a variety. But we prove that the subclass $\mathcal{M}_n$ of $\mathcal{M}$ whose the pre-period is less than or equal to n is a variety for every positive integer n; in fact, $\mathcal{M}_n$ is the variety satisfying the following laws:

- $f(a \vee b) = f(a) \vee f(b)$,
- $f(a \wedge b) = f(a) \wedge f(b)$,
- $f(0) = 0$,
- $f^n(1) = 0$

for all $a, b, c \in A$.

For a class $\mathcal{B}$ of algebras of the same type, *the variety generated by* $\mathcal{B}$ is the least variety which contains $\mathcal{B}$ and denoted by $V(\mathcal{B})$. In [4], G. Birkhoff proved that $\mathcal{K}$ is a variety if and only if $\mathcal{K} = V(Si(\mathcal{K}))$ where $Si(\mathcal{K})$ is the set of all subdirectly irreducible algebras in $\mathcal{K}$. By the result, every subvariety of $\mathcal{K}$ can be determined by a subset of $Si(\mathcal{K})$. Also, the class $\Lambda(\mathcal{K})$ of all subvarieties of $\mathcal{K}$ equipped with the order $\subseteq$ forms a complete lattice.

B. Jònsson proved in [15] that if $\mathcal{K}$ is a congruence-distributive variety generated by a finite set of finite algebras then $\Lambda(\mathcal{K})$ is a finite distributive lattice; besides, B.A. Davey [8] proved that $\Lambda(\mathcal{K})$ is isomorphic to the lattice $\mathcal{O}(\mathbf{Si}(\mathcal{K}))$ of all order ideals of $(Si(\mathcal{K}); \leq_{Si(\mathcal{K})})$ where an order on $Si(\mathcal{K})$ is defined by $\underline{\mathrm{A}} \leq_{Si(\mathcal{K})} \underline{\mathrm{B}}$ if and only if $\underline{\mathrm{A}} \in HS(\underline{\mathrm{B}})$.

It is known that every variety of lattice based-algebras is congruence distributive; so is $\mathcal{M}_n$ for all positive integers n. To describe the lattice $\Lambda(\mathcal{M}_n)$, it is interesting whether $\mathcal{M}_n$ is generated by a finite set of finite subdirectly irreducible algebras. We will prove the affirmative answer that the set $Si_F(\mathcal{M}_n)$ of all finite subdirectly irreducible algebras in $\mathcal{M}_n$ is finite (up to isomorphism) and then equipped with the result in [19] we prove that all subdirectly irreducible algebras in $\mathcal{M}_n$ are finite.

We organize this thesis into six chapters as follow:

In chapter 2, we summarize some basic concepts from several books which are useful in the sequel.

In chapter 3, we study general properties of BDLC-algebras and apply them to find a certain set of laws for varieties of BDLC-algebras; and then we characterize all their minimal non-identical congruences.

In chapter 4, we characterize all finite subdirectly irreducible BDLC algebras by using the results in Chapter 3; and then apply some results in [19] to prove that the varieties of BDLC-algebras has no infinite subdirectly irreducible algebras; moreover, we show that it is generated by a single subdirectly irreducible algebra.

In chapter 5, we apply the result in [8] to describe the lattice of all subvarieties of the varieties of BDLC-algebras.

In chapter 6, we summarize our main results in previous chapters for more insight.

To avoid a confusion in writing the thesis, let $\mathbb{N}$ be the set of all natural numbers, $\leq^*$ denote the natural order on $\mathbb{N} \cup \{0\}$ and $\leq$ denote the order of the lattice $\langle A; \vee, \wedge, 0, 1\rangle$.

Chapter 2

Basic Concepts

In this chapter, we provide some basic concepts which will be referred in the sequel. All theorems here are stated without proofs.

2.1 Ordered Sets

In this section, we introduce and present some basic properties of an ordered set.

Definition 2.1 Let P be a nonempty set. An *order* (or *partial order*) on P is a binary relation $\leq$ on P satisfying the following three conditions for all $x, y, z \in P$,

1. $x \leq x$, (reflexivity)
2. $x \leq y$ and $y \leq x$ imply $x = y$, (anti-symmetry)
3. $x \leq y$ and $y \leq z$ imply $x \leq z$. (transitivity)

A set P equipped with an order relation $\leq$ is said to be an *ordered set* (or *partially ordered set*) and denoted by $(P; \leq)$. Some authors use the shorthand *poset*. An ordered set $(Q; \leq')$ is called a *subordered set* of $(P; \leq)$ if $Q \subseteq P$ and $\leq'$ is the restriction of $\leq$ to $Q \times Q$, denoted by $\leq\!\downharpoonright_{Q\times Q}$.

An order relation $\leq$ on P gives rise to a relation $<$ of *strictly inequality*: $x < y$ in $\mathbf{P}$ if and only if $x \leq y$ and $x \neq y$. For each $x, y \in P$, we say that x is *comparable* with y if $x \leq y$ or $y \leq x$.

Definition 2.2 Let $\mathbf{P} = (P; \leq)$ be an ordered set.
(i) $\mathbf{P}$ is a *chain* if all pairs of elements of P are comparable.
(ii) $\mathbf{P}$ is an *antichain* if $x = y$ whenever $x \leq y$ for all $x, y \in P$; that is, no pairs of elements in P are comparable.

If $\mathbf{P} = (\{a_1, ..., a_t\}\,; \leq)$ is a finite chain with $a_1 < ... < a_t$ for some $t \in \mathbb{N}$, we denote $\mathbf{P}$ by $\{a_1 < ... < a_t\}$ or $\{a_t > ... > a_1\}$.

Example 2.3 *Examples of ordered sets arising in mathematics such as:*

1. *the set of real numbers equipped with the less than or equal relation* $(\mathbb{R}; \leq)$,
2. *the set of subsets of a given set* A *(power set of* A*) equipped with the inclusion* $(\mathscr{P}(A); \subseteq)$,
3. *the set of natural numbers equipped with the relation of divisibility* $(N; |)$.

Definition 2.4 Let $\mathbf{P}$ be an ordered set and let $x, y \in P$. We say that x is *covered by* y (or y *cover* x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $z = x$ for all $z \in P$ with $x \leq z < y$. The latter condition means that there is no element z of P with $x < z < y$.

Observe that if the universe P of $\mathbf{P}$ is finite, $x < y$ if and only if there exists a finite sequence of covering relations $x = x_0 \prec x_1 \prec \ldots \prec x_n = y$. Thus, in the finite case, the order relation determines, and is determined by, the covering relation.

Definition 2.5 Let $\mathbf{P}$ and $\mathbf{Q}$ be ordered sets and $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ be a function.

1. φ is called an *order-preserving* (or *monotone*) if $x \leq y$ in $\mathbf{P}$ implies $\varphi(x) \leq \varphi(y)$ in $\mathbf{Q}$.
2. φ is called an *order-embedding* if $x \leq y$ in $\mathbf{P}$ if and only if $\varphi(x) \leq \varphi(y)$ in $\mathbf{Q}$.
3. φ is called an *order-isomorphism* if it is an order-embedding mapping $\mathbf{P}$ onto $\mathbf{Q}$.

Whenever $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ is an order-embedding we will write $\varphi : \mathbf{P} \hookrightarrow \mathbf{Q}$. If there exists an order-isomorphism from $\mathbf{P}$ to $\mathbf{Q}$, we say that $\mathbf{P}$ is *isomorphic* to $\mathbf{Q}$ and denoted by $\mathbf{P} \cong \mathbf{Q}$

Definition 2.6 Let $\mathbf{P}$ be an ordered set and $Q \subseteq P$.
(i) Q is a *down-set* (alternative terms include *decreasing set* or *order ideal*) if $y \in Q$ whenever $x \in Q, y \in P$ and $y \leq x$.
(ii) Dually, Q is an *up-set* (alternative terms are *increasing set* or *order filter*) if $y \in Q$ whenever $x \in Q, y \in P$ and $y \geq x$.

Given an arbitrary subset Q of P and $x \in P$, we define

$$\downarrow Q = \{y \in P : (\exists x \in Q) y \leq x\} \text{ and } \uparrow Q = \{y \in P : (\exists x \in Q) y \geq x\}.$$

These are read " down Q "and " up Q ", respectively. It is easily checked that $\downarrow Q$ is the smallest down-set containing Q and that Q is a down-set if and only if $Q = \downarrow Q$, and dually for $\uparrow Q$. If $Q = \{x\}$ then we denote $\downarrow Q$ and $\uparrow Q$ by $\downarrow x$ and $\uparrow x$, respectively; that is, $\downarrow x = \{y \in P : y \leq x\}$ and $\uparrow x = \{y \in P : y \geq x\}$.

The family of all down-sets of P is denoted by $\mathcal{O}(\mathbf{P})$. It is proved that if P is finite then every nonempty set in $\mathcal{O}(\mathbf{P})$ can be written in the form $\downarrow B$ where B is a finite antichain in P.

Definition 2.7 Let $\mathbf{P}$ be an ordered set and $Q \subseteq P$.

1. $a \in Q$ is called a *maximal* element of Q if $a \leq x \in Q$ implies $a = x$ for all $x \in Q$.

2. $a \in Q$ is called a *minimal* element of Q if $a \geq x \in Q$ implies $a = x$ for all $x \in Q$.

3. $a \in Q$ is called the *greatest* (or *maximum*) element of Q if $a \geq x$ for every $x \in Q$, and in that case we write $a = \max Q$.

4. $a \in Q$ is called the *least* (or *minimum*) element of Q if $a \leq x$ for every $x \in Q$, and in that case we write $a = \min Q$.

5. **P** is said to be *bounded* if P has maximum and minimum elements; otherwise, **P** is said to be *unbounded*.

Example 2.8 *Let X be a set. The powerset $\mathscr{P}(X)$, consisting of all subsets of X, is ordered by the set inclusion: for $A, B \in \mathscr{P}(X)$, we define $A \leq B$ if and only if $A \subseteq B$. Moreover, X is the maximum element of $\mathscr{P}(X)$ and $\emptyset$ is the minimum element of $\mathscr{P}(X)$.*

2.2 Algebras

Definition 2.9 Let A be a set. For $n \in \mathbb{N}$, a function $f : A^n \to A$ is called an *n-ary operation* defined on A and is said to have *arity n*. An operation of arities one or two are often said to be *unary* or *binary*, respectively.

Definition 2.10 An *algebra* is a pair $\underline{A} = \langle A; F \rangle$ consisting of

- a nonempty set A which is called *the universe of* $\underline{A}$, and
- a set F of operations defined on A which is called *the set of the fundamental operations of* $\underline{A}$.

Sometimes, we may consider F as $\{f_j\}_{j \in J}$ for some index sets J. If $F = \{f_1, f_2, ..., f_t\}$ is finite for some positive integers t, we write $\underline{A} = \langle A; f_1, f_2, ..., f_t \rangle$. A *type* $\tau = (n_j)_{j \in J}$ of algebra is the sequence of all the arities of f_j. Avoiding of confusion, we denote $f_j^{\underline{A}}$ for n_j-ary operation of algebra $\underline{A}$ for all $j \in J$. If all elements in F are unary operations, $\underline{A} = \langle A; F \rangle$ is called a unary algebra. In particular, if F is a singleton set of a unary operation then $\underline{A}$ is a mono-unary algebra and we denote $\langle A; F \rangle$ by $\langle A; f \rangle$.

Groups and rings are examples of algebras of type (2) and $(2, 2)$. For a group, we may consider its identity e as a nullary operation of arity 0 which means a function from $\{\emptyset\}$ to e. And also, a function $^{-1}$,which map each element to its inverse, is a unary operation. In this case, a group is an algebra of type $(2, 1, 0)$.

Definition 2.11 Let $\underline{A} = \left\langle A; \left\{ f_j^{\underline{A}} \right\}_{j \in J} \right\rangle$ and $\underline{B} = \left\langle B; \left\{ f_j^{\underline{B}} \right\}_{j \in J} \right\rangle$ be algebras of the same type. $\underline{B}$ is called a *subalgebra* of $\underline{A}$, if the following conditions are satisfied:

1. $B \subseteq A$,

2. f_j^{B} is the restriction of the operation f_j^{A} to the set B, denoted by $f_j^{\mathrm{A}} \restriction_B$, for all $j \in J$.

Lemma 2.12 (Subalgebra Criterion)[13] Let $\underline{\mathrm{A}} = \left\langle A; \left\{ f_j^{\mathrm{A}} \right\}_{j \in J} \right\rangle$ be an algebra of type τ and let $B \subseteq A$ and $f_j^{\mathrm{B}} = f_j^{\mathrm{A}} \restriction_B$ for all $j \in J$. Then $\underline{\mathrm{B}} = \left\langle B; \left\{ f_j^{\mathrm{B}} \right\}_{j \in J} \right\rangle$ is a subalgebra of $\underline{\mathrm{A}}$ if and only if $f_j^{\mathrm{A}}(B^{n_j}) \subseteq B$ for all $j \in J$.

Definition 2.13 A binary relation θ on a set A is called an *equivalence relation* on A if the following three conditions hold for all $a, b, c \in A$:

1. $(a, a) \in \theta$, (reflexivity)

2. $(a, b) \in \theta$ implies $(b, a) \in \theta$, (symmetry)

3. $(a, b) \in \theta$ and $(b, c) \in \theta$ imply $(a, c) \in \theta$. (transitivity)

Definition 2.14 Let A be a set, let $\theta \subseteq A \times A$ be an equivalence relation on A, and let f be an n-ary operation on A. Then f is said to be *compatible* with θ, or to *preserve* θ or θ is *invariant with respect to* f, if for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$(a_1, b_1) \in \theta, \ldots, (a_n, b_n) \in \theta \text{ implies } (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta.$$

Definition 2.15 Let $\underline{\mathrm{A}}$ be an algebra. An equivalence relation θ on A is called a *congruence relation* on $\underline{\mathrm{A}}$ if all its fundamental operations are compatible with θ. We denote by $Con\ \underline{\mathrm{A}}$ the set of all congruence relations of the algebra $\underline{\mathrm{A}}$. In facts, $(Con\ \underline{\mathrm{A}}; \subseteq)$ is an ordered set.

For every algebra $\underline{\mathrm{A}}$, the equivalence relations

$$\Delta_A := \{(a, a) \mid a \in A\} \quad \text{and} \quad \nabla_A := A \times A$$

are congruence relations which are called the *identity relation* and the *full relation*, respectively.

Theorem 2.16 [13] *Let* $\{\theta_i : i \in I\} \subseteq Con\ \underline{\mathrm{A}}$. *Then* $\bigcap_{i\in I}\theta_i$ *is a congruence relation on* $\underline{\mathrm{A}}$.

Remark 2.17 [13] *In general, the union of two congruence relations of an algebra is not necessary a congruence relation since this does not hold even for equivalence relations; for example, let* $A = \{1, 2, 3\}$ *and define*

$$\theta_1 := \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\} \text{ and } \theta_2 := \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}.$$

Then θ_1 *and* θ_2 *are equivalence relations; but*

$$\theta_1 \cup \theta_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$$

is not an equivalence relation on A *since it is not transitive:*

$$(1, 2) \in \theta_1 \cup \theta_2 \text{ and } (2, 3) \in \theta_1 \cup \theta_2 \text{ but } (1, 3) \notin \theta_1 \cup \theta_2.$$

As in the subalgebra case, we can define a smallest congruence generated by the union. This motivates the following definition.

Definition 2.18 Let $\underline{\mathrm{A}}$ be an algebra and let θ be a binary relation on A. We define the congruence relation $\langle\theta\rangle_{Con\ \underline{\mathrm{A}}}$ on $\underline{\mathrm{A}}$ generated by θ to be the intersection of all congruence relations θ' on $\underline{\mathrm{A}}$ which contain θ:

$$\langle\theta\rangle_{Con\ \underline{\mathrm{A}}} := \cap\{\theta' : \theta' \in Con\ \underline{\mathrm{A}} \text{ and } \theta \subseteq \theta'\}.$$

Definition 2.19 Let $\underline{\mathrm{A}} = \left\langle A; \left\{f_j^{\mathrm{A}}\right\}_{j\in J}\right\rangle$ and $\underline{\mathrm{B}} = \left\langle B; \left\{f_j^{\mathrm{B}}\right\}_{j\in J}\right\rangle$ be algebras of the same type. A function $h : \underline{\mathrm{A}} \to \underline{\mathrm{B}}$ is called a *homomorphism* from $\underline{\mathrm{A}}$ into $\underline{\mathrm{B}}$ if for all $j \in J$,

$$h(f_j^{\mathrm{A}}(a_1, ..., a_{n_j})) = f_j^{\mathrm{B}}(h(a_1), ..., h(a_{n_j}))$$

for all $a_1, ..., a_{n_j} \in A$. A homomorphism from $\underline{\mathrm{A}}$ into itself is called an *endomorphism.* A surjective homomorphism is called an *epimorphism.* An injective homomorphism is called a *monomorphism* or an *embedding.* A bijective homomorphism is called an *isomorphism.*

Definition 2.20 Let $\{\underline{\mathrm{A_i}} : i \in I\}$ be a family of algebras of type τ. The *direct product* of the family $\{\underline{\mathrm{A_i}} : i \in I\}$ is defined as an algebra of type τ with the carrier set

$$\mathcal{A} := \{(a_i)_{i\in I} : a_i \in A_i \text{ for all } i \in I\}$$

and for each $j \in J$ the corresponding operations is defined by

$$f_j^{\mathcal{A}}((a_{1i})_{i\in I}, ..., (a_{n_j i})_{i\in I}) = (f_j^{\underline{\mathrm{A_i}}}(a_{1i}, ..., a_{n_j i}))_{i\in I}$$

We denote the direct product $\left\langle \mathcal{A}; \left\{f_j^{\mathcal{A}}\right\}_{j\in J} \right\rangle$ by $\prod\limits_{i\in I}\underline{\mathrm{A_i}}$. If $J = \{1, ..., n\}$ then $\prod\limits_{i\in I}\underline{\mathrm{A_i}}$ can be written as $\underline{\mathrm{A_1}} \times \cdots \times \underline{\mathrm{A_n}}$.

