DIGITAL PROBLEMS RELATED TO KAPREKAR CONSTANT AND MULTIPLICATION

By

MISS Niphawan PHOOPHA

A Thesis Proposal Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Graduate School, Silpakorn University

Academic Year 2020

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Title	Digital problems related to Kaprekar constant and multiplication
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NIPHAWAN PHOOPHA : DIGITAL PROBLEMS RELATED TO KAPREKAR CONSTANT AND MULTIPLICATION. THESIS ADVISOR : ASSOCIATE PROFESSOR PRAPANPONG PONGSRIIAM, Ph.D.

A function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ is called a digital function if $f(0) \in \{0, 1\}$ and $f(x) = f(a_k) + f(a_{k-1}) + \cdots + f(a_0)$ for each positive integer x having decimal representation as $x = (a_k a_{k-1} \ldots a_0)_{10}$ with $a_k \neq 0$. In this thesis, we show some interesting digital functions and give a proof of some mathematical memes.

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Niphawan PHOOPHA

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Chapter 1

Introduction

In this thesis, we study a new digit map that is a variation of Kaprekar operator and determine the new constants arising in the process of repeatedly applying the new digit map related to multiplication. For any $x \in \mathbb{N} \cup \{0\}$, we write the decimal expansion of x as

$$x = (a_k a_{k-1} \ldots a_1 a_0)_{10} = \sum_{0 \le j \le k} a_{k-j} 10^{k-j},$$

where $0 \le a_i \le 9$ for all $i = 0, 1, 2, \ldots, k$.

First, we introduce the reader to know about the Kaprekar constant. The Kaprekar operator K is defined by the following operation: take any positive integer x having four decimal digits which are not all equal and the leading digit is not zero, say $x = (a_3 a_2 a_1 a_0)_{10}$, $a_3 \neq 0$, and $a_i \neq a_j$ for some i, j, then rearrange a_3, a_2, a_1, a_0 as c_3, c_2, c_1, c_0 so that $c_3 \ge c_2 \ge c_1 \ge c_0$. Then

$$K(x) = (c_3 c_2 c_1 c_0)_{10} - (c_0 c_1 c_2 c_3)_{10}. \tag{1.1}$$

Observe that the second number on the right-hand side of (1.1) is obtained by reversing the decimal digits of the first. It is well known that no matter what x we start, after repeating this process at most 7 steps, we always obtain the number 6174, which is known as Kaprekar's constant. For example, suppose $x = 1000$.

Then

$$K(x) = 1000 - 1 = 999,$$

$$K^2(x) = K(K(x)) = K(999) = K(0999) = 9990 - 0999 = 8991,$$

$$K^3(x) = K(8991) = 9981 - 1899 = 8082,$$

$$K^4(x) = 8820 - 0288 = 8532,$$

$$K^5(x) = 8532 - 2358 = 6174,$$

and $K^m(x) = 6174$ for all $m \geq 6$. Here it is important to keep in mind that the Kaprekar operator operates on the positive integers having four digits not all equal. So the decimal representation of $K(x)$ with nonzero leading digit may has only 3 digits but to calculate $K(K(x))$, we must first write $K(x)$ as 4 digits number by adding 0 as the leading digit, as shown above in $K(999) = K(0999)$. We can generalize K to operate on any nonnegative integers as follows:

Definition 1.1 (Kaprekar operator on nonnegative integers)**.** Let $g : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be given by $g(0) = 0$ and if $x = (a_k a_{k-1} \ldots a_0)_{10}$, $a_k \neq 0$, and c_k, c_{k-1}, ..., c_0 is the permutation of a_k, a_{k-1}, ..., a_0 such that $c_k \geq c_{k-1} \geq \cdots \geq c_0$, then

$$g(x) = (c_k c_{k-1} \ldots c_1 c_0)_{10} - (c_0 c_1 \ldots c_{k-1} c_k)_{10}.$$

In addition, for the purpose of this thesis, if x is as above, we always write the decimal representation of $g(x)$ as $k+1$ digits number, say $g(x) = (b_k b_{k-1} \ldots b_0)_{10}$.

Another trick is as follows: take any positive integer having three digits, say $x = (a_2 a_1 a_0)_{10}$, where $a_2 \neq 0$, $0 \leq a_j \leq 9$ for all j, and $a_i \neq a_j$ for some

i, j. Then calculate $g(x)$, say $g(x) = b = (b_2b_1b_0)_{10}$. Then compute $f(b) = b + \text{reverse}(b) = (b_2b_1b_0)_{10} + (b_0b_1b_2)_{10}$. No matter what x we start with, we always obtain $f(b) = 1089$. We generalize this to the following operator.

Definition 1.2. Let f be the reverse and add operator and let $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be defined by $F = f \circ g$. In addition, to calculate $F(x) = f(g(x))$, we always keep the same convention in Definition 1.1 where the number of decimal digits of x and $g(x)$ are equal.

For example, suppose $x = 100$. Then $g(x) = 99 = 099$ and so $F(x) = f(099) = 990 + 099 = 1089$. By using a computer or a straightforward calculation, it is not difficult to notice the following pattern:

if $10 \le x < 10^2$, then $F(x) = 0$ or 99;

if $10^2 \le x < 10^3$, then $F(x) = 0$ or 1089;

if $10^3 \le x < 10^4$, then $F(x) = 0$, 10890, or 10989;

if $10^4 \le x < 10^5$, then $F(x) = 0$, 109890, 0 4or 109989.

In general result, which can read in Chapter 2. Moreover, it is an interesting open problem to determine whether or not a given number in the range of F is a Lychrel number. For more information on 6174 and the Kaprekar operator, see for instance in [6], [13], and [16]. For related articles on 1089 and 2178, see for example in [1], [2], [3], [4], [18], [19], and [22].

Next, we introduce the reader to know about the happy function. For each positive integer x, define $S(x)$ to be the sum of squares of the decimal digits of x. For example, $S(2) = 4$ and $S(123) = 1^2 + 2^2 + 3^2 = 14$. It is well known that [11] for any $x \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $S^{(n)}(x) \in \{1, 4\}$, where $S^{(n)}$ is the n-fold composition of S. The function is called the happy function and if $x \in \mathbb{N}$ and $S^{(n)}(x) = 1$ for some $n \in \mathbb{N}$, then x is called a happy number. Furthermore, we can generalize this concept to an (e, b)-happy function $S_{e,b}$ for e, $b \in \mathbb{N}$ and e, $b \geq 2$ by defining

$$S_{e,b}(x) = a_k^e + a_{k-1}^e + \cdots + a_0^e,$$

if $x = (a_k a_{k-1} \ldots a_0)_b = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_0$ is the b-adic expansion of x with $a_k \neq 0$ and $a_i \in \{0, 1, 2, \ldots, b-1\}$ for all $i = 0, 1, \ldots, k$. Then a similar result still holds: there exists a finite set $A \subseteq \mathbb{N}$ such that for any $x \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $S_{e,b}^{(n)}(x) \in A$. For example, if $(e, b) = (2, 10)$, then $A = \{1, 4\}$; and if $(e, b) = (3, 10)$, then $A = \{1, 55, 136, 153, 160, 370, 371, 407, 919\}$. For more details about this, see for instance in the articles by El-Sedy and Siksek [7], Grundman and Teeple [10], and the book by Guy [11].

On one hand, we may focus on the study of long strings of consecutive integers which are happy or (e, b)-happy as given by El-Sedy and Siksek [7], Pan [15], Zhou and Cai [23], Gilmer [8], Styer [17], and Chase [5]. On the other hand, we may consider generalizations of the concept of (e, b)-happy functions as in the work of Grundman [9], Chase [5], Swart et al. [21], Noppakaew, Phoopha, and Pongsriiam [14], and Subwattanachai and Pongsriiam [20]. In this thesis, we focus on the latter and continue the study from those articles [14, 20]. Let us consider the following functions.

Definition 1.3. (The sum of factorials of digits) Let $b \geq 2$ and let $f_b : \mathbb{N} \to \mathbb{N}$ be defined by

$$f_b(x) = a_k! + a_{k-1}! + \cdots + a_0!$$

if $x = (a_k a_{k-1} \ldots a_0)_b$ is the b-adic representation of x with $a_k \neq 0$.

Definition 1.4. (A power of sums of digits) Let e, $b \geq 2$ and let $g_{e,b} : \mathbb{N} \to \mathbb{N}$ be defined by

$$g_{e,b}(x) = (a_k + a_{k-1} + \cdots + a_0)^e$$

if $x = (a_k a_{k-1} \ldots a_0)_b$ is the b-adic representation of x with $a_k \neq 0$.

The functions f_b, $g_{e,b}$, and similar variations are natural examples of new digit maps falling outside the scope of Chase's definition and other articles on digit maps, yet similar results still hold. That is, if f is such a function, then we can explicitly determine a finite set $A \subseteq \mathbb{N}$ such that for every $x \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $f^{(n)}(x) \in A$. So we can study this result in Chapter 3. Furthermore, our results can be interpreted as solutions to certain Diophantine equations which explain some popular mathematical memes in Chapter 4.

