PASTING LEMMAS FOR GENERALIZED METRIC-PRESERVING FUNCTIONS

By

MISS Duangpon SIRIWAN

A Thesis Proposal Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

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Title	Pasting Lemmas for Generalized Metric-Preserving Functions
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DUANGPON SIRIWAN : PASTING LEMMAS FOR GENERALIZED METRIC-PRESERVING FUNCTIONS. THESIS ADVISOR : ASSISTANT PROFESSOR TAMMATADA KHEMARATCHATAKUMTHORN, Ph.D.

Let $f : [0, \infty) \to [0, \infty)$. We say that f is metric-preserving if for all metric spaces (X, d), $f \circ d$ is a metric on X. In addition, f is (g_1, g_2)-metric-preserving if $f \circ d$ is a generalized metric of type g_2 whenever d is a generalized metric of type g_1. In this thesis, we investigate some pasting lemmas for (g_1, g_2)-metric-preserving functions for certain types of (g_1, g_2).

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Duangpon SIRIWAN

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Chapter 1

Introduction

Let X be a nonempty set and $d : X \times X \to [0, \infty)$. Then d is a *metric* if d satisfies the following three conditions:

(M1) $\forall x, y \in X$, $d(x, y) = 0 \leftrightarrow x = y$,

(M2) $\forall x, y \in X$, $d(x, y) = d(y, x)$, and

(M3) $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

In 1944, Krasner [8] introduced ultrametric as follows: The function d is called an *ultrametric* if d satisfies (M1), (M2), and

(U3) for all $x, y, z \in X, d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

In 1989, Bakhtin [1] introduced b-metric as follows: The function d is said to be a *b-metric* if d satisfies (M1), (M2), and

(B3) there exists $s \geq 1$ such that

$$d(x, y) \leq s(d(x, z) + d(z, y)) \quad \text{for all} \quad x, y, z \in X.$$

It is easy to see that every ultrametric is a metric and every metric is a b-metric. The function $f : [0, \infty) \to [0, \infty)$ is said to be *metric-preserving* if for all metric spaces (X, d), $f \circ d$ is metric on X and let $\mathcal{M}$ be the set of all metric-preserving functions. The concept of metric preserving functions first appears in Wilson's article [11] and is thoroughly by many authors, see example, [2, 3, 4]. In 2014, Pongsriiam and Termwuttipong [9] introduced and investigated a variation of concept of metric-preserving functions where metrics are replaced by ultrametrics as follows.

Definition 1.1. [9] Let $f : [0, \infty) \to [0, \infty)$. Then

(i) f is *ultrametric-preserving* if for all ultrametric spaces (X, d), $f \circ d$ is an ultrametric,

(ii) f is *metric-ultrametric-preserving* if for all metric spaces (X, d), $f \circ d$ is an ultrametric,

(iii) f is *ultrametric-metric-preserving* if for all ultrametric spaces (X, d), $f \circ d$ is a metric, and

let $\mathcal{U}$ be the set of all ultrametric-preserving functions, $\mathcal{UM}$ the set of all ultrametric-metric-preserving functions, and $\mathcal{MU}$ the set of all metric-ultrametric-preserving functions.

In 2018, Khemaratchatakumthorn and Pongsriiam [6] also introduced and investigated a variation of concept of metric-preserving functions where metrics are replaced by b-metrics as follows.

Definition 1.2. [6] Let $f : [0, \infty) \to [0, \infty)$. Then

(i) f is *b-metric-preserving* if for all b-metric spaces (X, d), $f \circ d$ is a b-metric,

(ii) f is *metric-b-metric-preserving* if for all metric spaces (X, d), $f \circ d$ is a b-metric,

(iii) f is *b-metric-metric-preserving* if for all b-metric spaces (X, d), $f \circ d$ is a metric,

and let $\mathcal{B}$ the set of all b-metric-preserving functions, $\mathcal{MB}$ the set of all metric-b-metric-preserving functions, and $\mathcal{BM}$ the set of all b-metric-metric-preserving functions.

In 2020, Samphavat, Khemaratchatakumthorn, and Pongsriiam [10] also introduced and investigated a variation of concept of metric-preserving functions where metrics are replaced by b-metrics and ultrametric as follows.

Definition 1.3. [10] Let $f : [0, \infty) \to [0, \infty)$. Then

(i) f is *ultrametric-b-metric-preserving* if for all ultrametric spaces (X, d), $f \circ d$ is a b-metric,

(ii) f is *b-metric-ultrametric-preserving* if for all b-metric spaces (X, d), $f \circ d$ is a ultrametric, and

let $\mathcal{UB}$ the set of all ultrametric-b-metric-preserving functions and $\mathcal{BU}$ the set of all b-metric-ultrametric-preserving functions.

The relations between $\mathcal{M}$, $\mathcal{B}$, $\mathcal{MB}$, $\mathcal{BM}$, $\mathcal{U}$, $\mathcal{UM}$, $\mathcal{MU}$, $\mathcal{BU}$, $\mathcal{UB}$ are given as follows.

Proposition 1.4. [6, 7, 9, 10] *The following statements hold.*

(i) $\mathcal{MU} = \mathcal{BU} \subseteq \mathcal{BM} \subseteq \mathcal{M} \subseteq \mathcal{B} = \mathcal{MB} \subseteq \mathcal{UB}$.

(ii) $\mathcal{BU} = \mathcal{MU} \subseteq \mathcal{U} \subseteq \mathcal{UM} \subseteq \mathcal{UB}$.

(iii) $\mathcal{M} \subseteq \mathcal{UM}$.

They also summarized the subset relations in the following diagram (Figure 1.1). Note that $f \in A \Rightarrow f \in B$ means $f \in A$ implies $f \in B$. In addition, if there is no arrow from $f \in A$ to $f \in B$, it means that $A \nsubseteq B$.

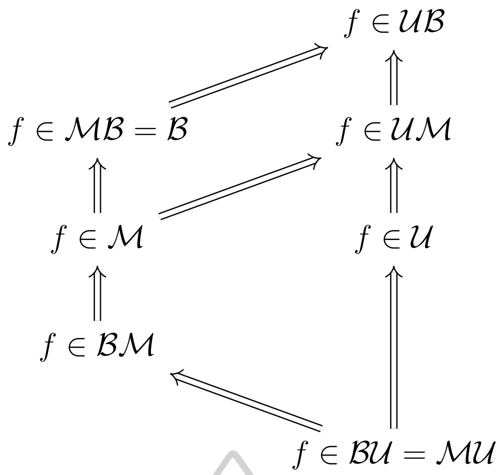

Figure 1.1: Subset Relations

It is well known that if $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are continuous and $g(b) = h(b)$, then the function $f : [a, c] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} g(x), & \text{if } x \in [a, b); \\ h(x), & \text{if } x \in [b, c) \end{cases}$$

is also continuous. This is usually called a *pasting lemma*. A version of a pasting lemma for metric-preserving functions is given by Doboš [5] but there is no pasting lemma for b-metric-preserving and other related functions in the literature.