Definition 2.21 An algebra $\underline{\mathrm{A}}$ of type τ is called *subdirectly irreducible* if $\bigcap\limits_{i\in I}\theta_i \neq \Delta_A$ for all $\theta_i \in Con(\underline{\mathrm{A}}) \setminus \{\Delta_A\}$ and $i \in I$.

Remark 2.22 [13] *It is easy to see that an algebra $\underline{\mathrm{A}}$ is subdirectly irreducible if and only if Δ_A has exactly one cover in the ordered set $(Con\,\underline{\mathrm{A}}; \subseteq)$ of all congruence relations on $\underline{\mathrm{A}}$. Then the ordered set $(Con\,\underline{\mathrm{A}}; \subseteq)$ has the form shown in the following figure.*

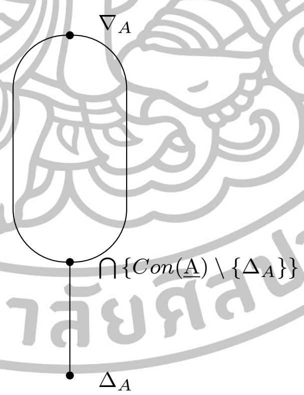

Figure 2 The ordered set $(Con\,\underline{\mathrm{A}}; \subseteq)$.

2.3 Terms and Term Operations

For each positive integer n, an n-element set $X_n = \{x_1, ..., x_n\}$ is called an *alphabet* and its elements are called *variables*. To every operation symbol f_j, we assign an integer $n_j \geq^* 0$, the arity of f_j. Let $\tau = (n_j)_{j\in J}$ be a type such that the set of operation symbols $\{f_j\}_{j\in J}$ is disjoint with X_n. Now we define the terms of type τ.

Definition 2.23 For each positive integer n, the *n-ary terms* of type τ are defined in the following inductive way:

1. every variable $x_j \in X_n$ is an n-ary term,

2. for $n_j \in \tau$, if $t_1, ..., t_{n_j}$ are n-ary terms and f_j is an n_j-ary operation symbol, then $f_j(t_1, ..., t_{n_j})$ is an n-ary term.

It follows immediately from the definition that every n-ary term is also k-ary if $k > n$; and we may write the n-ary term t in full by $t(x_1, x_2, ..., x_n)$. The set $W_\tau(X_n) = W_\tau(x_1, x_2, ..., x_n)$ of all n-ary terms is the smallest set which contains $x_1, x_2, ..., x_n$ and is closed under finite application of (2). The set of all terms of type τ over the alphabet $X := \{x_1, x_2, ...\}$ is defined as the union

$$W_\tau(X) = \bigcup_{n=1}^{\infty} W_\tau(X_n);$$

that is, $W_\tau(X)$ is the set of all terms of type τ over the countably infinite alphabet X. Let $\underline{A}$ be an algebra of type τ and let t be an n-ary term of type τ over X. Then the n-ary operation $t^{\underline{A}}$ on $\underline{A}$, which is called the *term operation on* $\underline{A}$, is induced by t via the following steps:

1. if $t = x_i$ then $t^{\underline{A}}$ is an n-ary projection on $\underline{A}$,

2. if $t = f_j(t_1, ..., t_{n_j})$ is n-ary term of type τ and $t_1^{\underline{A}}, ..., t_{n_j}^{\underline{A}}$ are term operations which are induced by $t_1, ..., t_{n_j}$ then $t^{\underline{A}} = f_j^{\underline{A}}(t_1^{\underline{A}}, ..., t_{n_j}^{\underline{A}})$

Theorem 2.24 [13] *Let* $\underline{A} = \left\langle A; \left\{f_j^{\underline{A}}\right\}_{j\in J}\right\rangle$ *and* $\underline{B} = \left\langle B; \left\{f_j^{\underline{B}}\right\}_{j\in J}\right\rangle$ *be algebras of type* τ *and let* n *be a positive integer.*

1. *If* $t \in W_\tau(X_n)$ *and* $\alpha : \underline{A} \to \underline{B}$ *is a homomorphism then*

$$\alpha(t^{\underline{A}}(a_1, a_2, ..., a_n)) = t^{\underline{B}}(\alpha(a_1), \alpha(a_2), ..., \alpha(a_n))$$

for all $a_1, a_2, ..., a_n \in A$.

2. *If* $S \subseteq A$ *then*

$$\langle S\rangle_{\underline{A}} = \left\{t^{\underline{A}}(a_1, a_2, ..., a_n) : t \in W_\tau(X_n), n \text{ is a positive integer and } a_1, a_2, ..., a_n \in S\right\}.$$

2.4 Identities and Varieties

In this section, we will introduce a popular class of algebras which is called a variety.

Definition 2.25 An *equation* of type τ is a pair of terms $(p, q) \in (W_\tau(X))^2$. Those such pairs are more commonly written as $p \approx q$.

Definition 2.26 An equation $p \approx q$ is said to be an *identity* of the algebra $\underline{A}$ of type τ if $p^{\underline{A}} = q^{\underline{A}}$; that is, if the term operations induced by p and q on the algebra $\underline{A}$ are equal. In this case we also say that the equation $p \approx q$ is *satisfied* by the algebra $\underline{A}$, and we write $\underline{A} \models p \approx q$.

If $p^{\underline{A}} = q^{\underline{A}}$ then we say that $\underline{A}$ *satisfies the law* $p^{\underline{A}} = q^{\underline{A}}$. For instance, we know that a group G is an algebra satisfying the associative identity $a \cdot (b \cdot c) \approx (a \cdot b) \cdot c$; that is, G satisfies the associative law: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in G$.

Let $Alg(\tau)$ be the class of all algebras of type τ. For any subset $\Sigma \subseteq (W_\tau(X))^2$ and any subclass $\mathcal{K} \subseteq Alg(\tau)$, let defined:

$$Mod\Sigma := \{\underline{A} \in Alg(\tau) : \forall p \approx q \in \Sigma, \underline{A} \models p \approx q\},$$

$$Id\mathcal{K} := \left\{p \approx q \in (W_\tau(X))^2 : \forall \underline{A} \in Alg(\tau), \underline{A} \models p \approx q\right\},$$

$S(\mathcal{K})$ is the class of all subalgebras of algebras from $\mathcal{K}$,
$H(\mathcal{K})$ is the class of all homomorphic images of algebras from $\mathcal{K}$,
$P(\mathcal{K})$ is the class of all direct products of families of algebras from $\mathcal{K}$.

Definition 2.27 A class $\mathcal{K} \subseteq Alg(\tau)$ is called a *variety* if $\mathcal{K}$ is closed under the operators H, S and P; that is, if $H(\mathcal{K}) \subseteq \mathcal{K}; S(\mathcal{K}) \subseteq \mathcal{K}$ and $P(\mathcal{K}) \subseteq \mathcal{K}$.

Theorem 2.28 [13] *For any class $\mathcal{K}$ of algebras of type τ, the class $HSP(\mathcal{K})$ is the least (with respect to set inclusion) variety which contains $\mathcal{K}$.*

For any class $\mathcal{K}$ of algebras of the same type, the variety $HSP(\mathcal{K})$ from Theorem 2.28 is called *the variety generated by* $\mathcal{K}$, it often denoted by $V(\mathcal{K})$. If $\mathcal{K}$ consists of a single algebra $\underline{A}$, we usually write $V(\underline{A})$ for the variety generated by $\underline{A}$.

Corollary 2.29 [13] *A class $\mathcal{K}$ of algebras of type τ is a variety if and only if $HSP(\mathcal{K}) = \mathcal{K}$.*

Theorem 2.30 [3] *A class $\mathcal{K}$ of algebras of type τ is a variety if and only if $\mathcal{K} = Mod(\Sigma)$ for some $\Sigma \subseteq (W_\tau(X))^2$.*

Theorem 2.31 [4] *A class $\mathcal{K}$ of algebras of type τ is a variety if and only if $\mathcal{K} = V(Si(\mathcal{K}))$ where $Si(\mathcal{K})$ is the set of all subdirectly irreducible algebras in $\mathcal{K}$.*

Definition 2.32 A variety $\mathcal{K}$ is *locally finite* if every finitely generated algebra in $\mathcal{K}$ is finite

2.5 Lattices

In this section, we give a definition and some properties of lattice.

Definition 2.33 Let $\mathbf{L}$ be an ordered set and let $S \subseteq L$. An element $x \in L$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. A *lower bound* is defined dually. The set of all upper bounds of S is denoted by S^u(read as 'S upper') and the set of all lower bounds of S is denoted by S^l(read as 'S lower'); that is,

$$S^u = \{x \in L : (\forall s \in S)\ s \leq x\} \ \ and \ \ S^l = \{x \in L : (\forall s \in S)\ s \geq x\}.$$

If S^u has the least element x then x is called the *least upper bound* of S or the *supremum* of S and is denoted by $supS$. Equivalently, x is the least upper bound of S if

1. x is an upper bound of S and

2. $x \leq y$ for all upper bound y of S.

Dually, if S^l has the greatest element x then x is called the *greatest lower bound* of S or the *infimum* of S and is denoted by $infS$.

Notation: We write $\vee S$ instead of $supS$ whenever $supS$ exists; for special case $S = \{x, y\}$, we write $x \vee y$ (read as 'x joins y'). Similarly we write $\wedge S$ or $x \wedge y$ (read as 'x meets y') instead of $infS$ whenever $infS$ exists.

Definition 2.34 Let $\mathbf{L}$ be a non-empty ordered set.

1. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in L$ then $\mathbf{L}$ is called a *lattice*.

2. If $\vee S$ and $\wedge S$ exist for all $S \subseteq L$ then $\mathbf{L}$ is called a *complete lattice*.

If $\mathbf{L}$ is a lattice then $\vee$ and $\wedge$ can be considered as binary operations on its universe; so, $\langle L; \vee, \wedge \rangle$ is an algebra. It is proved that $\langle L; \vee, \wedge \rangle$ satisfies the following identities:

- commutative : $x \vee y \approx y \vee x$ and $x \wedge y \approx y \wedge x$,

- associative : $(x \vee y) \vee z \approx x \vee (y \vee z)$ and $(x \wedge y) \wedge z \approx x \wedge (y \wedge z)$,

- idempotent : $x \vee x \approx x$ and $x \wedge x \approx x$,

- absorption : $x \vee (x \wedge y) \approx x$ and $x \wedge (x \vee y) \approx x$.

Conversely, if $\langle L; \vee, \wedge \rangle$ is an algebra satisfies those above four identities then L equipped with the order $\leq$ defined by

$$x \leq y \Leftrightarrow x = x \wedge y \text{ for all } x, y \in L$$

$$\Leftrightarrow y = x \vee y \text{ for all } x, y \in L$$

is a lattice.

By subalgebra criterion, if $\mathbf{L} = \langle L; \vee, \wedge \rangle$ is a lattice and $\emptyset \neq M \subseteq L$ then $\mathbf{M}$ is a *sublattice* of $\mathbf{L}$ if and only if $a \vee b \in M$ and $a \wedge b \in M$ for all $a, b \in M$.

Definition 2.35 Let $\mathbf{L}$ be a lattice.

(i) $\mathbf{L}$ is said to be *distributive* if it satisfies the distributive identity

$$a \wedge (b \vee c) \approx (a \wedge b) \vee (a \wedge c).$$

(ii) $\mathbf{L}$ is said to be *modular* if it satisfies the following condition:

$$a \wedge (b \vee c) = (a \wedge b) \vee c \text{ whenever } a \geq c \text{ for all } a, b, c \in L$$

which is equivalent to satisfying the modular identity $x \wedge (y \vee (x \wedge z)) \approx (x \wedge y) \vee (x \wedge z)$.

Theorem 2.36 [10] *The $M_3 - N_5$ Theorem:*
Let $\mathbf{L}$ *be a lattice. Then*
(i) $\mathbf{L}$ *is distributive if and only if* $\mathbf{L}$ *has no sublattices isomorphic to both* N_5 *and* M_3.
(ii) $\mathbf{L}$ *is modular if and only if* $\mathbf{L}$ *has no sublattices isomorphic to* N_5.

Theorem 2.37 [10] *If* $\mathbf{L}$ *is a distributive lattice then* $\mathbf{L}$ *is a modular lattice.*

Note that every chain is distributive.

Proposition 2.38 [13] *For every algebra* $\underline{\mathrm{A}}$*, the structure* $(Con\ \underline{\mathrm{A}}; \wedge, \vee)$ *with*
$\wedge : Con\ \underline{\mathrm{A}} \times Con\ \underline{\mathrm{A}} \longrightarrow Con\ \underline{\mathrm{A}}$ *define by* $(\theta_1, \theta_2) \longmapsto \theta_1 \cap \theta_2$,
$\vee : Con\ \underline{\mathrm{A}} \times Con\ \underline{\mathrm{A}} \longrightarrow Con\ \underline{\mathrm{A}}$ *define by* $(\theta_1, \theta_2) \longmapsto \langle \theta_1 \cup \theta_2 \rangle_{Con\ \underline{\mathrm{A}}}$
is a complete lattice, called the congruence lattice $Con(\underline{\mathrm{A}})$ *of* $\underline{\mathrm{A}}$.

The structure $(Con\ \underline{\mathrm{A}}; \wedge, \vee)$ is called the *congruence lattice* of $\underline{\mathrm{A}}$ and denote it by $Con(\underline{\mathrm{A}})$.

Definition 2.39 Let $\underline{\mathrm{A}}$ be an algebra.
(i) $\underline{\mathrm{A}}$ is *congruence-distributive* if $Con(\underline{\mathrm{A}})$ is distributive.
(ii) $\underline{\mathrm{A}}$ is *congruence-modular* if $Con(\underline{\mathrm{A}})$ is modular.

Definition 2.40 A variety $\mathcal{K}$ is *congruence distributive* if every algebra $\underline{\mathrm{A}} \in \mathcal{K}$ is congruence distributive.

Chapter 3

Algebraic Properties of BDLC-algebras

We begin this work with studying algebraic properties of BDLC-algebras such as general properties, subalgebras, product of algebras, homomorphic image of algebras and minimal algebras which are useful in the sequel. Then, we apply those results to show that the class $\mathcal{M}_n$ of all BDLC whose $\lambda(f) \leq^* n$ is a variety for all $n \in \mathbb{N}$; moreover, it can be described by identities. Besides in this chapter, we characterize all minimal (non-identical) congruences in $\mathrm{Con}\,(\underline{\mathrm{A}})$ for $\underline{\mathrm{A}} \in \mathcal{M}_n$ having no infinite chains.

3.1 Identities for Varieties of BDLC-algebras

In this section, we include general properties of BDLC-algebras and apply them to study algebraic properties of BDLC-algebras.

Proposition 3.1 *Let* $\underline{\mathrm{A}} := \langle A; \vee, \wedge, f, 0, 1\rangle$ *be BDLC.*

1. *If* $x \leq y$*, then* $f(x) \leq f(y)$ *for all* $x, y \in A$*,*

2. $\lambda(f)$ *is finite and* $f^{\lambda(f)}(A) = \{0\}$*,*

3. $\lambda(f)$ *is the least nonnegative integer such that* $f^{\lambda(f)}(1) = 0$*,*

4. $f^t(1) \leq f^k(1)$ *for all* $1 \leq^* k \leq^* t \leq^* \lambda(f)$*,*

5. *for each* $x \in A$ *and* $j \in \mathbb{N}$, $f^j(x) = x$ *if and only if* $x = 0$,

6. $\lambda(f_{\underline{\mathrm{A}}}) = \lambda(f_{\underline{\mathrm{B}}})$ *for all subalgebras* $\underline{\mathrm{B}}$ *of* $\underline{\mathrm{A}}$,

7. $\lambda(f_{\underline{\mathrm{B}}}) \leq^* \lambda(f_{\underline{\mathrm{A}}})$ *for all homomorphic images* $\underline{\mathrm{B}}$ *of* $\underline{\mathrm{A}}$.

Proof. (*1*) Let $x, y \in A$ with $x \leq y$. Then $x = x \wedge y$ which implies that $f(x) = f(x) \wedge f(y) \leq f(y)$.