Throughout this thesis, there is using a computer to calculate some numbers. Then we list some relevant codes in the last chapter. Moreover, we hope that this thesis will help explaining something related to 6174, 1089, and other similar magic numbers.

Chapter 2

Variation of Kaprekar operator and 1089

In this chapter, if $y \in \mathbb{R}$, then $\lfloor y \rfloor$ is the largest integer less than or equal to y and $\lceil y \rceil$ is the smallest integer larger than or equal to y; and unless stated otherwise, all other variables are nonnegative integers. Then we recall Definition 1.2 to introduce the general result.

Theorem 2.1. *Let $F = f \circ g$, $k \geq 2$, and $10^k \leq x < 10^{k+1}$. Let $x = (a_k a_{k-1} \ldots a_0)_{10}$, $a_k \neq 0$, and $0 \leq a_i \leq 9$ for all $i = 0, 1, \ldots, k$. If $k = 2$, then $F(x) = 0$ or 1089. Suppose that $k \geq 3$ and $c_k, c_{k-1}, \ldots, c_0$ is the permutation of $a_k, a_{k-1}, \ldots, a_0$ such that $c_k \geq c_{k-1} \geq \cdots \geq c_0$. If $a_i = a_j$ for all i, j, then $F(x) = 0$. Suppose that $a_i \neq a_j$ for some i, j and let $m = z(x)$ be the largest element of the set $\{j \in \{0, 1, \ldots, k\} \mid c_{k-j} > c_j\}$. Then*

$$F(x) = 10\underbrace{99\ldots9}_{y(x)}89\underbrace{00\ldots0}_{z(x)},$$

where $y(x) = k - 2 - z(x)$.

Proof. We first consider the case $k = 2$. Since $10^2 \leq x < 10^3$, it can be written in the decimal representation as $x = (a_2 a_1 a_0)_{10}$ where $a_2 \neq 0$ and $0 \leq a_i \leq 9$ for $i = 0, 1, 2$. If $a_2 = a_1 = a_0$, then $F(x) = 0$. So suppose that a_2, a_1, a_0 are not all the same and let c_2, c_1, c_0 be the permutation of a_2, a_1, a_0 such that $c_2 \geq c_1 \geq c_0$.

Then $c_2 > c_0$ and

$$\begin{aligned} g(x) &= (c_2c_1c_0)_{10} - (c_0c_1c_2)_{10} \\ &= (10^2c_2 + 10c_1 + c_0) - (10^2c_0 + 10c_1 + c_2) \\ &= 10^2(c_2 - c_0 - 1) + 10(9) + 10 - (c_2 - c_0) \\ &= (d_2d_1d_0)_{10}, \end{aligned}$$

where $d_2 = c_2 - c_0 - 1$, $d_1 = 9$, and $d_0 = 10 - (c_2 - c_0)$. Then it is easy to see that

$$F(x) = (d_2d_1d_0)_{10} + (d_0d_1d_2)_{10} = 1089.$$

Next, let $k \geq 3$, $10^k \leq x < 10^k$, and write $x = (a_ka_{k-1}\ldots a_0)_{10}$ where $a_k \neq 0$ and $0 \leq a_i \leq 9$ for all $i = 0, 1, \ldots, k$. If $a_i = a_j$ for all i, j, then $F(x) = 0$ and we are done. So suppose that $a_i \neq a_j$ for some i, j. Let c_k, c_{k-1}, $\ldots$, c_0 be the permutation of a_k, a_{k-1}, $\ldots$, a_0 such that $c_k \geq c_{k-1} \geq \cdots \geq c_0$. Then

$$\begin{aligned} g(x) &= (c_kc_{k-1}\ldots c_0)_{10} - (c_0c_1\ldots c_k)_{10} \\ &= \sum_{j=0}^{k} c_{k-j}10^{k-j} - \sum_{j=0}^{k} c_j10^{k-j} \\ &= \sum_{j=0}^{k} (c_{k-j} - c_j)10^{k-j}. \end{aligned} \tag{2.1}$$

Let $A = \{j \in \{0, 1, \ldots, k\} \mid c_{k-j} > c_j\}$. Since $c_k > c_0$, we see that $0 \in A$, and so $A \neq \varnothing$. Let m be the largest element of A. If $m \geq \lceil \frac{k}{2} \rceil$, then $k - m \leq k - \lceil \frac{k}{2} \rceil = \lfloor \frac{k}{2} \rfloor \leq m$, which implies $c_{k-m} \leq c_m$, which contradicts the fact that $m \in A$. Therefore $0 \leq m < \lceil \frac{k}{2} \rceil$. Since m is the largest element of A and

$c_k \geq c_{k-1} \geq \cdots \geq c_0$, we assert that the following relations hold:

$$c_{k-j} > c_j \quad \text{for} \quad 0 \leq j \leq m, \tag{2.2}$$

$$c_{k-j} \leq c_j \quad \text{for} \quad j > m, \tag{2.3}$$

$$c_{k-j} = c_j \quad \text{for} \quad m < j \leq \left\lfloor \frac{k}{2} \right\rfloor, \tag{2.4}$$

$$c_{k-j} = c_j \quad \text{for} \quad \left\lceil \frac{k}{2} \right\rceil \leq j < k-m, \tag{2.5}$$

$$c_{k-j} < c_j \quad \text{for} \quad k-m \leq j \leq k. \tag{2.6}$$

For (2.2), we know that $c_{k-m} > c_m$ and if $0 \leq j < m$, then $c_{k-j} \geq c_{k-m} > c_m \geq c_j$. So (2.2) is verified. By the choice of m, (2.3) follows immediately. If $j \leq \lfloor \frac{k}{2} \rfloor$, then $k - j \geq k - \lfloor \frac{k}{2} \rfloor = \lceil \frac{k}{2} \rceil \geq j$, and so $c_{k-j} \geq c_j$. This and (2.3) imply (2.4). Replacing j by $k-j$ in (2.4), we obtain (2.5). Changing j to $k-j$ in (2.2), we obtain (2.6).

Next, we divide the sum in (2.1) into 3 parts: $0 \leq j \leq m$, $m < j < k-m$, and $k-m \leq j \leq k$. By (2.4) and (2.5), the second part is zero. Therefore (2.1) becomes

$$g(x) = \sum_{0 \leq j \leq m} (c_{k-j} - c_j)10^{k-j} + \sum_{k-m \leq j \leq k} (c_{k-j} - c_j)10^{k-j}. \tag{2.7}$$

The terms $c_{k-j} - c_j$ in (2.7) are positive in the first sum and negative in the second sum. Then we write

$$\begin{aligned} 10^{k-m} &= \left(\sum_{m+1 \leq j \leq k-1} 9 \cdot 10^{k-j} \right) + 10 \\ &= \left(\sum_{m+1 \leq j \leq k-m-1} 9 \cdot 10^{k-j} \right) + \left(\sum_{k-m \leq j \leq k-1} 9 \cdot 10^{k-j} \right) + 10. \end{aligned}$$

Let $d_{k-m} = c_{k-m} - c_m - 1$ and $d_0 = 10 + c_0 - c_k$. Then

$$\begin{aligned}
(c_{k-m} - c_m)10^{k-m} + \sum_{k-m\le j\le k} (c_{k-j} - c_j)10^{k-j} \\
&= d_{k-m}10^{k-m} + 10^{k-m} + \sum_{k-m\le j\le k} (c_{k-j} - c_j)10^{k-j} \\
&= d_{k-m}10^{k-m} + \left(\sum_{m+1\le j\le k-m-1} 9 \cdot 10^{k-j} \right) \\
&\quad + \sum_{k-m\le j\le k-1} (9 + c_{k-j} - c_j)\, 10^{k-j} + d_0, \qquad (2.8)
\end{aligned}$$

where d_{k-m}, d_0, and the coefficients of 10^{k-j} in the above equation are nonnegative and are less than 10. Therefore (2.7) and (2.8) imply that we can write $g(x)$ in the decimal expansion as

$$g(x) = (d_k d_{k-1} \dots d_0)_{10} = \sum_{0\le j\le k} d_{k-j}10^{k-j},$$

where $0 \le d_i \le 9$ for all $i = 0, 1, 2, \dots, k$, and d_{k-j} satisfies the following relations:

$$d_{k-j} = c_{k-j} - c_j \quad \text{for} \quad 0 \le j < m, \tag{2.9}$$

$$d_{k-m} = c_{k-m} - c_m - 1, \tag{2.10}$$

$$d_{k-j} = 9 \quad \text{for} \quad m + 1 \le j \le k - m - 1, \tag{2.11}$$

$$d_{k-j} = 9 + c_{k-j} - c_j \quad \text{for} \quad k - m \le j \le k - 1, \tag{2.12}$$

$$d_0 = 10 + c_0 - c_k. \tag{2.13}$$

Since the decimal expansion of $g(x)$ has $k + 1$ digits, that of $f(g(x))$ has at most $k + 2$ digits. Then