Theorem 1.5. [5, p.26] *Let g, h be metric preserving. Let $r > 0$ be such that $g(r) = h(r)$. Define $f_{g,h,r} : [0, \infty) \to [0, \infty)$ as follows*

$$f_{g,h,r}(x) = \begin{cases} g(x), \textit{if } x \in [0, r), \\ h(x), \textit{if } x \in [r, \infty). \end{cases}$$

Suppose that g is increasing and concave. Then $f_{g,h,r}$ is metric preserving iff

$$\forall x, y \in [r, \infty) \ : \ |x - y| \le r \to |h(x) - h(y)| \le g(|x - y|).$$

In this thesis, we investigate pasting lemma by substituting continuous function or metric-preserving functions by generalized metric-preserving functions. This thesis is organized as follows: In Chapter 2, we recall some basic definitions and results concerning $\mathcal{M}$, $\mathcal{B}$, $\mathcal{MB}$, $\mathcal{BM}$, $\mathcal{U}$, $\mathcal{UM}$, $\mathcal{MU}$, $\mathcal{BU}$, $\mathcal{UB}$. In Chapter 3, we show pasting lemmas for functions in $\mathcal{B}$, $\mathcal{BM}$, $\mathcal{MU}$, $\mathcal{U}$, $\mathcal{UM}$, and $\mathcal{UB}$.

Chapter 2

Preliminaries and Lemmas

In this chapter, we recall some basic definitions and results concerning $\mathcal{M}$, $\mathcal{B}$, $\mathcal{MB}$, $\mathcal{BM}$, $\mathcal{U}$, $\mathcal{UM}$, $\mathcal{MU}$, $\mathcal{BU}$, $\mathcal{UB}$. Throughout this thesis let $f : [0, \infty) \to [0, \infty)$.

Definition 2.1. Let $I \subseteq [0, \infty)$. Then f is said to be *increasing* on I if $f(x) \leq f(y)$ for all $x, y \in I$ satisfying $x < y$, and f is said to be *strictly increasing* on I if $f(x) < f(y)$ for all $x, y \in I$ satisfying $x < y$.

Definition 2.2. The function f is said to be *amenable* if $f^{-1}(\{0\}) = 0$.

Definition 2.3. The function f is said to be *tightly bounded* on $(0, \infty)$ if there is $v > 0$ such that $f(x) \in [v, 2v]$ for all $x > 0$.

Definition 2.4. We say that f is *subadditive* if $f(a + b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$ and f is *quasi-subadditive* if there exists $s \geq 1$ such that $f(a + b) \leq s(f(a) + f(b))$ for all $a, b \in [0, \infty)$.

Definition 2.5. The function f is *concave* if

$$f((1 - t)x_1 + tx_2) \geq (1 - t)f(x_1) + tf(x_2)$$

for all $x_1, x_2 \in [0, \infty)$ and $t \in [0, 1]$.

Definition 2.6. A *triangle triplet* is a triple (a, b, c) of nonnegative real numbers for which

$$a \leq b + c, \quad b \leq a + c, \quad \text{and} \quad c \leq a + b,$$

or equivalently,

$$|a - b| \leq c \leq a + b.$$

Let $s \geq 1$ and $a, b, c \geq 0$. A triple (a, b, c) is a *s-triangle triplet* if

$$a \leq s(b + c), \quad b \leq s(a + c), \quad \text{and} \quad c \leq s(a + b).$$

A triple (a, b, c) of nonnegative real numbers is an *ultra-triangle triplet* if

$$a \leq \max\{a, b\} \quad b \leq \max\{c, a\} \quad \text{and} \quad c \leq \max\{b, c\}.$$

We let $\triangle$, $\triangle_s$, and $\triangle_\infty$ be the sets of all triangle triplets, s-triangle triplets and ultra-triangle triplets, respectively.

Next, we recall some results concerning metric-preserving functions.

Lemma 2.7. [2, 3, 5] *If f is amenable, subadditive and increasing on $[0, \infty)$, then $f \in \mathcal{M}$.*

Lemma 2.8. [2, 3, 5] *If f is amenable and tightly bounded, then $f \in \mathcal{M}$.*

Lemma 2.9. [2, 3, 5] *If $f \in \mathcal{M}$, then f is amenable and subadditive.*

Lemma 2.10. [2, 3, 5] *Let f be amenable. Then the following statements are equivalent.*

(i) $f \in \mathcal{M}$.

(ii) *For each $(a, b, c) \in \triangle$, $(f(a), f(b), f(c)) \in \triangle$.*

Next, we recall some results concerning b-metric and metric-preserving functions.

Lemma 2.11. [7] *Let f be amenable. Then the following statements are equivalent.*

(i) *$f \in \mathcal{B}$.*

(ii) *$f \in \mathcal{MB}$.*

(iii) *There exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \triangle_s$ for all $(a, b, c) \in \triangle$.*

Lemma 2.12. [6] *If $f \in \mathcal{B}$, then f is amenable and quasi-subadditive.*

Lemma 2.13. [6] *If $f \in \mathcal{BM}$ if and only if f is amenable and tightly bounded.*

Next, we recall some results concerning ultrametric and metric-preserving functions.

Lemma 2.14. [9] *If $f \in \mathcal{MU}$ if and only if f is amenable and constant on $(0, \infty)$.*

Lemma 2.15. [9] *If $f \in \mathcal{U}$ if and only if f is amenable and increasing.*

Lemma 2.16. [9] *Let f be amenable. Then the following statements are equivalent.*

(i) *$f \in \mathcal{UM}$.*

(ii) *For each $(a, b, c) \in \triangle_\infty$, $(f(a), f(b), f(c)) \in \triangle$.*

(iii) *For each $0 \leq a \leq b$, $f(a) \leq 2f(b)$.*

Next, we recall some results concerning b-metric, ultrametric and metric-preserving functions.