(*2*) Since f is connected, there are $n, m \in \mathbb{N} \cup \{0\}$ such that $f^n(1) = f^m(0) = 0$; and the result from (*1*) implies that $\mathrm{Im} f^n = \mathrm{Im} f^{n+1} = \{0\}$; so, $\lambda(f) \leq^* n$. By the property of $\lambda(f)$, we get $\mathrm{Im} f^{\lambda(f)} = \mathrm{Im} f^t$ for all $t \geq^* \lambda(f)$; and so, $f^{\lambda(f)}(A) = \mathrm{Im} f^{\lambda(f)} = \mathrm{Im} f^n = \{0\}$.

(*3*) Let $m \in \mathbb{N} \cup \{0\}$ such that $f^m(1) = 0$. By (*1*), we have $\mathrm{Im} f^m = f^m(A) = \{0\}$ which implies that $\mathrm{Im} f^m = \mathrm{Im} f^{m+1}$; so, $\lambda(f) \leq^* m$; that is, $\lambda(f)$ is the least nonnegative integer such that $f^{\lambda(f)}(1) = 0$.

(*4*) Since 1 is the greatest element with respect to $\leq$, we have $f^t(1) = f^k(f^{t-k}(1)) \leq f^k(1)$ for all $k, t \in \mathbb{N} \cup \{0\}$ with $k \leq^* t$.

(*5*) Let $x \in A$ and $j \in \mathbb{N}$. Since f preserves 0, if $x = 0$ then $f^j(x) = x$. Conversely, assume that $f^j(x) = x$. Since $j\lambda(f) \geq^* \lambda(f)$ and by (*4*), we have $x = f^{j\lambda(f)}(x) \leq f^{j\lambda(f)}(1) \leq f^{\lambda(f)}(1) = 0$ which implies that $x = 0$.

(*6*) Let $\underline{\mathrm{B}}$ be a subalgebra of $\underline{\mathrm{A}}$. Then $1_{\underline{\mathrm{A}}} = 1_{\underline{\mathrm{B}}}$; and so, $f^{\lambda(f_{\underline{\mathrm{A}}})}(1_{\underline{\mathrm{B}}}) = f^{\lambda(f_{\underline{\mathrm{A}}})}(1_{\underline{\mathrm{A}}}) = 0$. By (*3*), we get $\lambda(f_{\underline{\mathrm{B}}}) \leq^* \lambda(f_{\underline{\mathrm{A}}})$. Similarly, $\lambda(f_{\underline{\mathrm{A}}}) \leq^* \lambda(f_{\underline{\mathrm{B}}})$. Hence, $\lambda(f_{\underline{\mathrm{A}}}) = \lambda(f_{\underline{\mathrm{B}}})$.

(*7*) Let $\underline{\mathrm{B}} = h(\underline{\mathrm{A}})$ where $h : \underline{\mathrm{A}} \to \underline{\mathrm{B}}$ is a homomorphism. Since $1_{\underline{\mathrm{B}}} = h(1_{\underline{\mathrm{A}}})$, we have $f^{\lambda(f_{\underline{\mathrm{A}}})}(1_{\underline{\mathrm{B}}}) = h(f^{\lambda(f_{\underline{\mathrm{A}}})}(1_{\underline{\mathrm{A}}})) = h(0_{\underline{\mathrm{A}}}) = 0_{\underline{\mathrm{B}}}$ which implies that $\lambda(f_{\underline{\mathrm{B}}}) \leq^* \lambda(f_{\underline{\mathrm{A}}})$. ■

For a BDLC-algebra $\underline{\mathrm{A}}$, the chain $\left\{1 > f(1) > ... > f^{\lambda(f)}(1) = 0\right\}$ forms a subalgebra and is contained in every subalgebras of $\underline{\mathrm{A}}$. Hence, it is the smallest subalgebra of $\underline{\mathrm{A}}$; so, we denote it by $\underline{\mathrm{C_A}}$ and call it the *core BDLC-subalgebra of* $\underline{\mathrm{A}}$. If $\underline{\mathrm{A}} = \underline{\mathrm{C_A}}$; we call $\underline{\mathrm{A}}$, the *core BDLC-algebra*.

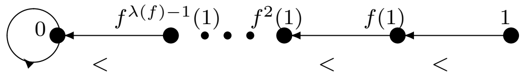

Figure 3 The core BDLC-algebra.

The following proposition shows some basic properties of BDLC-algebras which can be proved directly from properties of a homomorphism.

Proposition 3.2 *Let $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ be BDLC-algebras and $\phi : \underline{\mathrm{A}} \to \underline{\mathrm{B}}$ be a homomorphism.*

1. *$\phi(0) = 0$, $\phi(1) = 1$ and $\phi(f^t(1)) = f^t(1)$ for all $t \geq^* 1$,*
2. *$\phi(\underline{\mathrm{H}})$ is a subalgebra of $\underline{\mathrm{B}}$ for all subalgebras $\underline{\mathrm{H}}$ of $\underline{\mathrm{A}}$,*
3. *$\phi^{-1}(\underline{\mathrm{K}})$ is a subalgebra of $\underline{\mathrm{A}}$ for all subalgebras $\underline{\mathrm{K}}$ of $\underline{\mathrm{B}}$,*
4. *$\phi(C_A) = C_B$.*

Corollary 3.3 *If $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ are BDLC with $|C_A| <^* |C_B|$ then there are no homomorphisms between $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$.*

Recall that an algebra $\underline{\mathrm{A}}$ is minimal if $\underline{\mathrm{A}} \cong \underline{\mathrm{B}}$ whenever $\underline{\mathrm{B}}$ can be embedded in $\underline{\mathrm{A}}$ for all algebras $\underline{\mathrm{B}}$; or equivalently, $\underline{\mathrm{A}}$ has no proper subalgebras. Since $\underline{\mathrm{C_A}}$ is a subalgebra of $\underline{\mathrm{A}}$ for all BDLC-algebras $\underline{\mathrm{A}}$, it is obvious that $\underline{\mathrm{A}} = \underline{\mathrm{C_A}}$ whenever $\underline{\mathrm{A}}$ is minimal; and Proposition 3.2 implies the converse that $\underline{\mathrm{A}}$ is minimal whenever $\underline{\mathrm{A}} = \underline{\mathrm{C_A}}$.

Proposition 3.4 *All core BDLC-algebras are precisely minimal BDLC-algebras.*

Proof. Let $\underline{\mathrm{B}}$ be a BDLC-algebra which can be embedded in a core BDLC $\underline{\mathrm{A}}$. Then there is a monomorphism $\phi : \underline{\mathrm{B}} \to \underline{\mathrm{A}}$; so, by Proposition 3.2, $A = C_A = \phi(C_B) \subseteq \phi(B) \subseteq A$. Hence, ϕ is surjective. ∎

It is well known that the infinite direct product of connected unary algebras does not need to be connected; so, the class of all BDLC-algebras is not closed under the product. However, Proposition 3.1(2) implies that a direct product of BDLC-algebras whose $\lambda(f) \leq^* n$ for some fixed $n \in \mathbb{N}$ is BDLC.

Proposition 3.5 *Let $n \in \mathbb{N}$ and $\left\{\underline{\mathrm{A_i}} : i \in I\right\}$ be a family of BDLC-algebras whose $\lambda(f^{\underline{\mathrm{A_i}}}) \leq^* n$ for all $i \in I$. Then $\underline{\mathrm{A}} = \prod_{i\in I}\underline{\mathrm{A_i}}$ is a BDLC-algebra whose $\lambda(f^{\underline{\mathrm{A}}}) \leq^* n$; moreover, there exists $j \in I$ such that $\lambda(f^{\underline{\mathrm{A_j}}}) = \max\left\{\lambda(f^{\underline{\mathrm{A_i}}}) \in \mathbb{N} : i \in I\right\}$ and $\underline{\mathrm{C_{A_j}}} \cong \underline{\mathrm{C_A}}$.*

Proof. To show that the product $\underline{\mathrm{A}}$ is BDLC, it is left to prove that $f^{\underline{\mathrm{A}}}$ is connected. By Proposition 3.1(2), we have $(f^{\underline{\mathrm{A}}})^n(a)(i) = (f^{\underline{\mathrm{A_i}}})^n(a(i)) = 0 = (f^{\underline{\mathrm{A_i}}})^n(b(i)) = (f^{\underline{\mathrm{A}}})^n(b)(i)$ for all $a, b \in A$ and $i \in I$.

Note that $\left\{\lambda(f^{\underline{\mathrm{A_i}}}) \in \mathbb{N} : i \in I\right\}$ is bounded above by n; so, it is finite. Let $\lambda(f^{\underline{\mathrm{A_j}}}) = \max\left\{\lambda(f^{\underline{\mathrm{A_i}}}) \in \mathbb{N} : i \in I\right\}$ for some $j \in I$. Then, $(f^{\underline{\mathrm{A}}})^{\lambda(f^{\underline{\mathrm{A}}})}(1_{\underline{\mathrm{A}}}) = 0_{\underline{\mathrm{A}}}$ implies that $(f^{\underline{\mathrm{A_j}}})^{\lambda(f^{\underline{\mathrm{A}}})}(1_{\underline{\mathrm{A_j}}}) = 0_{\underline{\mathrm{A_j}}}$; so, $\lambda(f^{\underline{\mathrm{A_j}}}) \leq^* \lambda(f^{\underline{\mathrm{A}}})$ Since $\lambda(f^{\underline{\mathrm{A_j}}}) \geq^* \lambda(f^{\underline{\mathrm{A_i}}})$ for all $i \in I$, we have $(f^{\underline{\mathrm{A_i}}})^{\lambda(f^{\underline{\mathrm{A_j}}})}(1_{\underline{\mathrm{A_i}}}) = 0_{\underline{\mathrm{A_i}}}$ for all $i \in I$; so, $(f^{\underline{\mathrm{A}}})^{\lambda(f^{\underline{\mathrm{A_j}}})}(1_{\underline{\mathrm{A}}}) = 0_{\underline{\mathrm{A}}}$ which also implies that $\lambda(f^{\underline{\mathrm{A}}}) \leq^* \lambda(f^{\underline{\mathrm{A_j}}})$. Altogether, $\lambda(f^{\underline{\mathrm{A}}}) = \lambda(f^{\underline{\mathrm{A_j}}})$. Therefore, $\underline{\mathrm{C}_{\mathrm{A_j}}} \cong \underline{\mathrm{C}_{\mathrm{A}}}$. ■

From now on, let $n \in \mathbb{N}$ and $\mathcal{M}_n$ be the class of all BDLC whose $\lambda(f) \leq^* n$. By Proposition 3.5, we have $P(\mathcal{M}_n) \subseteq \mathcal{M}_n$. It is clear that a homomorphic image $\underline{\mathrm{B}}$ of a BDLC $\underline{\mathrm{A}}$ is BDLC and Proposition 3.1 implies that $\lambda(f^{\underline{\mathrm{B}}}) \leq^* \lambda(f^{\underline{\mathrm{A}}})$. Similarly, a subalgebra $\underline{\mathrm{B}}$ of a BDLC $\underline{\mathrm{A}}$ is BDLC whose $\lambda(f^{\underline{\mathrm{B}}}) = \lambda(f^{\underline{\mathrm{A}}})$.

Theorem 3.6 *$\mathcal{M}_n$ is a variety.*

Notice that $\bigcup_{i=1}^{\infty}\mathcal{M}_i$ is the class of all BDLC-algebras and $\mathcal{M}_i \subseteq \mathcal{M}_j$ for all $j \geq^* i$. Since the pre-period of the unary operation f of all algebras in $\mathcal{M}_n$ is less than or equal to n and by Proposition 3.1(2), we have $f^n(1) = 0$. And also, if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and f is an endomorphism on $\langle A; \vee, \wedge, 0\rangle$ with $f^n(1) = 0$ then $f^n(a) = 0$ for all $a \in A$ which implies that f is connected; so, $\langle A; \vee, \wedge, f, 0, 1\rangle$ belongs to $\mathcal{M}_n$. We have the following characterization.

Proposition 3.7 *An algebra $\underline{\mathrm{A}} := \langle A; \vee, \wedge, f, 0, 1\rangle$ is in $\mathcal{M}_n$ if and only if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and f is an endomorphism on $\langle A; \vee, \wedge, 0\rangle$ with $f^n(1) = 0$.*

We conclude the section by showing identities for the varieties of BDLC-algebras.

Theorem 3.8 *The variety $\mathcal{M}_n$ of BDLC-algebras is a class of BDL-algebras satisfying the following identities:*

$$f(x \vee y) \approx f(x) \vee f(y), f(x \wedge y) \approx f(x) \wedge f(y), f(0) \approx 0 \text{ and } f^n(1) \approx 0$$

3.2 The Congruence Lattice

It is well-known that a variety of BDL-algebras is congruence distributive; hence, also is $\mathcal{M}_n$. In this section, we will show a natural way of defining a unary operation on $\mathrm{Con}\,(\underline{\mathrm{A}})$ for $\underline{\mathrm{A}} \in \mathcal{M}_n$ so that $\underline{\mathrm{Con}\,(\underline{\mathrm{A}})} \in \mathcal{M}_n$ and we show a relationship between $\underline{\mathrm{C}_{\mathrm{A}}}$ and $\underline{\mathrm{C}_{\mathrm{Con}(\underline{\mathrm{A}})}}$.

Recall that for an algebra $\underline{\mathrm{A}}$, $\mathrm{Con}\,(\underline{\mathrm{A}})$ is a complete lattice having $\triangle_{\underline{\mathrm{A}}}$ and $\underline{\mathrm{A}} \times \underline{\mathrm{A}}$ as the least and the greatest element, $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$ and $\theta_1 \vee \theta_2 = \langle \theta_1 \cup \theta_2 \rangle$; besides, $\langle \theta_1 \cup \theta_2 \rangle = \bigcup\limits_{t \in \mathbb{N}} \underbrace{\theta_1 \circ \theta_2 \circ \cdots \circ \theta_1}_{t}$ and if $(a,b) \in \langle \theta_1 \cup \theta_2 \rangle$, there is $t \in \mathbb{N}$ such that $(a,b) \in \underbrace{\theta_1 \circ \theta_2 \circ \cdots \circ \theta_1}_{t}$; that is, $(a,c_1) \in \theta_1, (c_1,c_2) \in \theta_2, ..., (c_{t-1},b) \in \theta_1$ for some $c_1, ..., c_{t-1} \in A$.

Lemma 3.9 *Let* $\underline{\mathrm{A}} \in \mathcal{M}_n$.

1. *If* $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}})$ *then* $\bar{\theta} := \{(f(a), f(b)) : (a,b) \in \theta\} \cup \triangle_{\underline{\mathrm{A}}}$ *belongs to* $\mathrm{Con}\,(\underline{\mathrm{A}})$,

2. $\overline{\theta_1 \wedge \theta_2} = \bar{\theta_1} \wedge \bar{\theta_2}$ *and* $\overline{\theta_1 \vee \theta_2} = \bar{\theta_1} \vee \bar{\theta_2}$ *for all* $\theta_1, \theta_2 \in \mathrm{Con}\,(\underline{\mathrm{A}})$.

Proof. *(1)* Let $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}})$. Then $\bar{\theta}$ is an equivalence relation. To show that $\bar{\theta}$ preserves $\vee, \wedge$ and f, let $a_1, a_2, b_1, b_2 \in A$ with $(f(a_1), f(b_1)), (f(a_2), f(b_2)) \in \bar{\theta}$. Since θ is a congruence relation and f preserves $\vee$ and $\wedge$, we have

$$(f(a_1) \vee f(a_2), f(b_1) \vee f(b_2)) = (f(a_1 \vee a_2), f(b_1 \vee b_2)) \in \bar{\theta},$$

$$(f(a_1) \wedge f(a_2), f(b_1) \wedge f(b_2)) = (f(a_1 \wedge a_2), f(b_1 \wedge b_2)) \in \bar{\theta}$$

and

$$(f(f(a_1)), f(f(a_2))) \in \bar{\theta}.$$

Therefore, $\bar{\theta}$ is a congruence relation.

(2) Let $\theta_1, \theta_2 \in \mathrm{Con}\,(\underline{\mathrm{A}})$. It can be proved directly that $\overline{\theta_1 \wedge \theta_2} = \bar{\theta_1} \wedge \bar{\theta_2}$. Let $(x,y) \in \overline{\theta_1 \vee \theta_2}$. Then $(x,y) = (f(a), f(b))$ for some $(a,b) \in \theta_1 \vee \theta_2$; so, there exists $t \in \mathbb{N}$ such that $(a,b) \in \underbrace{\theta_1 \circ \theta_2 \circ \cdots \circ \theta_1}_{t}$;that is, $(a,c_1) \in \theta_1, (c_1,c_2) \in \theta_2, ..., (c_{t-1},b) \in \theta_1$ for some $c_1, ..., c_{t-1} \in A$. So, $(f(a), f(c_1)) \in \bar{\theta_1}, (f(c_1), f(c_2)) \in \bar{\theta_2}, ..., (f(c_{t-1}), f(b)) \in \bar{\theta_1}$ which implies that $(x,y) = (f(a), f(b)) \in \bar{\theta_1} \vee \bar{\theta_2}$. It follows that $\overline{\theta_1 \vee \theta_2} \subseteq \bar{\theta_1} \vee \bar{\theta_2}$. Similarly, we can prove that $\bar{\theta_1} \vee \bar{\theta_2} \subseteq \overline{\theta_1 \vee \theta_2}$. Therefore, $\overline{\theta_1 \vee \theta_2} = \bar{\theta_1} \vee \bar{\theta_2}$. ∎

By the above results and Proposition 3.7, if we define $g \subseteq (\mathrm{Con}\,(\underline{\mathrm{A}}))^2$ by $g(\theta) = \bar{\theta}$ for all $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}})$ then $\underline{\mathrm{Con}\,(\underline{\mathrm{A}})} := \langle \mathrm{Con}\,(\underline{\mathrm{A}})\,; \vee, \wedge, g, \triangle_{\underline{\mathrm{A}}}, \underline{\mathrm{A}} \times \underline{\mathrm{A}} \rangle \in \mathcal{M}_n$.