$$F(x) = f(g(x)) = (d_k d_{k-1} \dots d_0)_{10} + (d_0 d_1 \dots d_k)_{10} = (e_{k+1} e_k \dots e_0)_{10},$$

where $0 \le e_i \le 9$ for all $i = 0, 1, \ldots, k+1$. Recall the fact from an elementary arithmetic that $e_0 = d_0 + d_k - 10\varepsilon_0$ where $\varepsilon_0 = 0$ if $d_0 + d_k < 10$, and $\varepsilon_0 = 1$ if $d_0 + d_k \ge 10$. In addition, $e_j = d_j + d_{k-j} + \varepsilon_{j-1} - 10\varepsilon_j$ for $1 \le j \le k$, where $\varepsilon_{j-1} = 0$ if there is no carry in the addition in the $(j-1)$th position and $\varepsilon_{j-1} = 1$ otherwise; while $\varepsilon_j = 0$ if $d_j + d_{k-j} + \varepsilon_{j-1} < 10$, and $\varepsilon_j = 1$ if $d_j + d_{k-j} + \varepsilon_{j-1} \ge 10$. Moreover, $e_{k+1} = 0$ if there is no carry in the addition in the kth position and $e_{k+1} = 1$ otherwise. We now calculate $e_0, e_1, \ldots, e_k, e_{k+1}$ by using this fact and the relations in (2.9) to (2.13). We obtain

$$e_0 = d_0 + d_k - 10\varepsilon_0 = (10 + c_0 - c_k) + (c_k - c_0) - 10\varepsilon_0 = 10 - 10\varepsilon_0,$$

which implies $\varepsilon_0 = 1$ and $e_0 = 0$. Then

$$e_1 = d_1 + d_{k-1} + 1 - 10\varepsilon_1 = (9 + c_1 - c_{k-1}) + (c_{k-1} - c_1) + 1 - 10\varepsilon_1 = 10 - 10\varepsilon_1,$$

which implies $\varepsilon_1 = 1$ and $e_1 = 0$. In general, we replace j by $k-j$ in (2.12) to see that $d_j = 9 + c_j - c_{k-j}$ for $1 \le j \le m$; and if $\varepsilon_{j-1} = 1$ and $2 \le j \le m-1$, then

$$e_j = d_j + d_{k-j} + 1 - 10\varepsilon_j = (9 + c_j - c_{k-j}) + (c_{k-j} - c_j) + 1 - 10\varepsilon_j = 10 - 10\varepsilon_j,$$

which implies $\varepsilon_j = 1$ and $e_j = 0$. Applying this observation for $j = 2, 3, \ldots, m-1$, respectively, we obtain

$$\varepsilon_2 = 1, e_2 = 0, \varepsilon_3 = 1, e_3 = 0, \ldots, \varepsilon_{m-1} = 1, e_{m-1} = 0.$$

Then

$$\begin{aligned} e_m &= d_m + d_{k-m} + 1 - 10\varepsilon_m \\ &= (9 + c_m - c_{k-m}) + (c_{k-m} - c_m - 1) + 1 - 10\varepsilon_m = 9 - 10\varepsilon_m, \end{aligned}$$

which implies $\varepsilon_m = 0$ and $e_m = 9$. Then $e_{m+1} = d_{m+1} + d_{k-m-1} - 10\varepsilon_{m+1} = 9 + 9 - 10\varepsilon_{m+1}$, which implies $\varepsilon_{m+1} = 1$ and $e_{m+1} = 8$. In general, we replace j by $k - j$ in (2.11) to obtain $d_j = 9$ for $m + 1 \le j \le k - m - 1$; and if $\varepsilon_{j-1} = 1$ and $m + 2 \le j \le k - m - 1$, then

$$e_j = d_j + d_{k-j} + \varepsilon_{j-1} - 10\varepsilon_j = 9 + 9 + 1 - 10\varepsilon_j = 19 - 10\varepsilon_j,$$

which implies $\varepsilon_j = 1$ and $e_j = 9$. Applying this observation for $j = m + 2$, $m + 3$, $\ldots$, $k - m - 1$, respectively, we obtain

$$\varepsilon_{m+2} = 1, e_{m+2} = 9, \varepsilon_{m+3} = 1, e_{m+3} = 9, \ldots, \varepsilon_{k-m-1} = 1, e_{k-m-1} = 9.$$

Then

$$\begin{aligned} e_{k-m} &= d_{k-m} + d_m + 1 - 10\varepsilon_{k-m} \\ &= (c_{k-m} - c_m - 1) + (9 + c_m - c_{k-m}) + 1 - 10\varepsilon_{k-m} = 9 - 10\varepsilon_{k-m}, \end{aligned}$$

which implies $\varepsilon_{k-m} = 0$ and $e_{k-m} = 9$. Then

$$\begin{aligned} e_{k-m+1} &= d_{k-m+1} + d_{m-1} - 10\varepsilon_{k-m+1} \\ &= (c_{k-m+1} - c_{m-1}) + (9 + c_{m-1} - c_{k-m+1}) - 10\varepsilon_{k-m+1} \\ &= 9 - 10\varepsilon_{k-m+1}, \end{aligned}$$

which implies $\varepsilon_{k-m+1} = 0$ and $e_{k-m+1} = 9$. In general, we replace j by $k - j$ in (2.12) to obtain $d_j = 9 + c_j - c_{k-j}$ for $1 \le j \le m$; and if $\varepsilon_{k-j-1} = 0$ and $1 \le j < m$, then

$$e_{k-j} = d_{k-j} + d_j - 10\varepsilon_{k-j} = (c_{k-j} - c_j) + (9 + c_j - c_{k-j}) - 10\varepsilon_{k-j} = 9 - 10\varepsilon_{k-j},$$

which implies $\varepsilon_{k-j} = 0$ and $e_{k-j} = 9$. Applying this observation for $j = m-2$, $m-3$, ..., 1, respectively, we obtain

$$\varepsilon_{k-m+2} = 0, e_{k-m+2} = 9, \varepsilon_{k-m+3} = 0, e_{k-m+3} = 9, \ldots, \varepsilon_{k-1} = 0, e_{k-1} = 9.$$

Then

$$e_k = d_k + d_0 - 10\varepsilon_k = (c_k - c_0) + (10 + c_0 - c_k) - 10\varepsilon_k = 10 - 10\varepsilon_k,$$

which implies $\varepsilon_k = 1$ and $e_k = 0$. Then $e_{k+1} = 1$. To conclude, we obtain that $e_j = 0$ for $0 \le j < m$, $e_m = 9$, $e_{m+1} = 8$, $e_j = 9$ for $m+2 \le j \le k-1$, $e_k = 0$, and $e_{k+1} = 1$. This completes the proof. $\square$

Chapter 3

Happy Functions and Digit Maps

In this chapter, we first show the calculation related to f_b in Definition 1.3 and $g_{e,b}$ in Definition 1.4. After that we consider a similar function and give some calculations in less details. Our results are as follows.

Lemma 3.1. *Let $b \geq 2$ be integer. Then there exists an integer $M = M_b \geq 1$ such that*

$$(k+1)(b-1)! < b^k \text{ for all } k \geq M.$$

In particular, if $b = 10$, then we can choose $M = 7$.

Proof. By using a usual method in calculus, one can show that $b^k/(k+1) \to +\infty$ as $k \to +\infty$. So there is an integer $M \geq 1$ such that if $k \geq M$, then $b^k/(k+1)$ is larger than $(b-1)!$. This proves the first part. For the second part, we prove by induction that

$$(k+1)9! < 10^k \text{ for all } k \geq 7. \tag{3.1}$$

It is easy to see that (3.1) holds when $k = 7$. Suppose that $k \geq 7$ and (3.1) holds for k. Then

$$(k+2)9! < (10k+10)9! = 10(k+1)9! < 10^{k+1}.$$

Therefore (3.1) is verified and the proof is complete. $\square$

Remark 3.2. *By a similar method as in the proof of Lemma 3.1 for* $2 \leq b \leq 9$*, we can take* M_b *as follows:* $M_2 = 2$*,* $M_3 = 2$*,* $M_4 = 3$*,* $M_5 = 3$*,* $M_6 = 4$*,* $M_7 = 5$*,* $M_8 = 5$*, and* $M_9 = 6$*.*

Theorem 3.3. *Let* b *and* M *be the integers as given in Lemma 3.1. Then*

$$f_b(x) < x \text{ for all } x \geq b^M. \tag{3.2}$$

In particular, $f_{10}(x) < x$ *for all* $x \geq 10^7$*.*

Proof. Let $x \geq b^M$. Then $x = (a_k a_{k-1} \dots a_0)_b$ where $k \geq M$, $a_k \neq 0$, and $0 \leq a_i \leq b - 1$ for all $i = 0, 1, \dots, k$. By Lemma 3.1, we obtain

$$f_b(x) = a_k! + a_{k-1}! + \cdots + a_0! \leq (k+1)(b-1)! < b^k \leq a_k b^k \leq x.$$

This proves (3.2). The second part follows from (3.2) and Lemma 3.1. $\square$

Remark 3.4. *By Remark 3.2 and Theorem 3.3, we see that*

$$\begin{aligned}
&f_2(x) < x \text{ for all } x \geq 2^2, && f_3(x) < x \text{ for all } x \geq 3^2,\\
&f_4(x) < x \text{ for all } x \geq 4^3, && f_5(x) < x \text{ for all } x \geq 5^3,\\
&f_6(x) < x \text{ for all } x \geq 6^4, && f_7(x) < x \text{ for all } x \geq 7^5,\\
&f_8(x) < x \text{ for all } x \geq 8^5, && \text{and } f_9(x) < x \text{ for all } x \geq 9^6.
\end{aligned}$$

To obtain a finite set $A \subseteq \mathbb{N}$ satisfying $f_b^{(n)}(x) \in A$, we now only need to recall Theorem 1.2 of Noppakaew, Phoopha, and Pongsriiam [14]. Consider the following two conditions for a function $f : \mathbb{N} \to \mathbb{N}$:

(A) There exists $N_f \in \mathbb{N}$ such that $f(x) < x$ for all $x \geq N_f$.