Lemma 2.17. [10] *If $f \in \mathcal{UB}$, then f is amenable.*

Lemma 2.18. [10] *Let f be amenable. Then the following statements are equivalent.*

(i) $f \in \mathcal{UB}$.

(ii) *There exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \triangle_s$ for all $(a, b, c) \in \triangle_\infty$.*

(iii) *There exists $s' \geq 1$ such that $f(a) \leq s' f(b)$ whenever $0 \leq a \leq b$.*

Lemma 2.19. [5] *Let f be amenable. Then f is concave if and only if*

$$\forall t \geq 0 \ \forall x, y, z \in [0, t], \ x + t = y + z \rightarrow f(x) + f(t) \leq f(y) + f(z).$$

Chapter 3

Main Results

In this chapter, we give pasting lemmas for functions in $\mathcal{B}$, $\mathcal{BM}$, $\mathcal{MU}$, $\mathcal{U}$, $\mathcal{UM}$, and $\mathcal{UB}$.

Theorem 3.1. (A pasting lemma for functions in $\mathcal{B}$ and $\mathcal{MB}$) *Let $g, h : [0,\infty) \to [0,\infty)$, g, $h \in \mathcal{B}$, $r > 0$ and $g(r) = h(r)$. Define $f : [0,\infty) \to [0,\infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0,r), \\ h(x), & \text{if } x \in [r,\infty). \end{cases}$$

Suppose that g is increasing, concave, and

$$\forall x, y \in [r,\infty),\ |x-y| \leq r \to |h(x) - h(y)| \leq g(|x-y|).$$

Then $f \in \mathcal{B}$.

Proof. Since g, $h \in \mathcal{B}$, we obtain by Lemmas 2.11 and 2.12 that g is amenable,

$$\exists s_1 \geq 0\ \forall (a,b,c) \in \triangle,\ (g(a), g(b), g(c)) \in \triangle_{s_1} \quad \text{and}$$

$$\exists s_2 \geq 0\ \forall (a,b,c) \in \triangle,\ (h(a), h(b), h(c)) \in \triangle_{s_2}.$$

Let $s = \max\{s_1, s_2\} \geq 0$ and let $(a,b,c) \in \triangle$. Without loss of generality, we can assume that $0 \leq a \leq b \leq c \leq a+b$.

Case 1. $a, b, c \in [0,r)$. Then

$$(f(a), f(b), f(c)) = (g(a), g(b), g(c)) \in \triangle_{s_1} \subseteq \triangle_s.$$

Case 2. $a, b, c \in [r, \infty)$. Then

$$(f(a), f(b), f(c)) = (h(a), h(b), h(c)) \in \triangle_{s_2} \subseteq \triangle_s.$$

Case 3. $a, b \in [0, r)$ and $c \in [r, \infty)$. Then

$$f(a) = g(a) \leq g(b) = f(b) \leq f(b) + f(c) \leq s(f(b) + f(c)). \tag{3.1}$$

Since $|r - c| = c - r \leq a + b - r < r + r - r = r$,

$$|g(r) - h(c)| = |h(r) - h(c)| \leq g(|r - c|) = g(c - r).$$

Then

$$-g(c - r) \leq g(r) - h(c) \leq g(c - r). \tag{3.2}$$

Then $g(r) - g(c - r) \leq h(c)$. Since $c \leq a + b$, we obtain $c - r \leq a + b - r \leq a$. Since g is increasing, $g(c - r) \leq g(a)$. So $g(a) - g(c - r) \geq 0$. Then

$$\begin{aligned} f(b) = g(b) \leq g(r) &\leq g(r) + g(a) - g(c - r) = g(r) - g(c - r) + g(a) \\ &\leq h(c) + g(a) = f(c) + f(a) \leq s(f(c) + f(a)). \end{aligned} \tag{3.3}$$

Since g is concave, we can substitute $t = r$, $x = a + b - r$, $y = a$, $z = b$ in Lemma 2.19 to obtain $g(r) + g(a + b - r) \leq g(a) + g(b)$. By (3.2), we know that $h(c) \leq g(r) + g(c - r)$. Therefore

$$\begin{aligned} f(c) = h(c) &\leq g(r) + g(c - r) \leq g(r) + g(a + b - r) \\ &\leq g(a) + g(b) = f(a) + f(b) \leq s(f(a) + f(b)). \end{aligned} \tag{3.4}$$

From (3.1), (3.3), and (3.4), we conclude that $(f(a), f(b), f(c)) \in \triangle_s$.

Case 4. $a \in [0, r)$ and $b, c \in [r, \infty)$. Since $r \leq b + c$, $b \leq c \leq c + r$, and

$c \leq a + b \leq r + b$, we see that $(r, b, c) \in \triangle$. Since $h \in \mathcal{B}$, $(h(r), h(b), h(c)) \in \triangle_{s_2}$. Therefore

$$f(a) = g(a) \leq g(r) = h(r) \leq s_2(h(b) + h(c)) \leq s(h(b) + h(c)) = s(f(b) + f(c)). \tag{3.5}$$

Since $|b - c| = c - b \leq r$, we obtain $|h(b) - h(c)| \leq g(|b - c|) = g(c - b)$. Then $-g(c - b) \leq h(b) - h(c) \leq g(c - b)$. Therefore

$$f(b) = h(b) \leq g(c - b) + h(c) \leq g(a) + h(c) = f(a) + f(c) \leq s(f(a) + f(c)) \tag{3.6}$$

and

$$f(c) = h(c) \leq g(c - b) + h(b) \leq g(a) + h(b) = f(a) + f(b) \leq s(f(a) + f(b)). \tag{3.7}$$

From (3.5), (3.6), and (3.7), we obtain $(f(a), f(b), f(c)) \in \triangle_s$. In any case, $(f(a), f(b), f(c)) \in \triangle_s$, as required. Therefore $f \in \mathcal{B}$ and the proof is complete. $\square$

Theorem 3.2. (A pasting lemma for functions in $\mathcal{BM}$) *Let $g, h : [0, \infty) \to [0, \infty)$, $g, h \in \mathcal{BM}$, $r > 0$, and $g(r) = h(r)$. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Let $A = \sup_{x \in (0,\infty)} f(x)$ and $B = \inf_{x \in (0,\infty)} f(x)$. Then

(i) $A = \max\left\{\sup_{x \in (0,r)} g(x), \sup_{x \in [r,\infty)} h(x)\right\}$ *and*

$B = \min\left\{\inf_{x \in (0,r)} g(x), \inf_{x \in [r,\infty)} h(x)\right\}$,

and the following statements are equivalent

(ii) $f \in \mathcal{BM}$

(iii) $A \leq 2B$

(iv) $\sup_{x\in(0,r)} g(x) \leq 2\inf_{x\in[r,\infty)} h(x)$ *and* $\sup_{x\in[r,\infty)} h(x) \leq 2\inf_{x\in(0,r)} g(x)$.