Applying the proof in [21], one can see that $\mathrm{Con}\left(\underline{\mathrm{C_A}}\right) = \{\theta(0, c) : c \in C_A\}$ for all $\underline{\mathrm{A}} \in \mathcal{M}_n$ where $C_A \times C_A = \theta(0, 1)$ and $\triangle_{\underline{\mathrm{C_A}}} = \theta(0, 0)$. We will show that the map $\alpha : \underline{\mathrm{C_A}} \to \underline{\mathrm{Con}\left(\underline{\mathrm{C_A}}\right)}$, which is defined by $\alpha(c) = \theta(0, c)$ for all $c \in C_A$, is an isomorphism via the following proposition.

Proposition 3.10 *Let* $\underline{\mathrm{A}} \in \mathcal{M}_n$.

1. *For each* $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}}) \setminus \{\triangle_{\underline{\mathrm{A}}}\}$, *if* $(x, y) \in \theta$ *for some* $x < y$ *in* C_A *then* $(a, b) \in \theta$ *for all* $a, b \in A$ *with* $a \leq y$ *and* $b \leq y$,
2. $\theta(0, c) = \triangle_{\underline{\mathrm{A}}} \cup \{(x \vee a, x \vee b) : x \in A \text{ and } 0 \leq a, b \leq c\}$ *for all* $c \in C_A$,
3. $g(\theta(0, c)) = \theta(0, f(c))$ *for all* $c \in C_A$ *where* $g : \mathrm{Con}\,(\underline{\mathrm{A}}) \to \mathrm{Con}\,(\underline{\mathrm{A}})$ *is defined by* $g(\theta) = \bar{\theta}$ *for all* $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}})$.

Proof. *(1)* Let $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}}) \setminus \{\triangle_{\underline{\mathrm{A}}}\}$ and $(x, y) \in \theta$ for some $x, y \in C_A$ with $x < y$. Assume that $a, b \in A$ with $a \leq y$ and $b \leq y$ and let $k_y = \min\{j \in \mathbb{N} \cup \{0\} : f^j(a) = 0\}$. Since $x \leq f(y)$, we get $(f(y), y) = (f(y) \vee x, f(y) \vee y) \in \theta$. Since θ preserves f and θ is transitive, $(0, y) \in \theta$ which implies that $(a, y), (b, y) \in \theta$. Hence, $(a, b) \in \theta$.
(2) Let $c \in C_A$ and $\beta := \triangle_{\underline{\mathrm{A}}} \cup \{(x \vee a, x \vee b) : x \in A \text{ and } 0 \leq a, b \leq c\}$. It is clear that $\beta \in \mathrm{Con}\,(\underline{\mathrm{A}})$ and $\theta(0, c) \subseteq \beta$. Now, let $(u, v) \in \beta$. If $u \neq v$ then $(u, v) = (x \vee a, x \vee b)$ for some $x \in A$ and $0 \leq a, b \leq c$. So, $(a, b) \in \theta(0, c)$ which implies that $(u, v) \in \theta(0, c)$.
(3) Since $(0, c) \in \theta(0, c)$ and the definition of g, we have $(0, f(c)) \in g(\theta(0, c))$ which implies that $\theta(0, f(c)) \subseteq g(\theta(0, c))$. Let $(f(s), f(t)) \in g(\theta(0, c))$ for some $s, t \in A$ with $(s, t) \in \theta(0, c)$. If $s = t$, we are done. If $s \neq t$, we get by (2) that $(s, t) = (x \vee a, x \vee b)$ for some $x \in A$ and $0 \leq a, b \leq c$. So, $(s, t) = (x \vee a, x \vee b) \in \theta(0, c)$ which implies that $(f(s), f(t)) \in \theta(0, f(c))$. Hence, $g(\theta(0, c)) = \theta(0, f(c))$. ■

Theorem 3.11 $\underline{\mathrm{C_A}} \cong \underline{\mathrm{Con}\left(\underline{\mathrm{C_A}}\right)}$ *for all* $\underline{\mathrm{A}} \in \mathcal{M}_n$.

Proof. Let $\underline{A} \in \mathcal{M}_n$ and let $\alpha : \underline{C_A} \to \underline{\text{Con}\,(\underline{C_A})}$ be defined by $\alpha(c) = \theta(0, c)$ for all $c \in C_A$. Then α is a function which preserves 0 and 1 and α is onto. To show that α is one to one, let $c_1, c_2 \in C_A$ such that $\theta(0, c_1) = \theta(0, c_2)$. Since $(0, c_1) \in \theta(0, c_2)$ and Proposition 3.10 (2), we get $(0, c_1) = (x \vee a, x \vee b)$ for some $x \in A$ and $0 \leq a, b \leq c_2$; so, $x = 0$ which implies that $c_1 = b \leq c_2$. Similarly, we can prove that $c_2 \leq c_1$. Hence, α is one to one. It can be proved directly from Proposition 3.10 (2) that α preserves $\vee$ and $\wedge$. By Proposition 3.10 (3), $\alpha(f(c)) = \theta(0, f(c)) = g(\theta(0, c)) = g(\alpha(c))$; so, α preserves f. Therefore, α is an isomorphism. ■

By the definition of a core BDLC algebra and Proposition 3.10 (3), $C_{\text{Con}(\underline{A})} = \{\theta(0, c) : c \in C_A\}$. One can see that the map $\beta : \underline{\text{Con}\,(\underline{C_A})} \to \underline{C_{\text{Con}(\underline{A})}}$, which is defined by $\beta(\theta(0, c)) = \theta(0, c)$ for all $c \in C_A$, is an isomorphism. Notice that the map β is not an identity because the greatest element in $C_{\text{Con}(\underline{A})}$ is $\underline{A} \times \underline{A}$ but the greatest element in $\text{Con}\,(\underline{C_A})$ is $C_{\underline{A}} \times C_{\underline{A}}$. By Theorem 3.11, we have $\underline{C_A} \cong \underline{C_{\text{Con}(\underline{A})}}$.

3.3 Minimal Congruences

In this section, we characterize all minimal (non-identical) congruences in $\text{Con}\,(\underline{A})$ for $\underline{A} \in \mathcal{M}_n$ having no infinite chains. We begin with a summarization of some facts from lattice theory in Lemma 3.12 and Lemma 3.13 (one can see e.g. [10]) which are useful in the sequel.

Lemma 3.12 *Let $\langle A; \vee, \wedge \rangle$ be a distributive lattice and $a, b, c, d \in A$.*

1. *For each $t \in A$, if $a \vee t = b \vee t$ and $a \wedge t = b \wedge t$ then $a = b$.*

2. *If $a \prec b$, $c > a$ and $c \ngeq b$ then $b < b \vee c$ and $c \prec b \vee c$.*

3. *For each $z \in A$, if $a \prec b$ and $a \vee z \ngeq b$ then $a \vee z \prec b \vee z$.*

Lemma 3.13 *An ordered set $\langle A; \leq \rangle$ no contains infinite chains if and only if every non-empty subset T of A contains a maximal element and a minimal element.*

The following proposition shows a necessary condition of all minimal congruences on an algebra $\underline{\mathrm{A}} \in \mathcal{M}_n$ having no infinite chain. It is also a sufficient condition which will be shown in Theorem 3.16 via Lemma 3.15.

Proposition 3.14 *Let* $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}}) \setminus \{\triangle_{\underline{\mathrm{A}}}\}$ *be minimal. Then*

1. $\theta = \theta(c,d)$ *for some* $c, d \in A$ *with* $c \prec d$ *and* $f(c) = f(d)$,

2. *there are* $a \prec b \in A$ *such that* $\theta = \theta(a,b), f(a) = f(b)$ *and* $a \wedge x = b \wedge x$ *for all* $x \ngeq b$.

Proof. (*1*) Let $(p,q) \in \theta$ for some $p < q \in A$. Since $f^n(p) = f^n(q) = 0$, we let $t = \min\{j \in \mathbb{N} : f^j(p) = f^j(q)\}$ which implies that $f^{t-1}(p) < f^{t-1}(q)$. Let $c, d \in A$ with $c = f^{t-1}(p) \prec d \leq f^{t-1}(q)$. Then $f(d) = f^t(p) = f(c)$ and $(c,d) = (f^{t-1}(p) \wedge d, f^{t-1}(q) \wedge d) \in \theta$. Minimality of θ implies that $\theta = \theta(c,d)$.
(*2*) By (*1*), let $c, d \in A$ with $c \prec d$ and $f(c) = f(d)$. Let $T = \{t \in A : t < d \text{ and } t \nleq c\}$. If $T = \emptyset$, we choose $a = c$ and $b = d$. If $T \neq \emptyset$, let b be a minimal element of T. Then $b < d$ and $b \nleq c$. Let $a = c \wedge b$. Then $f(a) = f(b)$. If $a \prec s < b$ for some $s \in A$, then $b \vee c = d = s \vee c$ and $b \wedge c = a = s \wedge c$ which proves $b = s$, a contradiction; hence, $a \prec b$. Let $x \ngeq b$. Then $b \wedge x < b$ and the minimality of b implies that $b \wedge x \notin T$. Therefore, either $b \wedge x \nless d$ or $b \wedge x \leq c$; but, $b \wedge x < b < d$ and $c \prec d$ imply $b \wedge x \leq c$. Hence, $a \wedge x = b \wedge x$. So, $(a,b) = (c \wedge b, d \wedge b) \in \theta(c,d)$ and $(c,d) = (a \vee c, b \vee d) \in \theta(a,b)$ imply $\theta(c,d) = \theta(a,b)$. ■

Lemma 3.15 *Let* $a, b \in A$ *be those in Proposition 3.14 (2) and*

$$\gamma = \{(a \vee x, b \vee x) : x \in A\}$$

where $\gamma^{\smile}$ *is the inverse of* γ. *Then*

1. *if* $(p,q), (q,t) \in \gamma$ *(or* $\gamma^{\smile}$*) then* $q = t$ *(or* $p = q$*) for all* $p, q, t \in A$,

2. $\gamma \cup \gamma^{\smile} \cup \triangle_{\underline{\mathrm{A}}}$ *is an equivalence relation,*

3. $\gamma \cup \gamma^{\smile} \cup \triangle_{\underline{\mathrm{A}}}$ *is a congruence relation,*

4. $\theta(a,b) = \gamma \cup \gamma^{\smile} \cup \triangle_{\underline{A}}$.

Proof. *(1)* Let $(p,q),(q,t) \in \gamma$. Then $(p,q) = (a \vee x_1, b \vee x_1)$ and $(q,t) = (a \vee x_2, b \vee x_2)$ for some $x_1, x_2 \in A$; so, $a \vee x_2 = b \vee x_1 \geq b$ which implies that $b \vee x_2 \geq a \vee x_2 \geq b \vee x_2$. Hence, $q = t$. Similarly, if $(p,q),(q,t) \in \gamma^{\smile}$ then $p = q$.
(2) It is easily seen that $R := \gamma \cup \gamma^{\smile} \cup \triangle_{\underline{A}}$ is reflexive and symmetric. To show that R is transitive, let $(p,q),(q,t) \in R$. If $(p,q),(q,t) \in \gamma$ or $(p,q),(q,t) \in \gamma^{\smile}$ then $(p,t) \in R$ by (1). Assume that $(p,q) \in \gamma$ and $(q,t) \in \gamma^{\smile}$. Then $(p,q) = (a \vee x_1, b \vee x_1)$ and $(q,t) = (b \vee x_2, a \vee x_2)$ for some $x_1, x_2 \in A$; so, $b \vee x_1 = b \vee x_2$. If $x_1 \geq b$ or $x_2 \geq b$ then $p = q$ or $q = t$, respectively; so, $(p,t) \in R$. Assume that $x_1, x_2 \ngeq b$. Then $b \vee (a \vee x_1) = b \vee x_1 = b \vee x_2 = b \vee (a \vee x_2)$ and $b \wedge (a \vee x_1) = a \vee (b \wedge x_1) = a \vee (a \wedge x_1) = a = a \vee (a \wedge x_2) = a \vee (b \wedge x_2) = b \wedge (a \vee x_2)$. Distributivity of $\underline{A}$ implies that $a \vee x_1 = a \vee x_2$; that is, $(p,t) \in \triangle_{\underline{A}} \subseteq R$. In the case of $(p,q),(q,t) \in \triangle_{\underline{A}}$ and $(p,q) \in \gamma, (q,t) \in \triangle_{\underline{A}}$ are clear. Hence, R is transitive.
(3) It is left to show that $R := \gamma \cup \gamma^{\smile} \cup \triangle_{\underline{A}}$ preserves $\vee$, $\wedge$ and f. If $(p,q) \in \gamma \cup \gamma^{\smile} \cup \triangle_{\underline{A}}$ then $(f(p), f(q)) \in \triangle_{\underline{A}} \subseteq R$. It is clear that γ and $\gamma^{\smile}$ preserve $\vee$ and $\wedge$.

Let $(p,q) \in \gamma$ (or $\gamma^{\smile}$) and $(s,t) \in \triangle_{\underline{A}}$. Then $s = t$ and $(p,q) = (a \vee x, b \vee x)$ for some $x \in A$; and so, $(p \vee s, q \vee t) \in \gamma$. Note that $(p \wedge s, q \wedge t) = ((a \wedge s) \vee (x \wedge s), (b \wedge t) \vee (x \wedge t))$. If $s \geq b$, then $(p \wedge s, q \wedge t) \in \gamma$. If $s \ngeq b$, then $(p \wedge s, q \wedge t) \in \triangle_{\underline{A}}$.

Let $(p,q) \in \gamma$ and $(s,t) \in \gamma^{\smile}$. Then $(p,q) = (a \vee x, b \vee x)$ and $(s,t) = (b \vee x', a \vee x')$ for some $x, x' \in A$. So, $(p \vee s, q \vee t) \in \triangle_{\underline{A}}$. Since $p \wedge s = (a \vee x) \wedge (b \vee x')$ and $a < b$, we have $a \vee (x \wedge x') \leq (a \vee x) \wedge (b \vee x') \leq b \vee (x \wedge x')$. If $a \vee (x \wedge x') \geq b$ then $p \wedge s = a \vee (x \wedge x')$. Suppose that $a \vee (x \wedge x') \ngeq b$. Then Lemma 3.12(3) implies that $a \vee (x \wedge x') \prec b \vee (x \wedge x')$ and $(a \vee x) \wedge (b \vee x') = a \vee (x \wedge x')$ or $(a \vee x) \wedge (b \vee x') = b \vee (x \wedge x')$. If $(a \vee x) \wedge (b \vee x') = b \vee (x \wedge x')$ then $a \vee x = b \vee x$, a contradiction. Hence, $p \wedge s = a \vee (x \wedge x')$. Similarly, $q \wedge t = a \vee (x \wedge x')$. Therefore, $(p \wedge s, q \wedge t) \in \triangle_{\underline{A}}$. In any cases, $R \in \mathrm{Con}\,(\underline{A})$.
(4) By the definition of γ, we have $(a,b) \in \gamma \cup \gamma^{\smile} \cup \triangle_{\underline{A}}$. If $S \in \mathrm{Con}\,(\underline{A})$ contains (a,b) then $(a \vee x, b \vee x) \in S$ for all $x \in A$; so, $\gamma \cup \gamma^{\smile} \cup \triangle_{\underline{A}} \subseteq S$. Hence, $\theta(a,b) = \gamma \cup \gamma^{\smile} \cup \triangle_{\underline{A}}$.

■

Theorem 3.16 *A non-identical congruence θ on $\underline{\mathrm{A}}$ is minimal if and only if $\theta = \theta(c, d)$ for some $c, d \in A$ with $c \prec d$ and $f(c) = f(d)$.*

Proof. By Proposition 3.14, it is left to prove the converse. Let $a, b \in A$ be those in Proposition 3.14(2) whose $\theta = \theta(c, d) = \theta(a, b)$. Assume that $\beta \in \mathrm{Con}\,(\underline{\mathrm{A}}) \setminus \{\triangle_{\underline{\mathrm{A}}}\}$ is contained in $\theta(a, b)$. There are $s, t \in A$ such that $s \neq t$ and $(s, t) \in \beta \subseteq \theta(a, b)$. So, $(s, t) = (a \vee x, b \vee x) \in \beta$ for some $x \in A$. But $s \neq t$ implies $x \not\geq b$; so $((a \vee x) \wedge b, (b \vee x) \wedge b) = ((a \wedge b) \vee (x \wedge b), b) = (a \vee (x \wedge a), b) = (a, b) \in \beta$ which implies that $\beta = \theta(a, b)$. Therefore, $\theta(a, b)$ is minimal. ■

Chapter 4

All Subdirectly irreducible BDLC-algebras

By Birkhoff's theorem [4], every algebra can be represented by subdirectly irreducible algebras. In this chapter, we apply the results from Chapter 3 to characterize all finite subdirectly irreducible BDLC-algebras; and then apply some results in [19] to prove that the varieties $\mathcal{M}_n$ has no infinite subdirectly irreducible algebras for all $n \in \mathbb{N}$. Moreover, we can show that $\mathcal{M}_n$ is generated by a single subdirectly irreducible algebra.