(B) For each $x \in \mathbb{N}$, the sequence $\left(f^{(n)}(x)\right)_{n\geq 1}$ converges to a fixed point or eventually enters into a cycle. In addition, the number of all such fixed points and cycles is finite.

Then we have the following results.

Theorem 3.5. (Noppakaew, Phoopha, and Pongsriiam [14]) *If $f : \mathbb{N} \to \mathbb{N}$ satisfies the condition* (A)*, then f satisfies the condition* (B)*.*

Theorem 3.6. *Let $b \geq 2$ be an integer. Then there exists a finite set $A = A_b \subseteq \mathbb{N}$ such that for every $x \in \mathbb{N}$, there is an integer $n \geq 1$ such that $f_b^{(n)}(x) \in A$. In particular, if $b = 10$, then we can take $A = \{1, 2, 145, 40585, 169, 871, 872\}$. In fact, 1, 2, 145, 40585 are the fixed points of f_b and 169, 871, 872 are the elements of distinct cycles arising from the iteration $f_b^{(n)}(x)$ for any n, $x \in \mathbb{N}$.*

Proof. By Theorems 3.3 and 3.5, we see that f_b satisfies the condition (B). Then we choose A_b to be the set of all elements in the cycles and fixed points of f_b, so that A_b is a finite subset of $\mathbb{N}$. Let $x \in \mathbb{N}$ be given. We know that $f_b : \mathbb{N} \to \mathbb{N}$, so if $f_b^{(n)}(x)$ converges to a fixed point $y \in \mathbb{N}$ as $n \to \infty$, then it means that there is $N \in \mathbb{N}$ such that $f_b^{(n)}(x) = y$ for all $n \geq N$. So in particular, $f_b^{(N)}(x) \in A_b$. Moreover, if $f_b^{(n)}(x)$ eventually enters into a cycle as $n \to \infty$, then $f_b^{(n)}(x) \in A_b$ for some n. This proves the first part. For the second part, let $b = 10$, and let F_{10} be the set of fixed points of f_{10} and C_{10} the set of all cycles (which are not fixed points) occurring in the iteration $f_{10}^{(n)}(x)$ for any n, $x \in \mathbb{N}$. We assert that

$$F_{10} = \{1, 2, 145, 40585\} \quad \text{and}$$

$$C_{10} = \{(169, 363601, 1454), (871, 45361), (872, 45362)\}.$$

It is easy to check that if $x \in \{1, 2, 145, 40585\}$, then $f_{10}(x) = x$. Suppose $x \in \mathbb{N}$ and $f_{10}(x) = x$. By Theorem 3.3, we obtain $x < 10^7$. So we only need to check the integers x in $[1, 10^7)$ whether or not they satisfy $f_{10}(x) = x$. After a computation in a computer, we find that $f_{10}(x) = x$ if and only if $x \in \{1, 2, 145, 40585\}$. This gives the set F_{10}. Similarly, to determine the set C_{10}, it is enough to look for the cycles occurring in the sequence $\left(f^{(n)}(x)\right)$ where x runs over the integers in $[1, 10^7)$. After a straightforward verification, we obtain C_{10} as asserted.

Therefore we can take A to be the set consisting of 1, 2, 145, 40585, 169, 363601, 1454, 871, 45361, 872, 45362. But 169, 363601, 1454 are in the same cycle, so we need only one of them. For instance, if $f_{10}^{(n)}(x) = 169$, then $f_{10}^{(n+1)}(x) = 363601$, $f_{10}^{(n+2)}(x) = 1454$, and $f_{10}^{(n+3)}(x) = 169$. Similarly, we can choose just one of 871, 45361 and one of 872, 45362. Therefore we can take A to be the set consisting of 1, 2, 145, 40585, 169, 871, 872 as required. This completes the proof. $\square$

Remark 3.7. *By a similar method as in Theorem 3.6, we obtain for $2 \leq b \leq 9$ the set F_b of fixed points of f_b and the set C_b of cycles in the iteration $f_b^{(n)}(x)$ for any n, $x \in \mathbb{N}$ as follows. For $b = 2$, we only need to run a computation in a computer for x in $[1, 2^2)$ to obtain that $F_2 = \{1, 2\}$ and $C_2 = \varnothing$. Similarly, for $b = 3$, 4, 5, 6, 7, 8, 9, we run a computation, respectively, for $x \in [1, 3^2)$, $x \in [1, 4^3)$, $x \in [1, 5^3)$,*

$x \in [1, 6^4)$, $x \in [1, 7^5)$, $x \in [1, 8^5)$, $x \in [1, 9^6)$ to obtain

$$F_3 = \{1, 2\}, C_3 = \varnothing,$$

$$F_4 = \{1, 2, 7\}, C_4 = \{(3, 6)\},$$

$$F_5 = \{1, 2, 49\}, C_5 = \varnothing,$$

$$F_6 = \{1, 2, 25, 26\}, C_6 = \varnothing,$$

$$F_7 = \{1, 2\}, C_7 = \{(38, 126, 27, 726, 243, 864)\},$$

$$F_8 = \{1, 2\}, C_8 = \{(3, 6, 720, 10), (125, 5161)\},$$

$$F_9 = \{1, 2, 41282\},$$

and C_9 consists of exactly one cycle, namely,

$$(1450, 80642, 251, 40327, 10803, 5173, 15121, 1445, 45481, 41094, 735, 723, 80646, 969, 41043).$$

The calculation for $g_{e,b}$ is similar to that for f_b, but the well known Euler constant will appear in the proof. So to avoid confusion, we will write E to denote Euler's constant, while e is reserved for the integers appearing in the definition of $g_{e,b}$.

Lemma 3.8. *We have $81(k+1)^2 < 10^k$ for all $k \geq 4$, $729(k+1)^3 < 10^k$ for all $k \geq 6$, $6561(k+1)^4 < 10^k$ for all $k \geq 8$, $59049(k+1)^5 < 10^k$ for all $k \geq 10$. In general, if $e \geq 2$ is an integer, then*

$$9^e(k+1)^e < 10^k \quad \text{for all } k \geq e^2. \tag{3.3}$$

Proof. The first four inequalities can be straightforwardly proved by induction, so we leave the details to the reader. For (3.3), let $e \geq 2$ be an integer. Observe that it can be proved by induction that $9(n^2+1) < 10^n$ for all $n \geq 2$, so in particular $9(e^2+1) < 10^e$. This implies that (3.3) holds when $k = e^2$. Next, suppose that $k \geq e^2$ and (3.3) holds for k. Recall that the sequence $(a_n) = \left(\left(1+\frac{1}{n}\right)^n\right)$ is strictly increasing and converges to E, the Euler constant. From this and the fact that $k \geq e^2$, we obtain

$$\begin{aligned}\frac{(k+2)^e}{(k+1)^e} &= \left(1+\frac{1}{k+1}\right)^e \leq \left(1+\frac{1}{e^2+1}\right)^e < \left(1+\frac{1}{e^2+1}\right)^{e^2+1}\\ &= a_{e^2+1} \leq \sup\{a_n \mid n \in \mathbb{N}\} = \lim_{n\to\infty} a_n = E < 10.\end{aligned}$$

Then $9^e(k+2)^e < 9^e(10)(k+1)^e < 10^{k+1}$, by the induction hypothesis. So the proof is complete. $\square$

Lemma 3.8 will be used in the calculation in some examples. For a general result, we have the following theorem.