Proof. By Lemma 2.13, we see that $\inf_{x\in(0,r)} g(x)$, $\sup_{x\in(0,r)} g(x)$, $\inf_{x\in[r,\infty)} h(x)$, and $\sup_{x\in[r,\infty)} h(x)$ exist. Then $\sup_{x\in(0,\infty)} f(x)$ and $\inf_{x\in(0,\infty)} f(x)$ exist, and the statement (i) is obvious. Next assume that (ii) holds. By Lemma 2.13, there exists $v > 0$ such that $v \leq f(x) \leq 2v$ for all $x \in (0,\infty)$. Then $v \leq B \leq A \leq 2v$. Therefore $2B \geq 2v \geq A$, which proves (iii). Next, suppose (iii) holds. Then for each $x \in (0,\infty)$, we have

$$B = \inf_{x\in(0,\infty)} f(x) \leq f(x) \leq \sup_{x\in(0,\infty)} f(x) = A \leq 2B.$$

So f is tightly bounded. By Lemma 2.13, g and h are amenable. So f is also amenable. Applying Lemma 2.13 again, we obtain $f \in \mathcal{BM}$, as required. Hence (ii) and (iii) are equivalent. Next, we prove (iii) implies (iv). We have

$$\begin{aligned}\sup_{x\in(0,r)} g(x) &\leq \max\left\{\sup_{x\in(0,r)} g(x), \sup_{x\in[r,\infty)} h(x)\right\} = A \leq 2B \\ &= 2\min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\} \leq 2\inf_{x\in[r,\infty)} h(x),\end{aligned}$$

and similarly

$$\sup_{x\in[r,\infty)} h(x) \leq A \leq 2B \leq 2\inf_{x\in(0,r)} g(x),$$

which proves (iv). Finally, assume that (iv) holds.

Case 1 $\sup_{x\in(0,r)} g(x) \geq \sup_{x\in[r,\infty)} h(x)$. Then $A = \sup_{x\in(0,r)} g(x)$. Since $g \in \mathcal{BM}$, we can use an argument similar to the prove of (ii)$\Rightarrow$(iii) to obtain that

$$\sup_{x\in(0,r)} g(x) \leq 2 \inf_{x\in(0,r)} g(x).$$

By (iv),

$$\sup_{x\in(0,r)} g(x) \leq 2 \inf_{x\in[r,\infty)} h(x).$$

Therefore

$$A \leq \min\left\{2 \inf_{x\in(0,r)} g(x), 2 \inf_{x\in[r,\infty)} h(x)\right\} = 2\min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\} = 2B.$$

Case 2 $\sup_{x\in(0,r)} g(x) < \sup_{x\in[r,\infty)} h(x)$. Then $A = \sup_{x\in[r,\infty)} h(x)$. Similar to Case 1, since $h \in \mathcal{BM}$, we have $\sup_{x\in[r,\infty)} h(x) \leq 2\inf_{x\in[r,\infty)} h(x)$. By (iv), $\sup_{x\in[r,\infty)} h(x) \leq 2\inf_{x\in(0,r)} g(x)$. These imply $A \leq 2B$.

In any case, $A \leq 2B$, which proves (iii). So the proof is complete. □

Theorem 3.3. (A pasting lemma for functions in $\mathcal{MU}$ and $\mathcal{BU}$) *Let $g, h : [0,\infty) \to [0,\infty)$, g, $h \in \mathcal{MU}$, $r > 0$ and $g(r) = h(r)$. Define $f : [0,\infty) \to [0,\infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0,r), \\ h(x), & \text{if } x \in [r,\infty). \end{cases}$$

Then $f \in \mathcal{MU}$.

Proof. Since $g, h \in \mathcal{MU}$, by Lemma 2.14, g and h are amenable and constant on $(0,\infty)$. Since $g(r) = h(r)$ for all $r > 0$, we have $g(x) = g(r) = h(r) = h(x)$ for all $x > 0$. Then f is amenable and constant on $(0,\infty)$. Therefore $f \in \mathcal{MU}$. □

Theorem 3.4. (A pasting lemma for functions in $\mathcal{U}$) *Let* $g, h : [0, \infty) \to [0, \infty)$, g, $h \in \mathcal{U}$, $r > 0$ *and* $g(r) = h(r)$. *Define* $f : [0, \infty) \to [0, \infty)$ *by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Then $f \in \mathcal{U}$.

Proof. Since $g, h \in \mathcal{U}$, by Lemma 2.15, g and h are amenable and increasing. Since $g(r) = h(r)$ and h is increasing, we have $h(x) \geq g(r)$ for all $x \geq r$. Then f is increasing. Since g is amenable, so is f. Therefore $f \in \mathcal{U}$. □

Theorem 3.5. (A pasting lemma for functions in $\mathcal{UM}$) *Let* $g, h : [0, \infty) \to [0, \infty)$, $g, h \in \mathcal{UM}$, $r > 0$ *and* $g(r) = h(r)$. *Define* $f : [0, \infty) \to [0, \infty)$ *by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Then $f \in \mathcal{UM}$ *if and only if* $\sup_{x \in (0,r)} g(x) \leq 2 \inf_{x \in [r,\infty)} h(x)$.

Proof. We use Lemma 2.16 throughout the proof without further reference.