4.1 Finite Subdirectly irreducible BDLC-algebras

Recall that an algebra $\langle A; F\rangle$ is a *reduct* of an algebra $\langle A; F^*\rangle$ if $F \subseteq F^*$. We note that $\langle A; f\rangle$ is the reduct of $\underline{\mathrm{A}}$ for all BDLC-algebras $\underline{\mathrm{A}}$; so, $\mathrm{Con}\,(\underline{\mathrm{A}})$ is a sublattice of $\mathrm{Con}\,(\langle A, f\rangle)$. By the fact in [21], if $\underline{\mathrm{A}}$ is a core BDLC then $\mathrm{Con}\,(\langle A, f\rangle)$ is a chain; and also is $\mathrm{Con}\,(\underline{\mathrm{A}})$. Hence, every core BDLC-algebra is subdirectly irreducible. But, the converse is not always true; for instance, if $\underline{\mathrm{A}} := \langle\{0, f(1), x, 1\}\,; \vee, \wedge, f, 0, 1\rangle$ where $x \vee f(1) = 1, x \wedge f(1) = 0, f(x) = f(1)$ and $f^2(1) = 0$, then $\underline{\mathrm{A}} \in \mathcal{M}_2$ whose picture is shown in Figure 4. So, $\underline{\mathrm{A}}$ is not a core BDLC; but, it is subdirectly irreducible since $\mathrm{Con}\,(\underline{\mathrm{A}}) = \{\triangle_{\underline{\mathrm{A}}}, \theta(0, f(1)), \underline{\mathrm{A}} \times \underline{\mathrm{A}}\}$ is a chain.

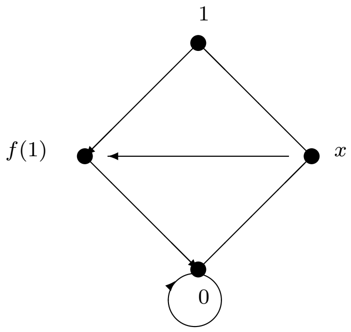

Figure 4 A subdirectly irreducible algebra $\underline{\mathrm{A}}$.

However, one can notice that $\mathrm{Con}\,(\underline{\mathrm{A}})$ of all above subdirectly irreducible algebras $\underline{\mathrm{A}}$ are chains. It is interesting whether $\mathrm{Con}\,(\underline{\mathrm{A}})$ is a chain for all finite subdirectly irreducible BDLC-algebras $\underline{\mathrm{A}}$. In this section, we prove an affirmative answer and characterize all finite subdirectly irreducible algebras in $\mathcal{M}$.

Lemma 4.1 *Let $\underline{\mathrm{A}}$ be a BDLC-algebra with $f(c) \prec c$ for all $c \in C_A \setminus \{0\}$.*

1. *$f(a) = 0$ if and only if either $a = 0$ or $a = f^{\lambda(f)-1}(1)$ for all $a \in A$,*

2. *If $x \notin C_A$ and $f^t(x) \in C_A$ for some $1 \leq^* t \leq^* \lambda(f) - 1$ then there exists $z \in C_A \cap f^{-t}(\{f^t(x)\})$ such that $x < z$ and x is not comparable with $f(z)$ and $f^{t+1}(z) = 0$ whenever $x \wedge f(z) \in C_A$.*

Proof. (*1*) Let $a \in A \setminus \{0\}$ with $f(a) = 0$ and we consider the case $a < 1$. Suppose that $a \neq b = f^{\lambda(f)-1}(1)$. Then $a \vee b$ is not comparable with $f^{-1}(b) \in C_A$ and $f(a \vee b) = f(b)$. By continuation this process in finite steps, there exists $a' \notin C_A$ such that $f(a') = f^3(1)$ and a' is not comparable with $f(1)$. Since $f(1) < a' \vee f(1) \leq 1$ and $f(1) \prec 1$, we get $a' \vee f(1) = 1$; so, $f^2(1) = f(1)$ which implies that $f(1) = 0$. Hence, $f(1) = 0 < a < 1$, a contradiction. Therefore $a = b = f^{\lambda(f)-1}(1)$. The converse follows directly from the properties of f.

(*2*) We prove by induction on t. If $x \notin C_A$ and $1 \neq f(x) \in C_A$, we choose $z \in C_A \cap f^{-1}(\{f(x)\})$. If $x \not< z$, we follow the proof of (1) to get $f(1) = 0$. So, $f(x) \leq f(1) = 0$ which implies that $f(x) = 0$. By (*1*), we have $x = 0$ or $x = f^{\lambda(f)-1}(1) \in C_A$, a contradiction. Therefore, $x < z$. Follows from $f(z) \prec z$, we get $x \not> f(z)$. If $x < f(z)$, then $f(z) = f(x) \leq f^2(z) < f(z)$, a contradiction. So, x is not comparable with $f(z)$.

Let $1 <^* t \leq^* \lambda(f)-1$. Suppose that the lemma is true for $t-1$. Assume that $x \notin C_A$ and $f^t(x) \in C_A$. If $f(x) \notin C_A$ then $f^{t-1}(f(x)) \in C_A$ implies that $f(x) < z'$ and $f(x)$ is not comparable with $f(z')$ for some $z' \in C_A \cap f^{-(t-1)}(\{f^{(t-1)}(f(x))\})$; and together with $f(x) \leq f(1)$, we have $z' \neq 1$. Choose $z \in C_A$ with $f(z) = z'$. Let $u = \min\{c \in C_A : x < c\}$. Then $x < u$. If $u < z$ then $x < u \leq f(z)$; so, $f(x) \leq f(z')$, a contradiction. Also, if $u > z$ then $f^2(u) = f(x) \vee f^2(u) = f(x \vee f(u)) = f(u)$; so, $f(x) \leq f(u) = 0$, a contradiction. Totally ordered of C_A implies $u = z$ and x is not comparable with $f(z)$. Also, the results implies the last statement. ■

Corollary 4.2 *If $x \notin C_A$ and $f(x) \in C_A$ then $x \wedge f^{\lambda(f)-1}(1) = 0$ and $x \vee f^{\lambda(f)-1}(1) = z$ for some $z \in C_A$ with $f(x) = f(z)$.*

Proposition 4.3 *If $\underline{\mathrm{A}}$ is BDLC with $f(c) \prec c$ for all $c \in C_A \setminus \{0\}$ then*

$$\bigcap\{\theta : \theta \in \mathrm{Con}\,(\underline{\mathrm{A}}) \setminus \{\triangle_{\underline{\mathrm{A}}}\}\} = \theta(0, f^{\lambda(f)-1}(1)).$$

Proof. Assume that $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}}) \setminus \{\triangle_{\underline{\mathrm{A}}}\}$ and $(x, y) \in \theta$ for some $x < y$. If $x, y \in C_A$, it follows by Proposition 3.10(1) that $(0, f^{\lambda(f)-1}(1)) \in \theta$.

Denote $z_a = \min\{c \in C_A : a \leq c\}$ for $a \in A$. If $x \in C_A$ and $y \notin C_A$ then $(f(z_y), z_y) \in \theta$ and by Proposition 3.10(1) we have $(0, f^{\lambda(f)-1}(1)) \in \theta$. Assume that $y \in C_A$ and $x \notin C_A$. Let $m = \min\{t \in \mathbb{N} : f^t(x) \in C_A\}$. Then $f^{m-1}(x) < f^{m-1}(y)$ and Corollary 4.2 imply that

$$(0, f^{\lambda(f)-1}(1)) = (f^{m-1}(x) \wedge f^{\lambda(f)-1}(1), f^{m-1}(y) \wedge f^{\lambda(f)-1}(1)) \in \theta.$$

Assume that $x, y \notin C_A$. If $z_x < z_y$ then $x < z_x \leq f(z_y)$; so, $(f(z_y), z_y) \in \theta$ which implies that $(0, f^{\lambda(f)-1}(1)) \in \theta$. We will show that if $z_x = z_y$, there exists $t \in C_A$ such that $t < z_x$ and t satisfies either $\{x \wedge t, y \wedge t\} \cap C_A \neq \emptyset$ or $\{x \wedge t, y \wedge t\} \cap C_A = \emptyset$ with $z_{x \wedge t} < z_{y \wedge t}$. Assume that $z_x = z_y$. Let $x_1 = x \wedge f(z_x)$ and $y_1 = z_{y \wedge f(z_x)}$. If $f(z_x)$ satisfies $\{x_1, y_1\} \cap C_A \neq \emptyset$ or $\{x_1, y_1\} \cap C_A = \emptyset$ with $z_{x_1} < z_{y_1}$ then we choose $t = f(z_x)$. If $\{x_1, y_1\} \cap C_A = \emptyset$ with $z_{x_1} = z_{y_1}$, we let $x_2 = x_1 \wedge f(z_{x_1})$ and $y_2 = y_1 \wedge f(z_{x_1})$. If $f(z_{x_1})$ satisfies $\{x_2, y_2\} \cap C_A \neq \emptyset$ or $\{x_2, y_2\} \cap C_A = \emptyset$ with $z_{x_2} < z_{y_2}$ then we choose $t = f(z_{x_1})$.

This process will stop before $z_{x_i} = z_{y_i} = f^{\lambda(f)-2}(1)$ for some $i \in \mathbb{N}$ because if $z_{x_i} = z_{y_i} = f^{\lambda(f)-2}(1)$ for some $i \in \mathbb{N}$ then $\{x_i, y_i, z_{x_i}, f^{\lambda(f)-1}(1), 0\}$ forms a subalgebra of $\underline{A}$ which contradicts to $\langle A; \vee, \wedge, 0, 1\rangle$ being a bounded distributive lattice. So, if $z_x = z_y$ then there exists $t \in C_A$ such that $t < z_x$ and t satisfies either $\{x \wedge t, y \wedge t\} \cap C_A \neq \emptyset$ or $\{x \wedge t, y \wedge t\} \cap C_A = \emptyset$ with $z_{x \wedge t} < z_{y \wedge t}$. In any cases, $(0, f^{\lambda(f)-1}(1)) \in \theta$. Therefore, $\theta(0, f^{\lambda(f)-1}(1)) \subseteq \bigcap \{\theta : \theta \in \mathrm{Con}\,(\underline{A}) \setminus \{\triangle_{\underline{A}}\}\}$. ■

From now on, we consider only finite BDLC-algebras.

Theorem 4.4 *The following statements are equivalent :*

1. $\underline{A}$ *is subdirectly irreducible,*
2. $f(c) \prec c$ *for all* $c \in C_A \setminus \{0\}$,
3. $\theta(0, f^{\lambda(f)-1}(1)) = \bigcup \{\theta(a, b) : a \prec b \text{ and } f(a) = f(b)\}$.

Proof. (*1*) $\Rightarrow$ (*2*) Suppose that there exists $t \geq^* 2$ such that $0 \prec x_1 \prec x_2 \prec \ldots \prec x_t = f^{\lambda(f)-1}(1)$. It follows by Theorem 3.16 that $\theta(0, x_1)$ and $\theta(x_1, x_2)$ are minimal. Therefore, $\theta(0, x_1) = \theta(x_1, x_2)$ which implies by Lemma 3.15 and $(x_1, x_2) \in \theta(0, x_1)$ that $(x_1, x_2) = (y, x_1 \vee y)$ for some $y \in A$; so, $x_1 = y$. Hence, $x_2 = x_1 \vee y = x_1$, a contradiction. Therefore, $t = 1$; and so, $0 \prec f^{\lambda(f)-1}(1)$.

Let $c \in C_A \setminus \{0\}$ and assume that $f(j) \prec j$ for all $j \in C_A \setminus \{0\}$ with $j < c$. Suppose that there exists $t \geq^* 2$ such that $f(c) \prec x_1 \prec x_2 \prec \ldots \prec x_t = c$. Then $f^2(c) \leq f(x_1) \leq f(c)$. Since $f^2(c) \prec f(c)$, either $f(x_1) = f(c)$ or $f(x_1) = f^2(c)$. If $f(x_1) = f(c)$ then a similarly proof implies $x_1 = x_2$, a contradiction. So, $f(x_1) = f^2(c)$. Theorem 3.16 implies that $\theta(f(c), x_1)$ is minimal; so, $\theta(f(c), x_1) = \theta(0, f^{\lambda(f)-1}(1))$. Hence, there exists $y \in A$ such that $(f(c), x_1) = (y, f^{\lambda(f)-1}(1) \vee y)$ which implies that $y = f(c)$ and $x_1 = f^{\lambda(f)-1}(1) \vee f(c) = f(c)$, a contradiction. Therefore, mathematical induction yields (2).

(*2*) $\Rightarrow$ (*3*) follows by Proposition 4.3 and $\theta(a, b)$ is minimal; and (*3*) $\Rightarrow$ (*1*) is proved directly from the definition of subdirectly irreducible. ■

Proposition 4.5 $\underline{A}$ *is simple if and only if* $|A| = 2$.

Proof. Assume that $\underline{\mathrm{A}}$ is simple. Then Theorem 4.4 implies that $f(c) \prec c$ for all $c \in C_A \setminus \{0\}$. Suppose that $\lambda(f) > 1$. Then $\theta(0, f^{\lambda(f)-1}(1)) \neq \triangle_{\underline{\mathrm{A}}}$; so, simplicity of $\underline{\mathrm{A}}$ implies $\theta(0, f^{\lambda(f)-1}(1)) = \underline{\mathrm{A}} \times \underline{\mathrm{A}}$. So, $(1, f(1)) \in \theta(0, f^{\lambda(f)-1}(1))$ which implies that $(1, f(1)) = (x, x \vee f^{\lambda(f)-1}(1))$ for some $x \in A$; i.e., $f(1) = 1 \vee f^{\lambda(f)-1}(1)$, a contradiction. Hence, $\lambda(f) = 1$. ■

We showed in Section 3.2 that for each $n \in \mathbb{N}$ and $\underline{\mathrm{A}} \in \mathcal{M}_n$, $\underline{\mathrm{Con}\,(\underline{\mathrm{A}})} = \langle \mathrm{Con}\,(\underline{\mathrm{A}})\,; \vee, \wedge, g, \triangle_{\underline{\mathrm{A}}}, \underline{\mathrm{A}} \times \underline{\mathrm{A}} \rangle \in \mathcal{M}_n$ where g is defined in Section 3.2. We now prove that $\underline{\mathrm{A}}$ is subdirectly irreducible if and only if $\underline{\mathrm{Con}\,(\underline{\mathrm{A}})} = \underline{\mathrm{C}_{\mathrm{Con}(\underline{\mathrm{A}})}}$.

Theorem 4.6 $\underline{\mathrm{A}}$ *is a subdirectly irreducible BDLC-algebra if and only if* $\mathrm{Con}\,(\underline{\mathrm{A}})$ *is the chain* $\{\theta(0, f^{\lambda(f)}(1)) \prec \theta(0, f^{\lambda(f)-1}(1)) \prec \ldots \prec \theta(0, f(1)) \prec \theta(0, 1)\}$.

Proof. Let $\theta \in \mathrm{Con}\,(\underline{\mathrm{A}})$ and $z = \max\{c \in C_A : (0, c) \in \theta\}$. Then $\theta(0, z) \subseteq \theta$. For each $a \in A$, let $z_a = \min\{c \in C_A : a \leq c\}$. Let $u, v \in A$ with $(u, v) \in \theta$. Then $z_u \leq z_v$ or $z_v \leq z_u$. If $z_u < z_v$ then $u \leq z_u \leq f(z_v)$; so, $(f(z_v), z_v) = (u \vee f(z_v), v \vee f(z_v)) \in \theta$ and Proposition 3.10(1) implies that $(u, v) \in \theta(0, z)$. Similarly, $(u, v) \in \theta(0, z)$ if $z_v < z_u$. If $z_u = z_v < z$ then by Proposition 3.10(1), we have $(u, v) \in \theta(0, z)$.