Theorem 3.9. *Let $e, b \geq 2$ be integers. Then the following statements hold.*

(i) *There exists an integer $M = M_{e,b} \geq 1$ such that $(k+1)^e(b-1)^e < b^k$ for all $k \geq M$.*

(ii) *$g_{e,b}(x) < x$ for all $x \geq b^M$.*

(iii) *$g_{e,b}$ satisfies the condition* (B) *and there exists a finite set $A = A_{e,b} \subseteq \mathbb{N}$ such that for every $x \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $g_{e,b}^{(n)}(x) \in A$.*

(iv) *Let $F_{e,b}$ and $C_{e,b}$ be the sets of fixed points of $g_{e,b}$ and the cycles arising in* *the sequence* $\left(g_{e,b}^{(n)}(x)\right)_{n\geq 1}$ *for any* $x \in \mathbb{N}$*. Then we have*

$F_{2,10} = \{1, 81\}, C_{2,10} = \{(169, 256)\}$,

$F_{3,10} = \{1, 512, 4913, 5832, 17576, 19683\}, C_{3,10} = \{(6859, 21952)\}$,

$F_{4,10} = \{1, 2401, 234256, 390625, 614656, 1679616\}, C_{4,10} = \{(104976, 531441)\}$,

$F_{5,10} = \{1, 17210368, 52521875, 60466176, 205962976\}$,

and $C_{5,10}$ *consists of the following cycles:*

$$(16807, 5153632, 9765625, 102400000), (6436343, 20511149), (28629151, 45435424).$$

Proof. Since e, b are already given, we obtain $b^k/(k+1)^e \to +\infty$ as $k \to \infty$, and so there exists $M \geq 1$ such that $b^k/(k+1)^e > (b-1)^e$ for all $k \geq M$. This proves (i). Suppose $x \geq b^M$. Then $x = (a_k a_{k-1} \ldots a_0)_b$ where $k \geq M$, $a_k \neq 0$, and $0 \leq a_i \leq b-1$ for all $i = 0, 1, \ldots, k$. Then by (i), we obtain

$$g_{e,b}(x) = (a_k + a_{k-1} + \cdots + a_0)^e \leq ((k+1)(b-1))^e < b^k \leq a_k b^k \leq x.$$

This proves (ii). Then (iii) follows from (ii), Theorem 3.5, and exactly the same argument as in Theorem 3.6. For (iv), to determine the set $F_{e,b}$ and $C_{e,b}$ for a particular pair of (e, b), we only need to apply Lemma 3.8 and run a computation on the integers in $[1, b^M)$ as in the proof of Theorem 3.6. If $e = 2$ and $b = 10$, we can take $M_{e,b} = 4$. After checking $\left(g_{e,b}^{(n)}(x)\right)_{n\geq 1}$ for x in the interval $[1, 10^4)$, we obtain $F_{2,10} = \{1, 81\}$, $C_{2,10} = \{(169, 256)\}$. If $e = 3$ and $b = 10$, we can take $M_{e,b} = 6$. Then running a computation for $g_{e,b}^{(n)}(x)$ where $n \in \mathbb{N}$ and $x \in [1, 10^6)$, we obtain

$$F_{3,10} = \{1, 512, 4913, 5832, 17576, 19683\} \quad \text{and} \quad C_{3,10} = \{(6859, 21952)\}.$$

Similarly, if $(e,b)=(4,10)$, then we take $M_{e,b}=8$; if $(e,b)=(5,10)$, then we take $M_{e,b}=10$. After running a computation in a computer, we obtain $F_{4,10}$, $C_{4,10}$, $F_{5,10}$, and $C_{5,10}$ as given above. So the proof is complete. $\square$

Observing that $3435=3^3+4^4+3^3+5^5$, we are interested in determining all numbers with this property. So we should consider $h(x)=a_k^{a_k}+a_{k-1}^{a_{k-1}}+\cdots+a_0^{a_0}$ if $x=(a_ka_{k-1}\ldots a_0)_{10}$ but there is a problem with this definition since 0^0 is not defined. One way to avoid this is to skip the zero digit and define $h(x)=b_m^{b_m}+b_{m-1}^{b_{m-1}}+\cdots+b_0^{b_0}$ if $x=(a_ka_{k-1}\ldots a_0)_{10}$ and b_m, b_{m-1}, …, b_0 are taken from a_k, a_{k-1}, …, a_0 but without zero. Equivalently, we can temporarily assign the value $0^0=0$ and study the following function.

Definition 3.10. Let $h:\mathbb{N}\cup\{0\}\to\mathbb{N}\cup\{0\}$ be defined by $h(0)=0$, $h(a)=a^a$ if $a\in\{1,2,\ldots,9\}$, and

$$h(x)=h(a_k)+h(a_{k-1})+\cdots+h(a_0)$$

if $x\geq 10$ and $x=(a_ka_{k-1}\ldots a_0)_{10}$ is the decimal representation of x with $a_k\neq 0$. Equivalently, we can assign $0^0=0$ and define h by

$$h(x)=a_k^{a_k}+a_{k-1}^{a_{k-1}}+\cdots+a_0^{a_0}$$

for each $x=(a_ka_{k-1}\ldots a_0)_{10}$.

The calculation for h can be done in the same way as that for f_b and $g_{e,b}$, so we skip the details and leave them to the reader. We have the following result.

Theorem 3.11. *The following statements hold.*

(i) $(k+1)\,9^9 < 10^k$ *for all* $k \geq 10$.

(ii) $h(x) < x$ *for all* $x \geq 10^{10}$.

(iii) h *satisfies the condition* (B) *and there exists a finite set* $A \subseteq \mathbb{N}$ *such that for every* $x \in \mathbb{N}$, *there is* $n \in \mathbb{N}$ *such that* $h^{(n)}(x) \in A$.

(iv) *The set of fixed points of* h *is* $\{1, 3435, 438579088\}$.

Proof. The statement (i) can be proved by induction. If $x \geq 10^{10}$, then we write $x = (a_k a_{k-1} \ldots a_0)_{10}$ with $k \geq 10$ and $a_k \neq 0$, and so

$$h(x) \leq 9^9(k+1) < 10^k \leq a_k 10^k \leq x.$$

Then (iii) follows from (ii), Theorem 3.5, and exactly the same argument as before. Then running a computation in a computer, we obtain (iv). $\square$

Chapter 4

Diophantine Equations and Proofs of Some Mathematical Memes

Many people have seen some fun fact in mathematics from memes which are distributed via social media worldwide. Memes can be discovered by anyone and can definitely be appreciated without proofs or explanations. Nevertheless, we show that our results can be interpreted as solutions to certain Diophantine equations and use them to explain or create some memes. For example, the only fixed points of f_{10} are 1, 2, 145, and 40585, and so the solutions in nonnegative integers a_k, a_{k-1}, ..., a_0 with $a_k \neq 0$ to the Diophantine equation

$$a_k! + a_{k-1}! + \cdots + a_0! = (a_k a_{k-1} \ldots a_0)_{10}$$

are given by the numbers 1, 2, 145, and 40585.

Corollary 4.1. $1 = 1!$, $2 = 2!$, $145 = 1! + 4! + 5!$, $40585 = 4! + 0! + 5! + 8! + 5!$, *and these are the only positive integers with this property. That is, a positive integer x is the sum of the factorials of all its decimal digits (except the leading zeros) if and only if $x = 1$, 2, 145, or* 40585.

Proof. Let $f_{10}(x)$ be the function in Theorem 3.3. We would like to find all $x \in \mathbb{N}$ such that $f_{10}(x) = x$. By Theorem 3.3, $f_{10}(x) < x$ for all $x \geq 10^7$. So we only need to find $x < 10^7$ such that $f_{10}(x) = x$, which can be done using a computer. $\square$

Corollary 4.2. $1 = (1)_9 = 1!$, $2 = (2)_9 = 2!$, $41282 = (62558)_9 = 6!+2!+5!+5!+8!$ *and these are the only positive integers with this property. That is, if* $x \in \mathbb{N}$, *then* x *is the sum of factorials of its digits (in base* 9*) if and only if* $x = (1)_9$, $(2)_9$, $(62558)_9$.

Proof. This follows immediately from Remark 3.7. $\square$

Corollary 4.3. *We have*

$$1 = 1^3, 512 = (5+1+2)^3, 4913 = (4+9+1+3)^3,$$

$$5832 = (5+8+3+2)^3, 17576 = (1+7+5+7+6)^3, 19683 = (1+9+6+8+3)^3,$$

and these are the only positive integers with this property. That is, if $x \in \mathbb{N}$, *then* x *is the cubes of the sum of its decimal digits if and only if* $x = 1$, 512, 4913, 5832, 17576, *or* 19683. *Similarly,*

$$1 = 1^4, 2401 = (2+4+0+1)^4,$$

$$234256 = (2+3+4+2+5+6)^4, 390625 = (3+9+0+6+2+5)^4,$$

$$614656 = (6+1+4+6+5+6)^4, 1679616 = (1+6+7+9+6+1+6)^4$$

are the only positive integers that are equal to the 4th power of the sum of their *decimal digits;*

$$1 = 1^5, 17210368 = (1+7+2+1+0+3+6+8)^5,$$

$$52521875 = (5+2+5+2+1+8+7+5)^5,$$

$$60466176 = (6+0+4+6+6+1+7+6)^5,$$

$$205962976 = (2+0+5+9+6+2+9+7+6)^5$$

are the only positive integers that are equal to the 5th power of the sum of their decimal digits.