Assume $f \in \mathcal{UM}$. Since $g(a) \leq 2g(r)$ for every $a \in (0, r)$, $\sup_{x \in (0,r)} g(x)$ exists. Since $h(b) \geq \frac{1}{2} h(r)$ for every $b \in [r, \infty)$, $\inf_{x \in [r,\infty)} h(x)$ exists. Let $x \in (0, r)$ and $y \in [r, \infty)$. Then $x \leq y$ and

$$g(x) = f(x) \leq 2f(y) = 2h(y).$$

Then $g(x) \leq 2h(y)$ for all $x \in (0, r)$. Hence $\sup_{x \in (0,r)} g(x) \leq 2h(y)$. Since $\sup_{x \in (0,r)} g(x) \leq 2h(y)$ for all $y \in [r, \infty)$, we have

$$\sup_{x \in (0,r)} g(x) \leq \inf_{y \in [r,\infty)} 2h(y) = 2 \inf_{y \in [r,\infty)} h(y).$$

For the converse, assume that $\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x)$. Let $0 \le a \le b$. If $a, b < r$, then $f(a) = g(a) \le 2g(b) = 2f(b)$. If $a, b \ge r$, then $f(a) = h(a) \le 2h(b) = 2f(b)$. So suppose that $a < r \le b$. Then

$$f(a) = g(a) \le \sup_{x\in(0,r)} g(x) \le 2 \inf_{x\in[r,\infty)} h(x) \le 2h(b) = 2f(b).$$

In any case, $f(a) \le 2f(b)$. Hence $f \in \mathcal{UM}$. This completes the proof. □

Theorem 3.6. (A pasting lemma for functions in $\mathcal{UB}$) *Let $g, h : [0,\infty) \to [0,\infty)$, $g, h \in \mathcal{UB}$, $r > 0$ and $g(r) = h(r)$. Define $f : [0,\infty) \to [0,\infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Then $f \in \mathcal{UB}$.

Proof. Since $g, h \in \mathcal{UB}$, by Lemma 2.18, we have

$$\exists s_1 \ge 1\ \forall\ 0 \le a \le b,\ g(a) \le s_1 g(b) \quad \text{and}$$
$$\exists s_2 \ge 1\ \forall\ 0 \le a \le b,\ h(a) \le s_2 h(b).$$

Since $g(a) \le s_1 g(r)$ for every $a \in (0, r)$, $\sup_{x\in(0,r)} g(x)$ exists. Since $h(b) \ge \frac{1}{s_2} h(r)$ for every $b \in [r,\infty)$, $\inf_{x\in[r,\infty)} h(x)$ exists and is positive. Then there exists $s_3 \ge 1$ such that

$$\sup_{x\in(0,r)} g(x) \le s_3 \inf_{x\in[r,\infty)} h(x).$$

To show that $f \in \mathcal{UB}$, we choose $s = \max\{s_1, s_2, s_3\}$. Let $0 \le a \le b$. If $a, b < r$, then $f(a) = g(a) \le s_1 g(b) \le sg(b) = sf(b)$. If $a, b \ge r$, then $f(a) = h(a) \le$

$s_2h(b) \leq sh(b) = sf(b)$. So suppose that $a < r \leq b$. Then

$$f(a) = g(a) \leq \sup_{x\in(0,r)} g(x) \leq s_3 \inf_{x\in[r,\infty)} h(x) \leq s \inf_{x\in[r,\infty)} h(x) \leq sh(b) = sf(b).$$

In any case, we have $f(a) \leq sf(b)$. Therefore $f \in \mathcal{UB}$, as desired, so the proof is complete. □

From the subset properties in Proposition 1.4, we immediately obtain the following theorems.

Theorem 3.7. *Let $g, h : [0,\infty) \to [0,\infty)$, $r > 0$ and $g(r) = h(r)$. Define $f : [0,\infty) \to [0,\infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0,r), \\ h(x), & \text{if } x \in [r,\infty). \end{cases}$$

Then

(i) *If $g, h \in \mathcal{MU}$, then $f \in \mathcal{BM}$.*

(ii) *If $g, h \in \mathcal{MU}$, then $f \in \mathcal{M}$.*

(iii) *If $g, h \in \mathcal{MU}$, then $f \in \mathcal{B}$.*

(iv) *If $g, h \in \mathcal{MU}$, then $f \in \mathcal{U}$.*

(v) *If $g, h \in \mathcal{MU}$, then $f \in \mathcal{UM}$.*

(vi) *If $g, h \in \mathcal{MU}$, then $f \in \mathcal{UB}$.*

Proof. This follows immediately from Proposition 1.4 and Theorem 3.3. □

Theorem 3.8. *Let $g, h : [0,\infty) \to [0,\infty)$, $r > 0$ and $g(r) = h(r)$. Define f :*

$[0, \infty) \to [0, \infty)$ *by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Then

(i) *If* $g, h \in \mathcal{U}$, *then* $f \in \mathcal{UM}$.

(ii) *If* $g, h \in \mathcal{U}$, *then* $f \in \mathcal{UB}$.

Proof. This follows immediately from Proposition 1.4 and Theorem 3.4. □

Theorem 3.9. *Let* $g, h : [0, \infty) \to [0, \infty)$, $r > 0$ *and* $g(r) = h(r)$. *Define* f : $[0, \infty) \to [0, \infty)$ *by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Then

(i) *If* $g, h \in \mathcal{MU}$, *then* $f \in \mathcal{U}$.

(ii) *If* $g, h \in \mathcal{MU}$, *then* $f \in \mathcal{UM}$.

(iii) *If* $g, h \in \mathcal{MU}$, *then* $f \in \mathcal{UB}$.

(iv) *If* $g \in \mathcal{MU}$ *and* $h \in \mathcal{U}$, *then* $f \in \mathcal{U}$.

(v) *If* $g \in \mathcal{MU}$ *and* $h \in \mathcal{U}$, *then* $f \in \mathcal{UM}$.

(vi) *If* $g \in \mathcal{MU}$ *and* $h \in \mathcal{U}$, *then* $f \in \mathcal{UB}$.

(vii) *If* $g \in \mathcal{U}$ *and* $h \in \mathcal{MU}$, *then* $f \in \mathcal{U}$.

(viii) *If* $g \in \mathcal{U}$ *and* $h \in \mathcal{MU}$, *then* $f \in \mathcal{UM}$.

(ix) *If* $g \in \mathcal{U}$ *and* $h \in \mathcal{MU}$, *then* $f \in \mathcal{UB}$.

Proof. This follows immediately from Proposition 1.4 and Theorem 3.4. □

Theorem 3.10. *Let $g, h : [0, \infty) \to [0, \infty)$, $r > 0$ and $g(r) = h(r)$. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Let A be one of the following sets : $\mathcal{MU}$, $\mathcal{BM}$, $\mathcal{M}$, $\mathcal{B}$, $\mathcal{U}$, $\mathcal{UM}$. Then if $g, h \in A$, then $f \in \mathcal{UB}$.