Suppose that $z_u = z_v \geq z$. Note by Proposition 3.10(1) that if $z_u = z_v = z$ then $(u, v) \in \theta(0, z)$. We will prove by the strong induction that for each $t \in \mathbb{N} \cup \{0\}$, if $(a, b) \in \theta$ with $z_a = z_b \geq z$ and $f^t(z_a) = z$ then $(a, b) \in \theta(0, z)$. Let $1 \leq^* t \leq^* \lambda(f)$ and suppose that the statement is true for any $1 \leq^* p \leq^* t$. Let $(a, b) \in \theta$, $z_a = z_b$ and $f^{t+1}(z_a) = z$. Assume that $a' = a \wedge f(z_a)$ and $b' = b \wedge f(z_a)$. If $z_{a'} < z_{b'}$ or $z_{b'} < z_{a'}$ then $(a', b') \in \theta(0, z)$. The absorbtion law and transitivity of $\theta(0, z)$ imply $(a, b) \in \theta(0, z)$. If $z_{a'} = z_{b'} \leq f(z_a)$ then $f^t(z_{b'}) \leq f^{t+1}(z_a) = z$ and by the induction hypothesis implies that $(a', b') \in \theta(0, z)$; so, $(a, b) \in \theta(0, z)$. By the fact above, if $z_u = z_v \geq z$ then $(u, v) \in \theta(0, z)$.

In any cases, $\theta \subseteq \theta(0, z)$. The converse is clear. ■

4.2 All Subdirectly irreducible BDLC-algebras

Recall that $\langle X\rangle_{\underline{\mathrm{A}}}$ is the smallest subalgebra of an algebra $\underline{\mathrm{A}}$ containing $X \subseteq A$; and $\langle X\rangle_{\underline{\mathrm{A}}} = \bigcup_{k=0}^{\infty} E^k(X)$ where

$$E^0(X) := X,$$

$$E(X) := X \cup \left\{ f_i^A(a_1, ..., a_{n_i}) : i \in I, a_1, ..., a_{n_i} \in X \right\},$$

$$\text{and} \quad E^{k+1}(X) := E(E^k(X)) \text{ where } k \in \mathbb{N}.$$

A variety $\mathcal{V}$ is *locally finite* if for each $\underline{\mathrm{A}} \in \mathcal{V}$, $\langle X\rangle_{\underline{\mathrm{A}}}$ is finite for all finite subset X of A. In [19], R.W. Quackenbush proved that if $\mathcal{V}$ is a locally finite variety and the set $Si_F(\mathcal{V})$ of all finite subdirectly irreducible algebras in $\mathcal{V}$ is finite (up to isomorphism) then $\mathcal{V}$ has no infinite subdirectly irreducible algebras. In this section, we apply the result in [19] to prove that $\mathcal{M}_n$ has no infinite subdirectly irreducible algebras for all $n \in \mathbb{N}$; especially, we show that it is generated by a single subdirectly irreducible.

It is well known that the variety of distributive lattice is locally finite. One can prove directly that for each $n \in \mathbb{N}$ and $\underline{\mathrm{A}} \in \mathcal{M}_n$, if $A = \langle\{a_1, ..., a_t\}\rangle_{\underline{\mathrm{A}}}$ for some $a_i \in A, 1 \leq^* i \leq^* t$ and $t \in \mathbb{N}$ then $A = \left\langle \bigcup_{i=0}^{n} f^i(\{a_1, ..., a_t, 0, 1\}) \right\rangle_{\langle A;\vee,\wedge\rangle}$ is finite.

Theorem 4.7 *$\mathcal{M}_n$ is locally finite for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and $\underline{\mathrm{A}} \in \mathcal{M}_n$ such that $A = \langle\{a_1, ..., a_t\}\rangle_{\underline{\mathrm{A}}}$ for some $a_i \in A, 1 \leq^* i \leq^* t$ and $t \in \mathbb{N}$. We will show that $E^k(\{a_1, ..., a_t\})$ is a subset of $\left\langle \bigcup_{i=0}^{n} f^i(\{a_1, ..., a_t, 0, 1\}) \right\rangle_{\langle A;\vee,\wedge\rangle}$ for all $k \in \mathbb{N} \cup \{0\}$ by induction on k.

Let $B = \left\langle \bigcup_{i=0}^{n} f^i(\{a_1, ..., a_t, 0, 1\}) \right\rangle_{\langle A;\vee,\wedge\rangle}$. It is clear that $\{a_1, ..., a_t\} \subseteq B$. Let $k \in \mathbb{N}$ and suppose that $E^k(\{a_1, ..., a_t\}) \subseteq B$. Note that $E^{k+1}(\{a_1, ..., a_t\}) = E^k(\{a_1, ..., a_t\}) \cup \left\{ x \vee y, x \wedge y, f(x), 0, 1 : x, y \in E^k(\{a_1, ..., a_t\}) \right\}$. By the assumption and the properties of B, it is left to prove that $f(x) \in B$ for all $x \in E^k(\{a_1, ..., a_t\})$. Let $x \in E^k(\{a_1, ..., a_t\}) \subseteq B$. Then $x = t^{(A;\vee,\wedge)}(b_1, ..., b_l)$ for some $b_1, ..., b_l \in \bigcup_{i=0}^{n} f^i(\{a_1, ..., a_t, 0, 1\})$ where $t^{\langle A;\vee,\wedge\rangle}$ is a term operation on $\langle A;\vee,\wedge\rangle$. So, $f(x) =$

$f(t^{(A;\vee,\wedge)}(b_1,...,b_l)) = t^{(A;\vee,\wedge)}(f(b_1),...,f(b_l))$. Since $b_1,...,b_l \in \bigcup_{i=0}^{n} f^i(\{a_1,...,a_t,0,1\})$, we have $f(x) \in B$; and so, $A \subseteq B$ which implies that A is finite. Hence, $\mathcal{M}_n$ is locally finite. Moreover in the similar proof, $\left\langle \bigcup_{i=0}^{n} f^i(\{a_1,...,a_t,0,1\}) \right\rangle_{\langle A;\vee,\wedge\rangle}$ is a subset of A. ■

We are now describing all finite subdirectly irreducible BDLC-algebras which show that the set $Si_F(\mathcal{M}_n)$ is finite for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$, define $\underline{\mathrm{A}}^{*n} := \langle A^n; \vee, \wedge, f, \mathbf{0}, \mathbf{1}\rangle$ whose $\langle A^n; \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ is the usual direct product of a BDL-algebra $\langle A; \vee, \wedge, 0, 1\rangle$ and $f : A^n \to A^n$ is defined by $f(a_1, a_2, ..., a_n) = (a_2, ..., a_n, 0)$ for all $a_i \in A$ and $1 \leq^* i \leq^* n$. Denote $\mathbf{0} := (\underbrace{0,...,0}_{n})$, $\mathbf{1} := (\underbrace{1,...,1}_{n})$ and $\underline{\mathrm{A}}^{*0}$ to be the trivial BDLC-algebra. One can see that f is an endomorphism on $\langle A^n; \vee, \wedge, \mathbf{0}\rangle$ with $f^n(\mathbf{1}) = \mathbf{0}$; it follows by Proposition 3.7 that $\underline{\mathrm{A}}^{*n}$ is BDLC whose $\lambda(f) = n$ if $A \neq \{0\}$. In particular, when $A = \{0,1\}$ we call it that an *n-cube BDLC-algebra* and denote by $\underline{2}^{*n}$.

Recall that $C_{2^{*n}} = \{\mathbf{1} > f(\mathbf{1}) > ... > f^n(\mathbf{1}) = \mathbf{0}\}$; so, the definition of f on $\underline{2}^{*n}$ implies that $f(c) \prec c$ for all $c \in C_{2^{*n}} \setminus \{\mathbf{0}\}$; hence, it follows by Theorem 4.4 that $\underline{2}^{*n}$ is a subdirectly irreducible algebra in $\mathcal{M}_n$. We will now prove that all algebras in $IS(\underline{2}^{*n})$ are subdirectly irreducible.

Proposition 4.8 *For $\underline{\mathrm{A}} \in \mathcal{M}_n$ and $\underline{\mathrm{B}} \in Si_F(\mathcal{M}_n)$, if $\underline{\mathrm{A}}$ can be embeded in $\underline{\mathrm{B}}$ then $\underline{\mathrm{A}} \in Si_F(\mathcal{M}_n)$.*

Proof. Let $\psi : \underline{\mathrm{A}} \to \underline{\mathrm{B}}$ be an embedding and let $c \in C_{\underline{\mathrm{A}}} \setminus \{0\}$. To show that $f(c) \prec c$, let $x \in A$ such that $f(c) \leq x < c$. Then $\psi(f(c)) \leq \psi(x) < \psi(c)$. Since $\psi(C_{\underline{\mathrm{A}}}) = C_{\underline{\mathrm{B}}}$ and $\underline{\mathrm{B}} \in Si_F(\mathcal{M}_n)$, we have $\psi(f(c)) \prec \psi(c)$; and so, $\psi(x) = \psi(f(c))$ which implies that $x = f(c)$. ■

Corollary 4.9 *$IS(\underline{2}^{*m}) \subseteq Si_F(\mathcal{M}_n)$ for all $m \leq^* n$.*

Proof. Let $m \in \mathbb{N}$ with $m \leq^* n$ and $\underline{\mathrm{A}} \in IS(\underline{2}^{*m})$. Then there is an embedding $\psi : \underline{\mathrm{A}} \to \underline{2}^{*m}$. Since $\underline{2}^{*m} \in Si_F(\mathcal{M}_m)$ and Proposition 4.8, we get $\underline{\mathrm{A}} \in Si_F(\mathcal{M}_m)$.

Note that $\mathcal{M}_m \subseteq \mathcal{M}_n$; so, $Si_F(\mathcal{M}_m) \subseteq Si_F(\mathcal{M}_n)$ which implies that $\underline{\mathrm{A}} \in Si_F(\mathcal{M}_n)$. ■

We are now giving another characterization of all finite subdirectly irreducible algebras in $\mathcal{M}_n$ by showing all elements in $Si_F(\mathcal{M}_n)$.

For $\underline{\mathrm{A}} \in \mathcal{M}_n$ and $a \in A$, let define $z_a := \min\{c \in C_{\underline{\mathrm{A}}} : a \leq c\}$ and $k_a := \min\{j \in \mathbb{N} \cup \{0\} : f^j(a) = 0\}$.

Lemma 4.10 *Let $\underline{\mathrm{A}} \in Si_F(\mathcal{M}_n)$. Then*

1. $k_a = k_{z_a}$ *for all* $a \in A$,
2. *for each* $0 \leq^* j \leq^* \lambda(f)-1$, *the map* χ_j *from* $\langle A; \vee, \wedge, 0, 1\rangle$ *to* $\langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ *defined by*

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in f^{-j}(\uparrow f^{\lambda(f)-1}(1)), \\ 0 & \text{if } x \notin f^{-j}(\uparrow f^{\lambda(f)-1}(1)). \end{cases}$$

is a homomorphism. Moreover, $\chi_j(f(x)) = \chi_{j+1}(x)$ *for all* $x \in A$.

Proof. (*1*) Let $a \in A$. If $a \in C_{\underline{\mathrm{A}}}$, we are done. Assume that $a \notin C_{\underline{\mathrm{A}}}$ and t is the least positive integer such that $f^t(a) \in C_{\underline{\mathrm{A}}}$. By Lemma 4.1, $f^t(z_a) = f^t(a)$; so, $0 = f^{k_a - t}(f^t(a)) = f^{k_a-t}(f^t(z_a)) = f^{k_a}(z_a)$ which implies that $k_{z_a} \leq^* k_a$. Similarly, $k_a \leq^* k_{z_a}$. Hence, $k_a = k_{z_a}$.

(*2*) Let $x, y \in A$ and $0 \leq j \leq \lambda(f) - 1$. Note that

$$\begin{aligned} \chi_j(x \wedge y) = 1 &\iff f^j(x \wedge y) \in \uparrow f^{\lambda(f)-1}(1) \\ &\iff f^j(x) \in \uparrow f^{\lambda(f)-1}(1) \text{ and } f^j(y) \in \uparrow f^{\lambda(f)-1}(1) \\ &\iff \chi_j(x) = 1 \text{ and } \chi_j(y) = 1 \\ &\iff \chi_j(x) \wedge \chi_j(y) = 1. \end{aligned}$$

Next, we will show that $f^j(x \vee y) \in \uparrow f^{\lambda(f)-1}(1)$ if and only if $f^j(x) \in \uparrow f^{\lambda(f)-1}(1)$ or $f^j(y) \in \uparrow f^{\lambda(f)-1}(1)$. It is clear that if $f^j(x) \in \uparrow f^{\lambda(f)-1}(1)$ or $f^j(y) \in \uparrow f^{\lambda(f)-1}(1)$ then $f^j(x \vee y) \in \uparrow f^{\lambda(f)-1}(1)$. Suppose that $f^j(x \vee y) \in \uparrow f^{\lambda(f)-1}(1)$ and $f^j(x), f^j(y) \notin \uparrow f^{\lambda(f)-1}(1)$. Then $f^j(x) \vee f^j(y) = (f^j(x) \vee f^{\lambda(f)-1}(1)) \vee f^j(y)$ and $f^j(x) \wedge f^j(y) =$

$(f^j(x) \vee f^{\lambda(f)-1}(1)) \wedge f^j(y)$ which imply that $f^j(x) = f^j(x) \vee f^{\lambda(f)-1}(1)$, a contradiction. So, $f^j(x) \in\uparrow f^{\lambda(f)-1}(1)$ or $f^j(y) \in\uparrow f^{\lambda(f)-1}(1)$. Therefore,

$$\begin{aligned}\chi_j(x \vee y) = 1 &\iff f^j(x \vee y) \in\uparrow f^{\lambda(f)-1}(1)\\ &\iff f^j(x) \in\uparrow f^{\lambda(f)-1}(1) \text{ or } f^j(y) \in\uparrow f^{\lambda(f)-1}(1)\\ &\iff \chi_j(x) \vee \chi_j(y) = 1.\end{aligned}$$

It is easy to see that $\chi_j(1) = 1$ and $\chi_j(0) = 0$. Hence, χ_j is a homomorphism.
By the definition of χ_j, we have

$$\begin{aligned}\chi_j(f(x)) = 1 &\iff f(x) \in f^{-j}(\uparrow f^{\lambda(f)-1}(1))\\ &\iff x \in f^{-(j+1)}(\uparrow f^{\lambda(f)-1}(1))\\ &\iff \chi_{j+1}(x) = 1.\end{aligned}$$

Hence, $\chi_j(f(x)) = \chi_{j+1}(x)$.

■

Theorem 4.11 *$\underline{\mathrm{A}} \in Si_F(\mathcal{M}_n)$ if and only if $\underline{\mathrm{A}} \in IS(\underline{2}^{*\lambda(f)})$.*

Proof. Let $\underline{\mathrm{A}}$ be a finite subdirectly irreducible algebra. Define a function $\phi : \underline{\mathrm{A}} \to \underline{2}^{*\lambda(f)}$ by $\phi(x) = (\chi_0(x), \chi_1(x), ..., \chi_{\lambda(f)-1}(x))$ for all $x \in A$ where χ_j is defined as in Lemma 4.10 for all $0 \leq^* j \leq^* \lambda(f)-1$. By Lemma 4.10, ϕ is a homomorphism. Next, we will show that ϕ is one to one. For each $t \in \mathbb{N}\cup\{0\}$, let P$(t)$ be the statement that for each $x, y \in A$, if $\phi(x) = \phi(y)$ and $|\{0 \leq^* j \leq^* \lambda(f) - 1 : f^j(x) \geq f^{\lambda(f)-1}(1)\}| \leq^* t$ then $x = y$. To show that P(0) is true, let $x, y \in A$. Assume that $\phi(x) = \phi(y)$ and $f^j(x), f^j(y) \ngeq f^{\lambda(f)-1}(1)$ for all $0 \leq^* j \leq^* \lambda(f)-1$. If $x \neq 0$ or $y \neq 0$ then Lemma 4.1 implies that $f^{k_x-1}(x) = f^{\lambda(f)-1}(1)$ or $f^{k_y-1}(y) = f^{\lambda(f)-1}(1)$, a contradiction. Hence, $x = 0 = y$.

Let $t \in \mathbb{N} \cup \{0\}$ and assume that P(t) is true. Let $x, y \in A$ such that $\phi(x) = \phi(y)$ and $|\{0 \leq^* j \leq^* \lambda(f) - 1 : f^j(x) \geq f^{\lambda(f)-1}(1)\}| \leq^* t + 1$. We may assume that $x, y \neq 0$. We will show that there exists $z \in A$ such that $x \vee z = y \vee z$ and $x \wedge z = y \wedge z$. Let $z = f(z_x)$. Since $\phi(x) = \phi(y)$ and the definition of ϕ, we have $f^j(x) \in\uparrow f^{\lambda(f)-1}(1)$ if and only if $f^j(y) \in\uparrow f^{\lambda(f)-1}(1)$ for all $0 \leq^* j \leq^* \lambda(f) - 1$.

If $k_x <^* k_y$ then $f^{k_y-1}(x) = 0 < f^{\lambda(f)-1}(1)$; and so, $f^{k_y-1}(y) \not\geq f^{\lambda(f)-1}(1)$, a contradiction. Hence, $k_x = k_y$ which implies that $k_{z_x} = k_{z_y}$. It follows that $z_x = z_y$; and so, $x \vee f(z_x) = y \vee f(z_y) = y \vee f(z_x)$.