Proof. This follows immediately from Theorem 3.9. □

Corollary 4.4. $1 = 1^1$, $3435 = 3^3+4^4+3^3+5^5$, $438579088 = 4^4+3^3+8^8+5^5+7^7+9^9+8^8+8^8$, *and these are the only positive integers with this property.*

Proof. This follows immediately from Theorem 3.11. □

Other known results in the literature can be used to produce fun fact or memes too. Here we rewrite the results of Grundman and Teeple [10], and Hargreaves and Siksek [12].

Corollary 4.5. (Grundman and Teeple [10], and Hargreaves and Siksek [12]) *We have*

$$1 = 1^3, 153 = 1^3+5^3+3^3, 370 = 3^3+7^3+0^3, 371 = 3^3+7^3+1^3, 407 = 4^3+0^3+7^3,$$

and these are the only positive integers with this property. That is, if $x \in \mathbb{N}$, then x is the sum of the cubes of its decimal digits if and only if $x = 1$, 153, 370, 371,

407. *Similarly,*

$$1 = 1^4, 1634 = 1^4 + 6^4 + 3^4 + 4^4, 8208 = 8^4 + 2^4 + 0^4 + 8^4, 9474 = 9^4 + 4^4 + 7^4 + 4^4,$$

are the only positive integers that are equal to the sum of the 4th powers of their decimal digits. In addition,

$$1 = 1^5, 4150 = 4^5 + 1^5 + 5^5 + 0^5, 4151 = 4^5 + 1^5 + 5^5 + 1^5,$$
$$54748 = 5^5 + 4^5 + 7^5 + 4^5 + 8^5, 92727 = 9^5 + 2^5 + 7^5 + 2^5 + 7^5,$$
$$93084 = 9^5 + 3^5 + 0^5 + 8^5 + 4^5, 194979 = 1^5 + 9^5 + 4^5 + 9^5 + 7^5 + 9^5$$

are the only positive integers that are equal to the sum of the 5th powers of their decimal digits.

Chapter 5

Computer Code for the Using MATLAB

In this chapter, we will explain the MATLAB code that we use various theorems.

For the first below code, we use it to calculate the general result of Definition 1.2.

```
clear all; % clears variables. It also clears a lot of
    other things from memory, such as breakpoints,
    persistent variables, and cached memory.
clc; % clears the command window
k=4; % digits numbers
fprintf('===== x is %d digit numbers =====\n', k+1);
cycle=[];
for x=10^k:10^(k+1)-1
    % split a number into its individual parts
    newx=rem(floor(x./(10.^(floor(log10(x)):-1:0))),10);
    % sort new number to descending and ascending
    desc=sort(newx,'descend');
    asc=sort(newx);
    % sum the individual digits
```

```
gx1=0; gx2=0;
for i=1:length(newx)
    gx1 = gx1 + desc(i)*10^(length(newx)-i);
    gx2 = gx2 + asc(i)*10^(length(newx)-i);
end
gx = gx1 - gx2;
% adding 0 when digits number is not equal to k.
if (gx < 10^k)
    gx = gx*10;
end
% split a number into its individual parts
y=rem(floor(gx./(10.^(floor(log10(gx)):-1:0))),10);
% sum the individual digits
fx1=0; fx2=0;
for i=1:length(y)
    fx1 = fx1 + y(i)*10^(length(y)-i);
    fx2 = fx2 + y(i)*10^(i-1);
end
fx = fx1 + fx2;
% printed any numbers yet?
if length(find(cycle==fx))==0
    fprintf('%d\n', fx);
```

```
            cycle=[cycle , fx ];
        end
end
```

Next, let $b = 10$ and we know that $M_{10} = 7$ in Lemma 3.1, so we use the code below to find the set F_{10} of fixed points of f_{10} and the set C_{10} of cycles in the iteration $f_{10}^{(n)}(x)$ for any $n \in \mathbb{N}$, $1 \leq x < 10^7$ in Theorem 3.6 as follows.

```
clear all; clc;
base=10; m=7; % m in Lemma 3.1
cycle=[];
for x=1:base^m - 1
    recal=1; kernel=[];
    while (recal>0)
        j=0; newx=x; number=0;
        % split number into its individual parts and sum
            factorials of all its digits
        while (j>=0)
            number = number + factorial(mod(newx,base));
            newx = floor(newx/base);
            if (newx == 0)
                j=-1;
            elseif (newx < base)
                number = number + factorial(newx);
```

```
        j=-1;
    end
end
% break for fixed point
if x == number
    if length(find(cycle==number))==0
        fprintf('%d\n',number);
        cycle=[cycle, number];
    end
    recal=-1;
% break or reloop for cycle
else
    if length(find(kernel==number))==0
        kernel=[kernel, number];
        x=number;
    else
        inxcy = find(kernel==number);
        if length(find(cycle==number))==0
            fprintf('(');
            for idx=inxcy:length(kernel)
                fprintf('%d, ',kernel(idx));
                cycle=[cycle, kernel(idx)];
```

```
                    end
                    fprintf(')\n');
                end
                recal=-1;
            end
        end
    end
end
```

In the same way, to calculate the results of Theorem 3.9, we just change the conditions of the calculation function but for finding fixed points and cycles are written the same.

```
clear all; clc;
base=10; m=6; % m in Theorem 3.9(i)
e=3; % e th power
cycle=[];
for x=1:base^m - 1
    recal=1; kernel=[];
    while (recal>0)
        % split number into its individual parts and power of sums in given base
        arrnumber = rem(floor(x./(base.^(floor(log(x)/log(base)):-1:0))),base);
```

```
        number = sum(arrnumber)^e;
        % break for fixed point
        if x == number
            if length(find(cycle==number))==0
                fprintf('%d\n',number);
                cycle=[cycle, number];
            end
            recal=-1;
        % break or reloop for cycle
        else
            if length(find(kernel==number))==0
                kernel=[kernel, number];
                x=number;
            else
                inxcy = find(kernel==number);
                if length(find(cycle==number))==0
                    fprintf('(');
                    for idx=inxcy:length(kernel)
                        fprintf('%d, ',kernel(idx));
                        cycle=[cycle, kernel(idx)];
                    end
                    fprintf(')\n');
```

```
                    end
                    recal=-1;
               end
          end
     end
end
```

Finally, we use the last code to calculate the last theorem, in which the principle of coding remains the same as the previous theorem.

```
clear all; clc;
base=10; m=10; % m in Theorem 3.11(i)
cycle=[];
for x=1:base^m - 1
     j=0; newx=x; number=0;
     % split number into its individual parts and sum of itself power
     while (j>=0)
          digit = mod(newx,base);
          if (digit ~= 0)
               number = number + digit^digit;
          end
          newx = floor(newx/base);
          if (newx == 0)
```

```
            j=-1;
        elseif (newx < base)
            number = number + newx^newx;
            j=-1;
        end
    end
    % break for fixed point
    if x == number
        if length(find(cycle==number))==0
            fprintf('%d\n',number);
            cycle=[cycle, number];
        end
    end
end
```

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Appendix

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Notes on 1089 and a Variation of the Kaprekar Operator

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Abstract

We study a variation of the Kaprekar operator $F(x)$ for all nonnegative integers x and show that the range of F consists of 0, 99, 1089, and the integers of the form $1099\ldots98900\ldots0$, where $99\ldots9$ and $00\ldots0$ may be long, short, or disappear.

1 Introduction and Statement of the Main Result

Throughout this article, if $y \in \mathbb{R}$, then $\lfloor y \rfloor$ is the largest integer less than or equal to y and $\lceil y \rceil$ is the smallest integer larger than or equal to y. Unless stated otherwise, all other variables are nonnegative integers. For any $x \in \mathbb{N} \cup \{0\}$, we write the decimal expansion of x as

$$x = (a_k a_{k-1} \ldots a_1 a_0)_{10} = \sum_{0 \le j \le k} a_{k-j} 10^{k-j},$$

where $0 \le a_i \le 9$ for all $i = 0, 1, 2, \ldots, k$.