Proof. This follows immediately from Proposition 1.4 and Theorem 3.6. □

Next, we give some examples to show that

(i) $g \in \mathcal{BM}$, $h \in \mathcal{BM}$ but $f \notin \mathcal{BM}$,

(ii) $g \in \mathcal{M}$, $h \in \mathcal{B}$ but $f \notin \mathcal{M}$,

(iii) $g \in \mathcal{B}$, $h \in \mathcal{B}$ but $f \notin \mathcal{M}$,

(iv) $g \in \mathcal{MU}$, $h \in \mathcal{UM}$ but $f \notin \mathcal{MU}$, and

(v) $g \in \mathcal{U}$, $h \in \mathcal{MU}$ but $f \notin \mathcal{MU}$.

Example 3.11. Let $g(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \in (0, 2), \\ 2, & \text{if } x \in [2, \infty) \end{cases}$ and $h(x) = \begin{cases} 0, & \text{if } x = 0, \\ 2, & \text{if } x \in (0, 2], \\ 3, & \text{if } x \in (2, \infty). \end{cases}$

Since g and h are amenable and tightly bounded, we have $g, h \in \mathcal{BM}$.

We will show that $f(x) = \begin{cases} g(x), & \text{if } x \in [0, 2), \\ h(x), & \text{if } x \in [2, \infty) \end{cases}$ is not tightly bounded.

$$\text{We have } f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \in (0,2), \\ 2, & \text{if } x = 2, \\ 3, & \text{if } x \in (2,\infty). \end{cases}$$

To show that f is not tightly bounded, let $a > 0$. Then $a \leq 1$ or $a > 1$.

Case 1. $a \leq 1$. Then $2a \leq 2$. Choose $x = 3$. So $f(x) = 3 > 2a$. Then $f(x) \notin [a, 2a]$.

Case 2. $a > 1$. Choose $x = 1$. Then $f(x) = 1 < a$, so $f(x) \notin [a, 2a]$.

In any case, $f(x) \notin [a, 2a]$, so f is not tightly bounded. This example show that $g, h \in \mathcal{BM}$ but $f \notin \mathcal{BM}$.

Example 3.12. Let $g(x) = x$ and $h(x) = x^2$ Then $g \in \mathcal{M}$ and $h \in \mathcal{B}$. We will show that $f(x) = \begin{cases} g(x), & \text{if } x \in [0,1), \\ h(x), & \text{if } x \in [1,\infty) \end{cases}$ is not metric-preserving function.

$$\text{We have } f(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ x^2, & \text{if } x \in [1,\infty). \end{cases}$$

Let $a = 3$, $b = 1$, and $c = 2$. We see that $(3,1,2) \in \triangle$. Then $f(3) = 9$ and $f(1) + f(2) = 5$. So $(f(3), f(1), f(2)) \notin \triangle$. Then $f \notin \mathcal{M}$. This example show that $g \in \mathcal{M}$ and $h \in \mathcal{B}$ but $f \notin \mathcal{M}$.

Since $\mathcal{M} \subseteq \mathcal{B}$, we also obtain example of $g \in \mathcal{B}$, $h \in \mathcal{B}$ but $f \notin \mathcal{M}$.

Example 3.13. Let $g(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0 \end{cases}$ and $h(x) = \begin{cases} x, & \text{if } x \leq 1, \\ \frac{1}{2}, & \text{if } x > 1. \end{cases}$

Since g is amenable and constant on $(0,\infty)$, $g \in \mathcal{MU}$.

By [9, Example 22], we have $h \in \mathcal{UM}$. We will show that $f(x) = \begin{cases} g(x), & \text{if } x \in [0,1), \\ h(x), & \text{if } x \in [1,\infty) \end{cases}$

is not ultrametric-metric-preserving function. We have $f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \in (0,1], \\ \frac{1}{2}, & \text{if } x \in (1,\infty). \end{cases}$

Since f is not constant on $(0,\infty)$, $f \notin \mathcal{MU}$. This example show that $g \in \mathcal{MU}$ and $h \in \mathcal{UM}$ but $f \notin \mathcal{MU}$.

Example 3.14. Let $g(x) = x$ and $h(x) = \begin{cases} 0, & \text{if } x = 0, \\ 2, & \text{if } x > 0. \end{cases}$ We see that $g \in \mathcal{U}$.

Since h is amenable and constant on $(0,\infty)$, $h \in \mathcal{MU}$. We will show that $f(x) = \begin{cases} g(x), & \text{if } x \in [0,2), \\ h(x), & \text{if } x \in [2,\infty) \end{cases}$ is not metric-ultrametric-preserving function. We have $f(x) = \begin{cases} x, & \text{if } x \in [0,2), \\ 2, & \text{if } x \in [2,\infty). \end{cases}$ Since f is not constant on $(0,\infty)$, $f \notin \mathcal{MU}$. This example show that $g \in \mathcal{U}$ and $h \in \mathcal{MU}$ but $f \notin \mathcal{MU}$.

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Appendix

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Pasting Lemmas for $b-$Metric Preserving and Related Functions

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Abstract

Previously ([7], [8]), we established some relations between $b-$metrics and metric-preserving functions. In this article, we give pasting lemmas for those functions.

1 Introduction

It is well known that if $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ are continuous and $g(b) = h(b)$, then the function $f : [a, c] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} g(x), & \text{if } x \in [a, b); \\ h(x), & \text{if } x \in [b, c] \end{cases}$$

is also continuous. This is usually called a pasting lemma. A version of a pasting lemma for metric-preserving functions is given by Doboš [6, p. 26] but there is no pasting lemma for $b-$metric-preserving and other related functions in the literature. So we provide such a lemma in this article. Let us recall the definitions and useful results on $b-$metrics and metric-preserving functions which were previously given in [7, 8] as follows:

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Definition 1.1. *Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a $b-$metric if it satisfies the following three conditions:*

(B1) *for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,*

(B2) *for all $x, y \in X$, $d(x, y) = d(y, x)$,*

(B3) *there exists $s \geq 1$ such that*

$$d(x, y) \leq s(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in X.$$

Definition 1.2. *The function $f : [0, \infty) \to [0, \infty)$ is called metric preserving if for all metric spaces (X, d), $f \circ d$ is a metric on X.*

The concept of b-metrics appears in many articles (for example in [3, 5, 7, 11]). We also refer the reader to [1, 2, 4, 6, 10] for more information on metric-preserving functions and to [9] for applications in fixed point theory. In connection with metric-preserving functions and $b-$metrics, Khemaratchatakumthorn and Pongsriiam [7] define the following notions:

Definition 1.3. *Let $f : [0, \infty) \to [0, \infty)$. We say that*

(i) *f is $b-$metric-preserving if for all $b-$metric spaces (X, d), $f \circ d$ is a $b-$metric on X,*

(ii) *f is metric-$b-$metric-preserving if for all metric spaces (X, d), $f \circ d$ is a $b-$metric on X, and*

(iii) *f is $b-$metric-metric-preserving if for all b-metric spaces (X, d), $f \circ d$ is a metric on X.*

We let $\mathcal{M}$ be the set of all metric-preserving functions, $\mathcal{B}$ the set of all $b-$metric-preserving functions, $\mathcal{MB}$ the set of all metric-$b-$metric-preserving functions, and $\mathcal{BM}$ the set of all $b-$metric-metric-preserving functions.