In fact, if $f^j(x \wedge f(z_x)) \geq f^{\lambda(f)-1}(1)$ then $f^j(x) \geq f^{\lambda(f)-1}(1)$ for all $0 \leq^* j \leq^* \lambda(f)-1$. So, $|\{0 \leq^* j \leq^* \lambda(f)-1 : f^j(x \wedge f(z_x)) \geq f^{\lambda(f)-1}(1)\}| \leq^* t+1$. Since $f^{k_x-1}(x \wedge f(z_x)) = 0$, we get $|\{0 \leq^* j \leq^* \lambda(f)-1 : f^j(x \wedge f(z_x)) \geq f^{\lambda(f)-1}(1)\}| \leq^* t$. It is easy to see that for each $\chi_j(x \wedge f(z_x)) = \chi_j(y \wedge f(z_x))$ for all $0 \leq^* j \leq^* \lambda(f)-1$ which implies that $\phi(x \wedge f(z_x)) = \phi(y \wedge f(z_x))$. By induction hypothesis, $x \wedge f(z_x) = y \wedge f(z_x)$. Distributivity of $\underline{\mathrm{A}}$ implies that $x = y$. Therefore, $\underline{\mathrm{A}}$ is isomorphic to $\phi(\underline{\mathrm{A}})$; that is, $\underline{\mathrm{A}} \in IS(\underline{2}^{*\lambda(f)})$. The converse follows by Corollary 4.9. ∎

Corollary 4.12 $Si_F(\mathcal{M}_n) = \bigcup_{m \leq^* n} IS(\underline{2}^{*m})$

Note that there are finite subalgebras of an m-cube BDLC-algebra for all $m \leq^* n$. By Corollary 4.12, $Si_F(\mathcal{M}_n)$ is finite (up to isomorphism). Since $\mathcal{M}_n$ is locally finite and the fact in [19], $\mathcal{M}_n$ contains no infinite subdirectly irreducible algebras; so, $Si(\mathcal{M}_n) = \bigcup_{m \leq^* n} IS(\underline{2}^{*m})$ where $Si(\mathcal{M}_n)$ is the set of all subdirectly irreducible algebras in $\mathcal{M}_n$. By Birkhoff's Theorem, $\mathcal{M}_n = V(\bigcup_{m \leq^* n} IS(\underline{2}^{*m}))$. Since $\bigcup_{m \leq n} IS(\underline{2}^{*m}) = IS(\bigcup_{m \leq n} \{\underline{2}^{*m}\})$, we get $\mathcal{M}_n = V(\bigcup_{m \leq n} IS(\underline{2}^{*m})) = V(\bigcup_{m \leq^* n} \{\underline{2}^{*m}\})$. Furthermore for each $m \leq^* n$, it is well-known that a map $h : \underline{2}^{*n} \to \underline{2}^{*m}$ defined by

$$h(a_1, a_2, ..., a_n) = (a_{n-m+1}, a_{n-m+2}, ..., a_n)$$

for all $a_j \in \{0, 1\}$ and $1 \leq^* j \leq^* n$ is a homomorphism. Hence, one can prove that

$$\mathcal{M}_n = V(\bigcup_{m \leq^* n} \{\underline{2}^{*m}\}) = V(H(\underline{2}^{*n})) = V(\underline{2}^{*n}).$$

Corollary 4.13 $\mathcal{M}_n = V(\underline{2}^{*n})$.

Chapter 5

The lattice of all subvarieties of $\mathcal{M}_n$

In [8], B.A. Davey applied Jónsson's Lemma [15] to prove that if $\mathcal{K} = V(A)$ is a congruence distributive variety generated by a finite set A of finite algebras then the lattice $\Lambda(\mathcal{K})$ of all subvarieties of $\mathcal{K}$ is a finite distributive lattice and it is isomorphic to the lattice $\mathcal{O}(\mathbf{Si}(\mathcal{K}))$ of all order ideals of $(Si(\mathcal{K}); \leq_{Si(\mathcal{K})})$ where an order on $Si(\mathcal{K})$ is defined by $\underline{\mathrm{A}} \leq_{Si(\mathcal{K})} \underline{\mathrm{B}}$ if and only if $\underline{\mathrm{A}} \in HS(\underline{\mathrm{B}})$. It is well-known that a variety of lattice based-algebras is a congruence distributive variety; so is $\mathcal{M}_n$ for all $n \in \mathbb{N}$. Hence, the fact in [8] implies that for each $n \in \mathbb{N}$, $\Lambda(\mathcal{M}_n) \cong \mathcal{O}(\mathbf{P}'_\mathbf{n})$ where $\mathbf{P}'_\mathbf{n} = (Si(\mathcal{M}_n); \leq_{P'_n})$ and the order on $Si(\mathcal{M}_n)$ is defined by $\underline{\mathrm{A}} \leq_{P'_n} \underline{\mathrm{B}}$ if and only if $\underline{\mathrm{A}} \in HS(\underline{\mathrm{B}})$.

In this chapter, we show a method of drawing the diagram of the ordered set $\mathbf{P}'_\mathbf{n}$ which is a useful tool to describe the lattice $\Lambda(\mathcal{M}_n)$ for all $n \in \mathbb{N}$. For specification $n = 3$, we describe the diagram of $\Lambda(\mathcal{M}_3)$ via the diagram of the lattice $\mathcal{O}(\mathbf{Si}(\mathcal{M}_\mathbf{3}))$ and this idea can be extended to the lattice $\Lambda(\mathcal{M}_n)$ for all $n \in \mathbb{N}$.

5.1 The lattice of subvarieties of $\mathcal{M}_n$

Let $n \in \mathbb{N}$. We know from Chapter 4 that $Si(\mathcal{M}_n) = \displaystyle\bigcup_{m \leq^* n} IS(\underline{2}^{*m})$ which is infinite; so, its diagram is so complicated. However, the ordered set $\mathbf{P}'_\mathbf{n}$ can be considered as an ordered set $\mathbf{P_n} = (\bigcup_{m \leq^* n} S(\underline{2}^{*m}); \leq_{P_n})$ where $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$ if and only if $\underline{\mathrm{A}} \in HS(\underline{\mathrm{B}})$ which is shown in the following proposition.

Proposition 5.1 $\mathbf{P_n'} \cong \mathbf{P_n}$.

Proof. Define a function $\alpha : \mathbf{P_n'} \to \mathbf{P_n}$ by $\alpha(\underline{\mathrm{A}}') = \underline{\mathrm{A}}$ for all $\underline{\mathrm{A}}' \in P_n'$ where $\underline{\mathrm{A}} \in S(\underline{2}^{*m})$ with $\underline{\mathrm{A}}' \cong \underline{\mathrm{A}}$. It is easy to see that α is onto. We will show that α is an order-embedding; that is, for each $\underline{\mathrm{A}}', \underline{\mathrm{B}}' \in P_n'$, $\underline{\mathrm{A}}' \leq_{P_n'} \underline{\mathrm{B}}'$ if and only if $\alpha(\underline{\mathrm{A}}') \leq_{P_n} \alpha(\underline{\mathrm{B}}')$. Let $\underline{\mathrm{A}}', \underline{\mathrm{B}}' \in P_n'$. Assume that $\underline{\mathrm{A}}' \leq_{P_n'} \underline{\mathrm{B}}'$. We will prove that $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$ where $\underline{\mathrm{A}}, \underline{\mathrm{B}} \in S(\underline{2}^{*m})$ with $\underline{\mathrm{A}}' \cong \underline{\mathrm{A}}$ and $\underline{\mathrm{B}}' \cong \underline{\mathrm{B}}$. Since $\underline{\mathrm{A}}' \leq_{P_n'} \underline{\mathrm{B}}'$, there exists a homomorphism $h' : \underline{\mathrm{C}}' \to \underline{\mathrm{A}}'$ such that $\underline{\mathrm{A}}' = h'(\underline{\mathrm{C}}')$ for some subalgebras $\underline{\mathrm{C}}'$ of $\underline{\mathrm{B}}'$. Let $h = \alpha \circ h' \circ \phi^{-1} \downharpoonright_{\phi(\underline{\mathrm{C}}')}$ where $\phi : \underline{\mathrm{B}}' \to \underline{\mathrm{B}}$ is an isomorphism. Then h is a homomorphism; and so, $h(\phi(\underline{\mathrm{C}}')) = \alpha \circ h' \circ \phi^{-1} \downharpoonright_{\phi(\underline{\mathrm{C}}')} (\phi(\underline{\mathrm{C}}')) = \alpha \circ h'(\underline{\mathrm{C}}') = \alpha(\underline{\mathrm{A}}') = \underline{\mathrm{A}}$. Hence, $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$.

Conversely, assume that $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$. Then there exists a homomorphism $h : \underline{\mathrm{C}} \to \underline{\mathrm{A}}$ such that $\underline{\mathrm{A}} = h(\underline{\mathrm{C}})$ for some subalgebras $\underline{\mathrm{C}}$ of $\underline{\mathrm{B}}$. Let $h' = \varphi^{-1} \circ h \circ \phi \downharpoonright_{\phi^{-1}(\underline{\mathrm{C}})}$ where $\varphi : \underline{\mathrm{A}}' \to \underline{\mathrm{A}}$ and $\phi : \underline{\mathrm{B}}' \to \underline{\mathrm{B}}$ are isomorphisms. Then h' is a homomorphism; and so, $h'(\phi^{-1}(\underline{\mathrm{C}})) = \varphi^{-1} \circ h \circ \phi \downharpoonright_{\phi^{-1}(\underline{\mathrm{C}})} (\phi^{-1}(\underline{\mathrm{C}})) = \varphi^{-1} \circ h(\underline{\mathrm{C}}) = \varphi^{-1}(\underline{\mathrm{A}}) = \underline{\mathrm{A}}'$ which implies that $\underline{\mathrm{A}}' \leq_{P_n'} \underline{\mathrm{B}}'$. Therefore, $\alpha : \mathbf{P_n'} \to \mathbf{P_n}$ is an isomorphism. ■

One can see that it is not easy to check directly that $\underline{\mathrm{A}} \in HS(\underline{\mathrm{B}})$; that is, $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$ for all $\underline{\mathrm{A}}, \underline{\mathrm{B}} \in \bigcup_{m \leq^* n} S(\underline{2}^{*m})$. So, we are interested in simplifying the condition of the order $\leq_{P_n}$.

Proposition 5.2 *For $l, m \in \mathbb{N}$ with $l \leq^* m$, if $\underline{\mathrm{A}}$ is a subalgebra of $\underline{2}^{*m}$ then there exists a unique homomorphism h_{m-l}^m from $\underline{\mathrm{A}}$ to $\underline{2}^{*l}$.*

Proof. Let $l, m \in \mathbb{N}$ with $l \leq^* m$. Assume that $\underline{\mathrm{A}}$ is a subalgebra of $\underline{2}^{*m}$ and let $i = m - l$. Define $h_i^m : \underline{\mathrm{A}} \to \underline{2}^{*m-i}$ by $h_i^m(a_1, a_2, ..., a_m) = (a_{i+1}, a_{i+2}, ..., a_m)$ for all $a_j \in \{0, 1\}$ and $1 \leq^* j \leq^* m$. It is clear that $h_i^m(\mathbf{0}) = \mathbf{0}$, $h_i^m(\mathbf{1}) = \mathbf{1}$ and h_i^m preserves $\vee$ and $\wedge$; besides, h_i^m preserves f since $h_i^m(f(a_1, a_2, ..., a_m)) = h_i^m(a_2, ..., a_m, 0) = (a_{i+2}, ..., a_m, 0) = f(a_{i+1}, ..., a_m) = f(h_i^m(a_1, a_2, ..., a_m))$ for all $(a_1, a_2, ..., a_m) \in A$. Hence, h_i^m is a homomorphism.

Now we are proving the uniqueness of h_i^m. Assume that $\alpha : \underline{\mathrm{A}} \to \underline{2}^{*m-i}$ is a homomorphism. We first show that for each $a \in \{0,1\}$ and $1 \leq^* j \leq^* m$, $\alpha(\underbrace{0,...,0}_{j}, a, 0, ..., 0) = (\underbrace{0,...,0}_{j-i}, a, 0, ..., 0)$ if $j >^* i$ and $\alpha(\underbrace{0,...,0}_{j}, a, 0, ..., 0) = (0, 0, ..., 0)$ if $j \leq^* i$. Let $1 \leq^* j \leq^* m$. If $a = 0$, then we are done. If $a = 1$, we let $x = \alpha(\underbrace{0,...,0}_{j}, 1, 0, ..., 0)$ and $y = (\underbrace{0,...,0}_{j-i}, 1, 0, ..., 0)$. If $j >^* i$ then

$$
\begin{aligned}
x \vee \alpha(f^{m-j}(\mathbf{1})) &= \alpha(\underbrace{0,...,0}_{j}, 1, 0, ..., 0) \vee \alpha(\underbrace{1,...,1}_{j}, \underbrace{0,...,0}_{m-j}) \\
&= \alpha(\underbrace{1,1,...,1}_{j+1}, \underbrace{0,...,0}_{m-j-1}) \\
&= \alpha(f^{m-j-1}(\mathbf{1})) \\
&= f^{m-j-1}(\alpha(\mathbf{1})) \\
&= (\underbrace{1,1,...,1}_{j-i+1}, \underbrace{0,...,0}_{m-j-1}) \\
&= (\underbrace{0,...,0}_{j-i}, 1, 0, ..., 0) \vee (\underbrace{1,...,1}_{j-i}, \underbrace{0,...,0}_{m-j}) \\
&= y \vee \alpha(f^{m-j}(\mathbf{1}))
\end{aligned}
$$

and

$$
\begin{aligned}
x \wedge \alpha(f^{m-j}(\mathbf{1})) &= \alpha(\underbrace{0,...,0}_{j}, 1, 0, ..., 0) \wedge \alpha(\underbrace{1,...,1}_{j}, \underbrace{0,...,0}_{m-j}) \\
&= \alpha(\mathbf{0}) \\
&= (\underbrace{0,...,0}_{j-i}, 1, 0, ..., 0) \wedge (\underbrace{1,...,1}_{j-i}, \underbrace{0,...,0}_{m-j}) \\
&= y \wedge \alpha(f^{m-j}(\mathbf{1})).
\end{aligned}
$$

Distributivity of the lattice $\underline{2}^{*m-i}$ implies that $x = y$. Similarly, $x = y$ if $j \leq^* i$. Since α preserves $\vee$, it follows that $\alpha(a_1, a_2, ..., a_m) = (a_{i+1}, a_{i+2}, ..., a_m) = h_i^m(a_1, a_2, ..., a_m)$ for all $(a_1, a_2, ..., a_m) \in A$. ■

The following theorem can be proved directly by Proposition 5.2.

Theorem 5.3 *For $l, m \leq^* n$ and $\underline{\mathrm{A}} \in S(\underline{2}^{*l})$ and $\underline{\mathrm{B}} \in S(\underline{2}^{*m})$, $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$ if and only if*

1. *$l \leq^* m$, and*

2. *there is a subalgebra $\underline{\mathrm{C}}$ of $\underline{\mathrm{B}}$ such that $\underline{\mathrm{A}} = h_{m-l}^m(\underline{\mathrm{C}})$.*

Proof. Let $\underline{\mathrm{A}}, \underline{\mathrm{B}} \in \bigcup_{m \leq^* n} S(\underline{2}^{*m})$ such that $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$. Since $\underline{\mathrm{A}}, \underline{\mathrm{B}} \in \bigcup_{m \leq^* n} S(\underline{2}^{*m})$, there exist $l, m \in \mathbb{N}$ such that $\underline{\mathrm{A}} \in S(\underline{2}^{*l})$ and $\underline{\mathrm{B}} \in S(\underline{2}^{*m})$. Since $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{B}}$, there is a homomorphism $h : \underline{\mathrm{C}} \to \underline{\mathrm{A}}$ such that $h(\underline{\mathrm{C}}) = \underline{\mathrm{A}}$ for some subalgebras $\underline{\mathrm{C}}$ of $\underline{\mathrm{B}}$. By Corollary 3.3, we have $|C_{\underline{\mathrm{C}}}| \geq^* |C_{\underline{\mathrm{A}}}|$ which implies that $\lambda(f_{\underline{\mathrm{C}}}) \geq^* \lambda(f_{\underline{\mathrm{A}}})$; and so, $m \geq^* l$. By Proposition 5.2, we get $\underline{\mathrm{A}} = h(\underline{\mathrm{C}}) = h^m_{m-l}(\underline{\mathrm{C}})$. The converse is clear by the definition of the order $\leq_{P_n}$. ■

We are now going to show the picture of the ordered set $\mathbf{P_n}$. Since $P_n = \bigcup_{m \leq^* n} S(\underline{2}^{*m})$, we first focus on its subordered set $(S(\underline{2}^{*m}); \leq_{P_n}\lfloor_{S(\underline{2}^{*m})})$ for all $m \leq^* n$. By Theorem 5.3, the order $\leq_{P_n}\lfloor_{S(\underline{2}^{*m})}$ is the inclusion $\subseteq$ on $S(\underline{2}^{*m})$ for all $m \leq^* n$.