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The Kaprekar operator K is defined by the following operation: take any positive integer x having four decimal digits which are not all equal and the leading digit is not zero, say $x = (a_3a_2a_1a_0)_{10}$, $a_3 \neq 0$, and $a_i \neq a_j$ for some i, j, then rearrange a_3, a_2, a_1, a_0 as c_3, c_2, c_1, c_0 so that $c_3 \geq c_2 \geq c_1 \geq c_0$. Then

$$K(x) = (c_3c_2c_1c_0)_{10} - (c_0c_1c_2c_3)_{10}. \tag{1.1}$$

Observe that the second number on the right-hand side of (1.1) is obtained by reversing the decimal digits of the first. It is well known that no matter what x we start with, after repeating this process at most 7 steps, we always obtain the number 6174. For example, suppose $x = 1000$. Then

$$\begin{aligned}
&K(x) = 1000 - 1 = 999,\\
&K^2(x) = K(K(x)) = K(999) = K(0999) = 9990 - 0999 = 8991,\\
&K^3(x) = K(8991) = 9981 - 1899 = 8082,\\
&K^4(x) = 8820 - 0288 = 8532,\\
&K^5(x) = 8532 - 2358 = 6174,
\end{aligned}$$

and $K^m(x) = 6174$ for all $m \geq 6$. Here, it is important to keep in mind that the Kaprekar operator operates on the positive integers having four digits not all equal. So the decimal representation of $K(x)$ with nonzero leading digit may have only 3 digits but, to calculate $K(K(x))$, we must first write $K(x)$ as 4 digits number by adding 0 as the leading digit, as shown above in $K(999) = K(0999)$. We can generalize K to operate on any nonnegative integers as follows:

Definition 1.1 (Kaprekar operator on nonnegative integers). *Let* $g : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ *be given by* $g(0) = .0$ *If* $x = (a_ka_{k-1}\dots a_0)_{10}$, $a_k \neq 0$, *and* c_k, c_{k-1}, $\dots$, c_0 *is the permutation of* a_k, a_{k-1}, $\dots$, a_0 *such that* $c_k \geq c_{k-1} \geq \dots \geq c_0$, *then*

$$g(x) = (c_kc_{k-1}\dots c_1c_0)_{10} - (c_0c_1\dots c_{k-1}c_k)_{10}.$$

In addition, for the purpose of this article, if x *is as above, then we always write the decimal representation of* $g(x)$ *as* $k+1$ *digits number, say* $g(x) = (b_kb_{k-1}\dots b_0)_{10}$.

Another trick is as follows: take any positive integer having three digits, say $x = (a_2a_1a_0)_{10}$, where $a_2 \neq 0$, $0 \leq a_j \leq 9$ for all j, and $a_i \neq a_j$ for some i, j. Then calculate $g(x)$, say $g(x) = b = (b_2b_1b_0)_{10}$. Then compute $f(b) = b + \text{reverse}(b) = (b_2b_1b_0)_{10} + (b_0b_1b_2)_{10}$. No matter what x we start with, we always obtain $f(b) = 1089$. We generalize this to the following operator:

Definition 1.2. *Let f be the reverse and add an operator. Let $F : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ be defined by $F = f \circ g$. In addition, to calculate $F(x) = f(g(x))$, we always keep the same convention in Definition 1.1, where the number of decimal digits of x and $g(x)$ are equal.*

For example, suppose $x = 100$. Then $g(x) = 99 = 099$ and so $F(x) = f(099) = 990 + 099 = 1089$. By using a computer or a straightforward calculation, it is not difficult to notice the following pattern:

if $10 \leq x < 10^2$, then $F(x) = 0$ or 99;
if $10^2 \leq x < 10^3$, then $F(x) = 0$ or 1089;
if $10^3 \leq x < 10^4$, then $F(x) = 0$, 10890, or 10989;
if $10^4 \leq x < 10^5$, then $F(x) = 0$, 109890, or 109989.

In general, we have the following result.

Theorem 1.3. *Let $F = f \circ g$, $k \geq 2$, and $10^k \leq x < 10^{k+1}$. Let $x = (a_k a_{k-1} \ldots a_0)_{10}$, $a_k \neq 0$, and $0 \leq a_i \leq 9$ for all $i = 0, 1, \ldots, k$. If $k = 2$, then $F(x) = 0$ or 1089. Suppose that $k \geq 3$ and $c_k, c_{k-1}, \ldots, c_0$ is the permutation of $a_k, a_{k-1}, \ldots, a_0$ such that $c_k \geq c_{k-1} \geq \cdots \geq c_0$. Let $m = z(x)$ be the largest element of the set $\{j \in \{0, 1, \ldots, k\} \mid c_{k-j} > c_j\}$. If $a_i = a_j$ for all i, j, then $F(x) = 0$. If $a_i \neq a_j$ for some i, j, then*

$$F(x) = 10\underbrace{99\ldots9}_{y(x)}89\underbrace{00\ldots0}_{z(x)},$$

where $y(x) = k - 2 - z(x)$.

Although the result is easy to observe for $k = 2, 3, 4$, it is more difficult when k is large. As far as we know, there is no proof for a general k. We hope that this article will help explain something related to 6174, 1089, and other similar magic numbers. Finally, it is an interesting open problem to determine whether or not a given number in the range of F is a Lychrel number. We leave this problem for the interested reader. For more information on 6174 and the Kaprekar operator, see for instance in [5], [6], and [7]. For related articles on 1089 and 2178, see for example [1], [2], [3], [4], [8], [9], and [10].

2 Proof of the Main Result

Proof. We first consider the case $k = 2$. Since $10^2 \le x < 10^3$, it can be written in the decimal representation as $x = (a_2a_1a_0)_{10}$, where $a_2 \neq 0$ and $0 \le a_i \le 9$ for $i = 0, 1, 2$. If $a_2 = a_1 = a_0$, then $F(x) = 0$. So suppose that a_2, a_1, a_0 are not all the same and let c_2, c_1, c_0 be the permutation of a_2, a_1, a_0 such that $c_2 \ge c_1 \ge c_0$. Then $c_2 > c_0$ and

$$\begin{aligned} g(x) &= (c_2c_1c_0)_{10} - (c_0c_1c_2)_{10} \\ &= (10^2c_2 + 10c_1 + c_0) - (10^2c_0 + 10c_1 + c_2) \\ &= 10^2(c_2 - c_0 - 1) + 10(9) + 10 - (c_2 - c_0) \\ &= (d_2d_1d_0)_{10}, \end{aligned}$$

where $d_2 = c_2 - c_0 - 1$, $d_1 = 9$, and $d_0 = 10 - (c_2 - c_0)$. Then it is easy to see that

$$F(x) = (d_2d_1d_0)_{10} + (d_0d_1d_2)_{10} = 1089.$$

Next, let $k \ge 3$, $10^k \le x < 10^k$, and write $x = (a_ka_{k-1}\dots a_0)_{10}$, where $a_k \neq 0$ and $0 \le a_i \le 9$ for all $i = 0, 1, \dots, k$. If $a_i = a_j$ for all i, j, then $F(x) = 0$ and we are done. So suppose that $a_i \neq a_j$ for some i, j. Let $c_k, c_{k-1}, \dots, c_0$ be the permutation of $a_k, a_{k-1}, \dots, a_0$ such that $c_k \ge c_{k-1} \ge \dots \ge c_0$. Then

$$\begin{aligned} g(x) &= (c_kc_{k-1}\dots c_0) - (c_0c_1\dots c_k)_{10} \\ &= \sum_{j=0}^{k} c_{k-j}10^{k-j} - \sum_{j=0}^{k} c_j10^{k-j} \\ &= \sum_{j=0}^{k} (c_{k-j} - c_j)10^{k-j}. \end{aligned} \tag{2.2}$$

Let $A = \{j \in \{0, 1, \dots, k\} \mid c_{k-j} > c_j\}$. Since $c_k > c_0$, we see that $0 \in A$, and so $A \neq \varnothing$. Let m be the largest element of A. If $m \ge \lceil \frac{k}{2} \rceil$, then $k - m \le k - \lceil \frac{k}{2} \rceil = \lfloor \frac{k}{2} \rfloor \le m$, which implies $c_{k-m} \le c_m$ which contradicts the fact that $m \in A$. Therefore, $0 \le m < \lceil \frac{k}{2} \rceil$. Since m is the largest element of

A and $c_k \geq c_{k-1} \geq \cdots \geq c_0$, we assert that the following relations hold:

$$c_{k-j} > c_j \quad \text{for} \quad 0 \leq j \leq m, \tag{2.3}$$

$$c_{k-j} \leq c_j \quad \text{for} \quad j > m, \tag{2.4}$$

$$c_{k-j} = c_j \quad \text{for} \quad m < j \leq \left\lfloor \frac{k}{2} \right\rfloor, \tag{2.5}$$

$$c_{k-j} = c_j \quad \text{for} \quad \left\lceil \frac{k}{2} \right\rceil \leq j < k - m, \tag{2.6}$$

$$c_{k-j} < c_j \quad \text{for} \quad k - m \leq j \leq k. \tag{2.7}$$

For (2.3), we know that $c_{k-m} > c_m$ and if $0 \leq j < m$, then $c_{k-j} \geq c_{k-m} > c_m \geq c_j$. So (2.3) is verified. By the choice of m, (2.4) follows immediately. If $j \leq \lfloor \frac{k}{2} \rfloor$, then $k - j \geq k - \lfloor \frac{k}{2} \rfloor = \lceil \frac{k}{2} \rceil \geq j$, and so $c_{k-j} \geq c_j$. This and (2.4) imply (2.5). Replacing j by $k - j$ in (2.5), we obtain (2.6). Changing j to $k - j$ in (2.3), we obtain (2.7).