From [7, Theorem 15 and Example 16] and [8, Theorem 3.1], we have the following theorem.

Theorem 1.4. [7, 8] *We have $\mathcal{BM} \subseteq \mathcal{M} \subseteq \mathcal{B} = \mathcal{MB}$, $\mathcal{M} \not\subseteq \mathcal{BM}$, and $\mathcal{B} \not\subseteq \mathcal{M}$.*

2 Preliminaries and Lemmas

In order to prove our main theorem, we need to recall some basic definitions and results in [7].

Let $f : [0, \infty) \to [0, \infty)$ and let $I \subseteq [0, \infty)$. Then f is said to be increasing on I if $f(x) \leq f(y)$ for all $x, y \in I$ satisfying $x < y$, and f is said to be strictly increasing on I if $f(x) < f(y)$ for all $x, y \in I$ satisfying $x < y$. The notion of decreasing or strictly decreasing functions is defined similarly.

The function f is said to be amenable if $f^{-1}(0) = \{0\}$, and f is said to be tightly bounded on $(0, \infty)$ if there is $v > 0$ such that $f(x) \in [v, 2v]$ for all $x > 0$. We say that f is concave if $f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2)$ for all $x_1, x_2 \in [0, \infty)$ and $t \in [0, 1]$. In addition, we say that f is quasi-subadditive if there exists $s \geq 1$ such that $f(a + b) \leq s(f(a) + f(b))$ for all $a, b \in [0, \infty)$.

Definition 2.1. *A triangle triplet is a triple (a, b, c) of nonnegative real numbers for which*

$$a \leq b + c,\ b \leq a + c,\ \text{and}\ c \leq a + b,$$

or, equivalently,

$$|a - b| \leq c \leq a + b.$$

Let $s \geq 1$ and $a, b, c \geq 0$. A triple (a, b, c) is an s-triangle triplet if

$$a \leq s(b + c),\ b \leq s(a + c),\ \text{and}\ c \leq s(a + b).$$

Let Δ and Δ_s be the sets of all triangle triplets and $s-$triangle triplets, respectively.

Next, we recall results concerning $b-$metrics and metric-preserving functions. Again, we let $f : [0, \infty) \to [0, \infty)$ throughout.

Lemma 2.2. [7] *$f \in \mathcal{BM}$ if and only if f is amenable and tightly bounded.*

Lemma 2.3. [7] *If $f \in \mathcal{B}$, then f is amenable and quasi-subadditive.*

Lemma 2.4. [7, 8] *Suppose f is amenable. Then $f \in \mathcal{B}$ if and only if there exists $s \geq 1$ such that $(f(a), f(b), f(c)) \in \Delta_s$ for all $(a, b, c) \in \Delta$.*

Lemma 2.5. [6, p. 12] *Let f be amenable. Then f is concave if and only if for all $t \geq 0$ and $x, y, z \in [0, t]$ if $x + t = y + z$, then $f(x) + f(t) \leq f(y) + f(z)$.*

3 Main Results

We begin with a pasting lemma for functions in $\mathcal{B}$. We see that a slight modification from those in $\mathcal{M}$ is enough. In addition, by Theorem 1.4, this also gives a pasting lemma for functions in $\mathcal{MB}$ as follows.

Theorem 3.1. (A pasting lemma for functions in $\mathcal{B}$ and $\mathcal{MB}$) *Let $g, h \in \mathcal{B}$, $r > 0$, and $g(r) = h(r)$. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Suppose that g is increasing, concave, and

$$\forall x, y \in [r, \infty), |x - y| \le r \Rightarrow |h(x) - h(y)| \le g(|x - y|).$$

Then $f \in \mathcal{B}$.

Proof. Since $g, h \in \mathcal{B}$, by Lemmas 2.3 and 2.4 there are $s_1, s_2 \ge 1$ such that

$$(g(a), g(b), g(c)) \in \Delta_{s_1} \text{ and } (h(a), h(b), h(c)) \in \Delta_{s_2} \text{ for every } (a, b, c) \in \Delta.$$

Let $s = \max\{s_1, s_2\}$ and let $(a, b, c) \in \Delta$. Without loss of generality, assume $0 \le a \le b \le c \le a + b$. If $a, b, c \in [0, r)$, then $(f(a), f(b), f(c)) = (g(a), g(b), g(c)) \in \Delta_{s_1} \subseteq \Delta_s$. If $a, b, c \in [r, \infty)$, then $(f(a), f(b), f(c)) = (h(a), h(b), h(c)) \in \Delta_{s_2} \subseteq \Delta_s$. So it remains to consider the cases where a, b, c are not in the same interval. If $c \in [0, r)$, then $a, b \in [0, r)$ too. So there are two cases left to consider as follows.

Case 1. $a, b \in [0, r)$ and $c \in [r, \infty)$. Then

$$f(a) = g(a) \le g(b) = f(b) \le f(b) + f(c) \le s(f(b) + f(c)). \tag{3.1}$$

Since $|r - c| = c - r \le a + b - r < r + r - r = r$,

$$|g(r) - h(c)| = |h(r) - h(c)| \le g(|r - c|) = g(c - r).$$

Then

$$-g(c - r) \le g(r) - h(c) \le g(c - r). \tag{3.2}$$

Then $g(r) - g(c - r) \le h(c)$. Since $c \le a + b$, $c - r \le a + b - r \le a$. Since g is increasing, $g(c - r) \le g(a)$ and therefore

$$\begin{aligned} f(b) = g(b) \le g(r) &\le g(r) + g(a) - g(c - r) = (g(r) - g(c - r)) + g(a) \\ &\le h(c) + g(a) = f(c) + f(a) \\ &\le s(f(c) + f(a)). \end{aligned} \tag{3.3}$$

Since g is concave, we can substitute $t = r$, $x = a+b-r$, $y = a$, $z = b$ in Lemma 2.5 to obtain $g(a+b-r)+g(r) \leq g(a)+g(b)$. By (3.2), $h(c) \leq g(r)+g(c-r)$. Therefore

$$\begin{aligned} f(c) = h(c) \leq g(r)+g(c-r) &\leq g(r)+g(a+b-r) \\ &\leq g(a)+g(b) = f(a)+f(b) \\ &\leq s(f(a)+f(b)). \end{aligned} \tag{3.4}$$

From (3.1), (3.3), and (3.4), we conclude that $(f(a), f(b), f(c)) \in \Delta_s$.