Proposition 5.4 *$S(\underline{2}^{*m}) = h^n_{n-m}(S(\underline{2}^{*n}))$ for all $m \leq^* n$.*

Proof. Let $m \leq^* n$. It is clear that $h^n_{n-m}(S(\underline{2}^{*n})) \subseteq S(\underline{2}^{*m})$. Conversely, let $\underline{\mathrm{A}}$ be a subalgebra of $\underline{2}^{*m}$. Then

$$B := \{(x_1, ..., x_{n-m}, a_1, a_2, ..., a_m) : x_1, ..., x_{n-m} \in \{0, 1\} \text{ and } (a_1, a_2, ..., a_m) \in A\}$$

is a subalgebra of $\underline{2}^{*n}$; hence, Proposition 5.2 implies that there exists a homomorphism h^n_{n-m} from $\underline{\mathrm{B}}$ to $\underline{2}^{*m}$; so, $h^n_{n-m}(\underline{\mathrm{B}}) = \underline{\mathrm{A}}$. Therefore, $S(\underline{2}^{*m}) \subseteq h^n_{n-m}(S(\underline{2}^{*n}))$. ■

Next, we consider a condition of covering of all elements in $P_n = \bigcup_{m \leq n} S(\underline{2}^{*m})$ which is shown in the following theorem.

Theorem 5.5 *$\underline{\mathrm{A}} \prec_{P_n} \underline{\mathrm{B}}$ in $\bigcup_{m \leq^* n} S(\underline{2}^{*m})$ if and only if there exists $m \in \mathbb{N}$ such that either*

1. *$\underline{\mathrm{A}} \prec_{S(\underline{2}^{*m})} \underline{\mathrm{B}}$, or*
2. *$\underline{\mathrm{A}} \in S(\underline{2}^{*m-1})$ and $\underline{\mathrm{B}} \in S(\underline{2}^{*m})$ where $\underline{\mathrm{B}}$ is a minimal of $\{\underline{\mathrm{D}} \in S(\underline{2}^{*m}) : h^m_1(\underline{\mathrm{D}}) = \underline{\mathrm{A}}\}$.*

Proof. Let $\underline{\mathrm{A}}, \underline{\mathrm{B}} \in \bigcup_{m \leq^* n} S(\underline{2}^{*m})$ with $\underline{\mathrm{A}} \prec_{P_n} \underline{\mathrm{B}}$. Then there exist $l, m \leq^* n$ such that $\underline{\mathrm{A}} \in S(\underline{2}^{*l})$ and $\underline{\mathrm{B}} \in S(\underline{2}^{*m})$. By Theorem 5.3, we get $l \leq^* m$ and there is a subalgebra $\underline{\mathrm{C}}$ of $\underline{\mathrm{B}}$ such that $\underline{\mathrm{A}} = h^m_{m-l}(\underline{\mathrm{C}})$. If $m - l >^* 1$ then $\underline{\mathrm{A}} = h^m_{m-l}(\underline{\mathrm{C}}) = h^{m-1}_{m-l-1}(h^m_1(\underline{\mathrm{C}})) <_{P_n} h^m_1(\underline{\mathrm{C}}) <_{P_n} \underline{\mathrm{C}}$ which contradicts to $\underline{\mathrm{A}} \prec_{P_n} \underline{\mathrm{B}}$. So, $m - l \leq^* 1$.

If $l = m$ then $\underline{\mathrm{A}} \prec_{S(\underline{2}^{*m})} \underline{\mathrm{B}}$. If $l = m-1$ then $\underline{\mathrm{A}} \in S(\underline{2}^{*m-1})$. Let $\underline{\mathrm{D}}$ be a proper subalgebra of $\underline{\mathrm{B}}$ such that $h^m_1(\underline{\mathrm{D}}) = \underline{\mathrm{A}}$. Then $\underline{\mathrm{A}} = h^m_1(\underline{\mathrm{D}}) <_{P_n} h^m_1(\underline{\mathrm{B}}) <_{P_n} \underline{\mathrm{B}}$, a contradiction. Hence, $\underline{\mathrm{B}}$ is minimal.

Conversely, if $\underline{\mathrm{A}} \prec_{S(\underline{2}^{*m})} \underline{\mathrm{B}}$ then $\underline{\mathrm{A}} \prec_{P_n} \underline{\mathrm{B}}$. Assume that $\underline{\mathrm{A}} \in S(\underline{2}^{*m-1})$ and $\underline{\mathrm{B}} \in S(\underline{2}^{*m})$ where $\underline{\mathrm{B}}$ is a minimal of $\{\underline{\mathrm{D}} \in S(\underline{2}^{*m}) : h^m_1(\underline{\mathrm{D}}) = \underline{\mathrm{A}}\}$. Let $\underline{\mathrm{C}} \in \bigcup_{m \leq^* n} S(\underline{2}^{*m})$ such that $\underline{\mathrm{A}} \leq_{P_n} \underline{\mathrm{C}} <_{P_n} \underline{\mathrm{B}}$. Then $\underline{\mathrm{C}} \in S(\underline{2}^{*t})$ for some $t \leq^* n$; hence, Theorem 5.3 implies that $t = m-1$ or $t = m$.

If $t = m$, the minimality of $\underline{\mathrm{B}}$ implies that $\underline{\mathrm{C}} = \underline{\mathrm{B}}$, a contradiction. So, $t = m - 1$ which implies that $A \subseteq C$. By Theorem 5.3 and $\underline{\mathrm{C}} <_{P_n} \underline{\mathrm{B}}$, we have $\underline{\mathrm{C}} = h^m_1(\underline{\mathrm{D}})$ for some subalgebra $\underline{\mathrm{D}}$ of $\underline{\mathrm{B}}$; so, $\underline{\mathrm{C}}$ is a subalgebra of $\underline{\mathrm{A}}$. Therefore, $\underline{\mathrm{A}} = \underline{\mathrm{C}}$ which implies that $\underline{\mathrm{A}} \prec_P \underline{\mathrm{B}}$. ■

5.2 The lattice of subvarieties of $\mathcal{M}_3$

In this section, we will follow the concepts in Section 5.1 to show all elements in $\mathcal{O}(\mathbf{P_3})$ where $\mathbf{P_3} = (\bigcup_{m \leq 3} S(\underline{2}^{*m}); \leq_{P_3})$. By the fact in [8], one can see that $\mathcal{O}(\mathbf{P_3})$ is isomorphic to the lattice $\Lambda(\mathcal{M}_3)$. Firstly, we find all elements in $S(\underline{2}^{*3})$.

We see that $A_1 = \{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$,

$A_2 = \{(0,0,0), (0,1,0), (1,0,0), (1,1,0), (1,1,1)\}$,

$A_3 = \{(0,0,0), (0,1,0), (0,1,1), (1,0,0), (1,1,0), (1,1,1)\}$,

$A_4 = \{(0,0,0), (0,1,0), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

and $A_5 = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

are all subalgebras of $\underline{2}^{*3}$ and the diagram of the lattice $(S(\underline{2}^{*3}); \subseteq)$ is shown in Figure 5.

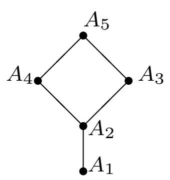

Figure 5 The lattice $(S(\underline{2}^{*3}); \subseteq)$.

By Proposition 5.4, we have $S(\underline{2}^{*2}) = h_1^3(S(\underline{2}^{*3}))$. So, $h_1^3(A_1) = h_1^3(A_2) = h_1^3(A_4) = \{(0,0),(1,0),(1,1)\}$ and $h_1^3(A_3) = h_1^3(A_5) = \{(0,0),(0,1),(1,0),(1,1)\}$ are all subalgebras of $\underline{2}^{*2}$.

Let $A_6 = \{(0,0),(1,0),(1,1)\}$ and $A_7 = \{(0,0),(0,1),(1,0),(1,1)\}$. The diagram of the lattice $(S(\underline{2}^{*2});\subseteq)$ is shown in Figure 6.

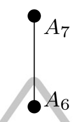

Figure 6 The lattice $(S(\underline{2}^{*2});\subseteq)$.

Similarly, $S(\underline{2}^{*1}) = h_1^2(S(\underline{2}^{*2}))$; so, $h_1^2(A_6) = h_1^2(A_7) = \{0,1\}$ is the only subalgebra of $\underline{2}^{*1}$ and let denote it by A_8. And the trivial $A_9 = \{0\}$ is the only subalgebra of $\underline{2}^{*0}$.

By Theorem 5.5, we have $\underline{A_6} \prec_{P_3} \underline{A_1}$, $\underline{A_7} \prec_{P_3} \underline{A_3}$ and $\underline{A_8} \prec_{P_3} \underline{A_6}$.

The following figure shows the ordered set $\mathbf{P_3} = (\bigcup_{m\leq 3} S(\underline{2}^{*m}), \leq_{P_3})$.

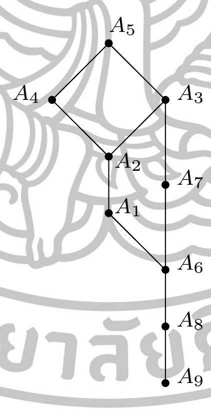

Figure 7 The ordered set $\mathbf{P_3} = (\bigcup_{m\leq 3} S(\underline{2}^{*m}); \leq_{P_3})$.

Since every nonempty set in $\mathcal{O}(\mathbf{P_3})$ is in the form $\downarrow B$ where B is a finite antichain in $\bigcup_{m\leq 3} S(\underline{2}^{*m})$; so, we can find all elements in $\mathcal{O}(\mathbf{P_3})$ as follows: $\downarrow\{A_1\}$, $\downarrow\{A_2\}$, $\downarrow\{A_3\}$, $\downarrow\{A_4\}$, $\downarrow\{A_5\}$, $\downarrow\{A_6\}$, $\downarrow\{A_7\}$, $\downarrow\{A_8\}$, $\downarrow\{A_9\}$, $\downarrow\{A_1, A_7\}$, $\downarrow\{A_2, A_7\}$, $\downarrow\{A_3, A_4\}$ and $\downarrow\{A_4, A_7\}$.

In facts, for each order set $(Q;\leq_Q)$ and $X, Y \subseteq Q$, $\downarrow X \subseteq \downarrow Y$ if and only if for each $x \in X$ there exists $y \in Y$ such that $x \leq_Q y$. So, the diagram of the order ideal $\mathcal{O}(\mathbf{P_3})$ is shown in the following Figure 8.

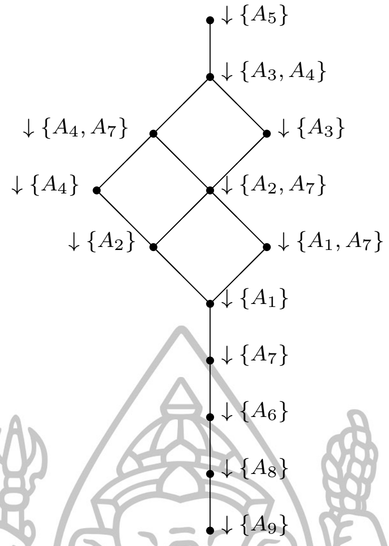

Figure 8 The order ideal $\mathcal{O}(\mathbf{P_3})$.

Chapter 6

Conclusion

For each $n \in \mathbb{N}$, the class $\mathcal{M}_n$ of all BDLC-algebras whose $\lambda(f) \leq^* n$ is a variety determined by identities; in addition, it can be generated by a single algebra.

Theorem 6.1 *For each $n \in \mathbb{N}$, the variety $\mathcal{M}_n$ is a class of BDL-algebras satisfying the following identities:*

$$f(x \vee y) \approx f(x) \vee f(y), f(x \wedge y) \approx f(x) \wedge f(y), f(0) \approx 0$$

$$\textit{and } f^n(1) \approx 0.$$

Theorem 6.2 $\mathcal{M}_n = V(\underline{2}^{*n})$ *for all $n \in \mathbb{N}$.*

Moreover, $\underline{2}^{*n}$ is a subdirectly irreducible algebra in $\mathcal{M}_n$ for all $n \in \mathbb{N}$ and every subdirectly irreducible algebra is an isomorphic copy of a subalgebra of $\underline{2}^{*m}$ for some $m \leq^* n$.

Theorem 6.3 $Si(\mathcal{M}_n) = \bigcup_{m \leq^* n} IS(\underline{2}^{*m})$ *for all $n \in \mathbb{N}$.*

Applying Theorem 6.2 together with the result in [8], we obtain a tool for drawing the diagram of the lattice $\Lambda(\mathcal{M}_n)$ of all subvarieties of $\mathcal{M}_n$ for all $n \in \mathbb{N}$.

If $n = 3$, the lattice $\Lambda(\mathcal{M}_3)$ is shown in the following figure.

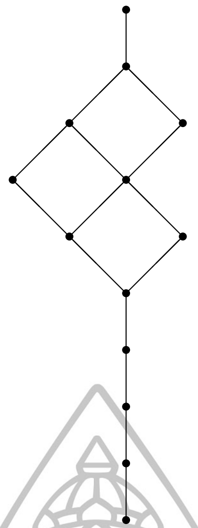

Figure 9 The lattice $\Lambda(\mathcal{M}_3)$.

In fact, all BDLC are dualisable. For a future work, it is interesting to find duality for BDLC-algebras by applying NU-duality theorem.

Bibliography

[1] J. Berman, Distributive lattices with an additional unary operation, Aequationes Math., 16, 165–171, 1977.

[2] G. Birkhoff, Applications of lattice algebra, Proc. Camb. Phil. Soc., 30, 115–122, 1934.

[3] G. Birkhoff, On the structure of abstract algebras, Proc. Camb. Phil. Soc., 31, 433–454, 1935.

[4] G. Birkhoff, Subdirect unions in universal algebra, Bull. Amer. Math. Soc., 50, 764–768, 1944.

[5] G. Birkhoff and M. Ward, A characterization of boolean algebras, Ann. of Math., 40, 609–610, 1939.

[6] G. Boole, The Mathematical Analysis of Logic, London, 1847.

[7] D.M. Clark and B.A. Davey, Natural Dualities for the Working Algebraist, Cambridge Mathematical Textbooks, New York, 1998.

[8] B.A. Davey, On the lattice of subvariety, Houston J. Math, 5 (2), 183–192, 1979.

[9] B.A. Davey, Dualities for stone algebras, double stone algebras and relative stone algebras, Coll. Math, 46, 1–14, 1982.

[10] B.A. Davey and H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, New York, 1990.

[11] B.A. Davey and H. Werner, Dualities and equivalences for varieties of algebras, Contributions to Lattice Theory (Szeged, 1980) (A.P. Huhn and E.T. Schmidt, eds), Colloq. Math. Soc. J*á*nos Bolyai 33, North-Holland, Amsterdam, New York, 101–275, 1983.

[12] K. Denecke and C. Ratanaprasert, Hyperidentities in order-primal algebra, Journal of Applied Algebra and Discrete Structure, 2, 101–118, 2004.

[13] K. Denecke and S.L. Wismath, Universal Algebra and Applications in Theoretical Computer Science, Chapman & Hall, CRC Press, Boca Raton, London, New York, Washington DC, 2002.

[14] M.S. Goldberg, Distributive ockham algebras: free algebras and injectivity, Bull. Austral. Math. Soc., 24, 161–203, 1981.

[15] B. J*ò*nsson, Algebras whose congruence lattices are distributive, Math. Scand., 21, 110–121, 1967.

[16] G.C. Moisil, Recherches sur l'alg*è*bre de la logique, Ann. Sci. Univ. Jassy., 22, 1–118, 1936.

[17] H.A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, Bull. London Math. Soc., 2, 186–190, 1970.

[18] H.A. Priestley, Varieties of distributive lattices with unary operations I, J. Austral. Math. Soc., 63, 165–207, 1997.

[19] R.W. Quackenbush , Equational classes generated by finite algebras, Algebra Universalis, 1, 265–266, 1971.

[20] C. Ratanaprasert and K. Denecke, Unary operations with long pre-periods, Discrete Mathematics, 308(21), 4998–5005, 2008.

[21] C. Ratanaprasert and S. Thiranantanakorn, Modular and distributive congruence lattices of monounary algebras, Bull. Malays. Math. Sci. Soc.,36, 291–298, 2013.

[22] A. Urguhart, Distributive lattices with a dual homomorphic operation, Studia Logica, 38, 201–209, 1979.

[23] G.H. Wenzel, Subdirect irreducibility and equational compactness in unary algebras, Arch. Math., 21, 256–264, 1970.

[24] M. Yoeli, Subdirectly irreducible unary algebra, Mathematical Monthly, 957–960, 1967.

[25] D. Zupnik, Cayley functions, Semigroup Forum, 3, 349–358, 1972.

APPENDIX

Publications and Presentation

Publications

A. Charoenpol and C. Ratanaprasert, A distributive lattice-based algebra : the BDLC, FJMS., 100(1), 135–145, 2016.

A. Charoenpol and C. Ratanaprasert, All subdirectly irreducible BDLC-algebras, FJMS., 100(3), 477–490, 2016.

A. Charoenpol and C. Ratanaprasert, The lattice of all subvarieties of $\mathcal{M}_n$, FJMS, submitted.

Presentation

A. Charoenpol, SK-algebras, International Conference on Discrete Mathematics and Applied Sciences, University of the Thai Chamber of Commerce (UTCC), Bangkok, Thailand, May 21-23, 2014.

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