Case 2. $a \in [0, r)$ and $b, c \in [r, \infty)$. Since $r \leq b+c$, $b \leq c \leq c+r$, and $c \leq a+b \leq r+b$, we see that $(r, b, c) \in \Delta$. Then $(h(r), h(b), h(c)) \in \Delta_{s_2}$. Therefore

$$\begin{aligned} f(a) = g(a) \leq g(r) = h(r) &\leq s_2(h(b)+h(c)) \\ &\leq s(h(b)+h(c)) = s(f(b)+f(c)). \end{aligned} \tag{3.5}$$

Since $|b-c| = c-b \leq r$, $|h(b)-h(c)| \leq g(|b-c|) = g(c-b)$. Then $-g(c-b) \leq h(b)-h(c) \leq g(c-b)$ and therefore

$$\begin{aligned} f(b) = h(b) &\leq g(c-b)+h(c) \leq g(a)+h(c) \\ &= f(a)+f(c) \leq s(f(a)+f(c)), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} f(c) = h(c) &\leq g(c-b)+h(b) \leq g(a)+h(b) \\ &= f(a)+f(b) \leq s(f(a)+f(b)). \end{aligned} \tag{3.7}$$

From (3.5), (3.6), and (3.7), we obtain $(f(a), f(b), f(c)) \in \Delta_s$. In all cases, $(f(a), f(b), f(c))$ is in Δ_s, as required. Consequently, $f \in \mathcal{B}$ and the proof is complete. □

It remains to consider functions in $\mathcal{BM}$.

Theorem 3.2. (A pasting lemma for functions in $\mathcal{BM}$) *Let $g, h \in \mathcal{BM}$, $r > 0$, and $g(r) = h(r)$. Define $f : [0, \infty) \to [0, \infty)$ by*

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0, r), \\ h(x), & \text{if } x \in [r, \infty). \end{cases}$$

Let $A = \sup_{x \in (0,\infty)} f(x)$ and $B = \inf_{x \in (0,\infty)} f(x)$. Then

(i) $A = \max\left\{\sup_{x\in(0,r)} g(x), \sup_{x\in[r,\infty)} h(x)\right\}$ *and*

$B = \min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\}$,

and the following statements are equivalent

(ii) $f \in \mathcal{BM}$

(iii) $A \le 2B$

(iv) $\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x)$ *and* $\sup_{x\in[r,\infty)} h(x) \le 2\inf_{x\in(0,r)} g(x)$.

Proof. By Lemma 2.2, it follows that $\inf_{x\in(0,r)} g(x)$, $\sup_{x\in(0,r)} g(x)$, $\inf_{x\in[r,\infty)} h(x)$, and $\sup_{x\in[r,\infty)} h(x)$ exist. Then $\sup_{x\in(0,\infty)} f(x)$ and $\inf_{x\in(0,\infty)} f(x)$ exist, and the statement (i) is obvious. Next, assume that (ii) holds. By Lemma 2.2, there exists $v > 0$ such that $v \le f(x) \le 2v$ for all $x \in (0,\infty)$. Then $v \le B \le A \le 2v$. Therefore $2B \ge 2v \ge A$, which proves (iii). Now, suppose (iii) holds. Then for each $x \in (0,\infty)$, we have

$$B = \inf_{x\in(0,\infty)} f(x) \le f(x) \le \sup_{x\in(0,\infty)} f(x) = A \le 2B.$$

So f is tightly bounded. By Lemma 2.2, g and h are amenable. So f is also amenable. Applying Lemma 2.2 again, we obtain $f \in \mathcal{BM}$, as required. Hence (ii) and (iii) are equivalent. Next, we prove (iii) implies (iv). We have

$$\begin{aligned}\sup_{x\in(0,r)} g(x) &\le \max\left\{\sup_{x\in(0,r)} g(x), \sup_{x\in[r,\infty)} h(x)\right\} = A \le 2B\\ &= 2\min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\} \le 2\inf_{x\in[r,\infty)} h(x),\end{aligned}$$

and, similarly,

$$\sup_{x\in[r,\infty)} h(x) \le A \le 2B \le 2\inf_{x\in(0,r)} g(x),$$

which proves (iv). Finally, assume that (iv) holds.

Case 1. $\sup_{x\in(0,r)} g(x) \ge \sup_{x\in[r,\infty)} h(x)$. Then $A = \sup_{x\in(0,r)} g(x)$. Since $g \in \mathcal{BM}$, we can use an argument similar to the prove of (ii)$\Rightarrow$(iii) to obtain

$$\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in(0,r)} g(x).$$

By (iv),

$$\sup_{x\in(0,r)} g(x) \le 2\inf_{x\in[r,\infty)} h(x).$$

Therefore

$$A \leq \min\left\{2\inf_{x\in(0,r)} g(x), 2\inf_{x\in[r,\infty)} h(x)\right\}$$
$$= 2\min\left\{\inf_{x\in(0,r)} g(x), \inf_{x\in[r,\infty)} h(x)\right\} = 2B.$$

Case 2. $\sup_{x\in(0,r)} g(x) < \sup_{x\in[r,\infty)} h(x)$. Then $A = \sup_{x\in[r,\infty)} h(x)$. Similar to Case 1, since $h \in \mathcal{BM}$, we have $\sup_{x\in[r,\infty)} h(x) \leq 2\inf_{x\in[r,\infty)} h(x)$. By (iv), $\sup_{x\in[r,\infty)} h(x) \leq 2\inf_{x\in(0,r)} g(x)$. These imply $A \leq 2B$.

In all cases, $A \leq 2B$, which proves (iii). So the proof is complete. □

Pasting lemmas for other functions will be given in a future article.

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