Next, we divide the sum in (2.2) into 3 parts: $0 \leq j \leq m$, $m < j < k-m$, and $k - m \leq j \leq k$. By (2.5) and (2.6), the second part is zero. Therefore, (2.2) becomes

$$g(x) = \sum_{0 \leq j \leq m} (c_{k-j} - c_j) 10^{k-j} + \sum_{k-m \leq j \leq k} (c_{k-j} - c_j) 10^{k-j}. \tag{2.8}$$

The terms $c_{k-j} - c_j$ in (2.8) are positive in the first sum and negative in the second. Then we write

$$\begin{aligned} 10^{k-m} &= \left(\sum_{m+1 \leq j \leq k-1} 9 \cdot 10^{k-j} \right) + 10 \\ &= \left(\sum_{m+1 \leq j \leq k-m-1} 9 \cdot 10^{k-j} \right) + \left(\sum_{k-m \leq j \leq k-1} 9 \cdot 10^{k-j} \right) + 10. \end{aligned}$$

Let $d_{k-m} = c_{k-m} - c_m - 1$ and $d_0 = 10 + c_0 - c_k$. Then

$$\begin{aligned} &(c_{k-m} - c_m) 10^{k-m} + \sum_{k-m \leq j \leq k} (c_{k-j} - c_j) 10^{k-j} \\ &\qquad = d_{k-m} 10^{k-m} + 10^{k-m} + \sum_{k-m \leq j \leq k} (c_{k-j} - c_j) 10^{k-j} \\ &\qquad = d_{k-m} 10^{k-m} + \left(\sum_{m+1 \leq j \leq k-m-1} 9 \cdot 10^{k-j} \right) \\ &\qquad\quad + \sum_{k-m \leq j \leq k-1} (9 + c_{k-j} - c_j)\, 10^{k-j} + d_0, \end{aligned} \tag{2.9}$$

where d_{k-m}, d_0, and the coefficients of 10^{k-j} in the above equation are nonnegative and are less than 10. Therefore, (2.8) and (2.9) imply that we can write $g(x)$ in the decimal expansion as:

$$g(x) = (d_k d_{k-1} \ldots d_0)_{10} = \sum_{0 \le j \le k} d_{k-j} 10^{k-j},$$

where $0 \le d_i \le 9$ for all $i = 0, 1, 2, \ldots, k$, and d_{k-j} satisfies the following relations:

$$d_{k-j} = c_{k-j} - c_j \quad \text{for} \quad 0 \le j < m, \tag{2.10}$$
$$d_{k-m} = c_{k-m} - c_m - 1, \tag{2.11}$$
$$d_{k-j} = 9 \quad \text{for} \quad m+1 \le j \le k-m-1, \tag{2.12}$$
$$d_{k-j} = 9 + c_{k-j} - c_j \quad \text{for} \quad k-m \le j \le k-1, \tag{2.13}$$
$$d_0 = 10 + c_0 - c_k. \tag{2.14}$$

Since the decimal expansion of $g(x)$ has $k+1$ digits, that of $f(g(x))$ has at most $k+2$ digits. Then

$$F(x) = f(g(x)) = (d_k d_{k-1} \ldots d_0)_{10} + (d_0 d_1 \ldots d_k)_{10} = (e_{k+1} e_k \ldots e_0)_{10},$$

where $0 \le e_i \le 9$ for all $i = 0, 1, \ldots, k+1$. From elementary arithmetic, recall the fact that $e_0 = d_0 + d_k - 10\varepsilon_0$, where $\varepsilon_0 = 0$ if $d_0 + d_k < 10$, and $\varepsilon_0 = 1$ if $d_0 + d_k \ge 10$. In addition, $e_j = d_j + d_{k-j} + \varepsilon_{j-1} - 10\varepsilon_j$ for $1 \le j \le k$, where $\varepsilon_{j-1} = 0$ if there is no carry in the addition in the $(j-1)$th position and $\varepsilon_{j-1} = 1$ otherwise; while $\varepsilon_j = 0$ if $d_j + d_{k-j} + \varepsilon_{j-1} < 10$, and $\varepsilon_j = 1$ if $d_j + d_{k-j} + \varepsilon_{j-1} \ge 10$. Moreover, $e_{k+1} = 0$ if there is no carry in the addition in the kth position and $e_{k+1} = 1$ otherwise. We now calculate $e_0, e_1, \ldots, e_k, e_{k+1}$ by using this fact and the relations in (2.10) to (2.14). We obtain

$$e_0 = d_0 + d_k - 10\varepsilon_0 = (10 + c_0 - c_k) + (c_k - c_0) - 10\varepsilon_0 = 10 - 10\varepsilon_0,$$

which implies $\varepsilon_0 = 1$ and $e_0 = 0$. Then

$$e_1 = d_1 + d_{k-1} + 1 - 10\varepsilon_1 = (9 + c_1 - c_{k-1}) + (c_{k-1} - c_1) + 1 - 10\varepsilon_1 = 10 - 10\varepsilon_1,$$

which implies $\varepsilon_1 = 1$ and $e_1 = 0$. In general, we replace j by $k-j$ in (2.13) to get $d_j = 9 + c_j - c_{k-j}$ for $1 \le j \le m$; and if $\varepsilon_{j-1} = 1$ and $2 \le j \le m-1$, then

$$e_j = d_j + d_{k-j} + 1 - 10\varepsilon_j = (9 + c_j - c_{k-j}) + (c_{k-j} - c_j) + 1 - 10\varepsilon_j = 10 - 10\varepsilon_j,$$

which implies $\varepsilon_j = 1$ and $e_j = 0$. Applying this observation for $j = 2, 3, \ldots, m-1$, respectively, we obtain

$$\varepsilon_2 = 1, e_2 = 0, \varepsilon_3 = 1, e_3 = 0, \ldots, \varepsilon_{m-1} = 1, e_{m-1} = 0.$$

Then

$$\begin{aligned} e_m &= d_m + d_{k-m} + 1 - 10\varepsilon_m \\ &= (9 + c_m - c_{k-m}) + (c_{k-m} - c_m - 1) + 1 - 10\varepsilon_m = 9 - 10\varepsilon_m, \end{aligned}$$

which implies $\varepsilon_m = 0$ and $e_m = 9$. Then $e_{m+1} = d_{m+1} + d_{k-m-1} - 10\varepsilon_{m+1} = 9 + 9 - 10\varepsilon_{m+1}$, which implies $\varepsilon_{m+1} = 1$ and $e_{m+1} = 8$. In general, we replace j by $k - j$ in (2.12) to obtain $d_j = 9$ for $m + 1 \leq j \leq k - m - 1$; and if $\varepsilon_{j-1} = 1$ and $m + 2 \leq j \leq k - m - 1$, then

$$e_j = d_j + d_{k-j} + \varepsilon_{j-1} - 10\varepsilon_j = 9 + 9 + 1 - 10\varepsilon_j = 19 - 10\varepsilon_j,$$

which implies $\varepsilon_j = 1$ and $e_j = 9$. Applying this observation for $j = m + 2, m + 3, \ldots, k - m - 1$, respectively, we obtain

$$\varepsilon_{m+2} = 1, e_{m+2} = 9, \varepsilon_{m+3} = 1, e_{m+3} = 9, \ldots, \varepsilon_{k-m-1} = 1, e_{k-m-1} = 9.$$

Then

$$\begin{aligned} e_{k-m} &= d_{k-m} + d_m + 1 - 10\varepsilon_{k-m} \\ &= (c_{k-m} - c_m - 1) + (9 + c_m - c_{k-m}) + 1 - 10\varepsilon_{k-m} = 9 - 10\varepsilon_{k-m}, \end{aligned}$$

which implies $\varepsilon_{k-m} = 0$ and $e_{k-m} = 9$. Then

$$\begin{aligned} e_{k-m+1} &= d_{k-m+1} + d_{m-1} - 10\varepsilon_{k-m+1} \\ &= (c_{k-m+1} - c_{m-1}) + (9 + c_{m-1} - c_{k-m+1}) - 10\varepsilon_{k-m+1} \\ &= 9 - 10\varepsilon_{k-m+1}, \end{aligned}$$

which implies $\varepsilon_{k-m+1} = 0$ and $e_{k-m+1} = 9$. In general, we replace j by $k - j$ in (2.13) to obtain $d_j = 9 + c_j - c_{k-j}$ for $1 \leq j \leq m$; and if $\varepsilon_{k-j-1} = 0$ and $1 \leq j < m$, then

$$e_{k-j} = d_{k-j} + d_j - 10\varepsilon_{k-j} = (c_{k-j} - c_j) + (9 + c_j - c_{k-j}) - 10\varepsilon_{k-j} = 9 - 10\varepsilon_{k-j},$$

which implies $\varepsilon_{k-j} = 0$ and $e_{k-j} = 9$. Applying this observation for $j = m-2, m - 3, \ldots, 1$, respectively, we obtain

$$\varepsilon_{k-m+2} = 0, e_{k-m+2} = 9, \varepsilon_{k-m+3} = 0, e_{k-m+3} = 9, \ldots, \varepsilon_{k-1} = 0, e_{k-1} = 9.$$

Then

$$e_k = d_k + d_0 - 10\varepsilon_k = (c_k - c_0) + (10 + c_0 - c_k) - 10\varepsilon_k = 10 - 10\varepsilon_k,$$

which implies $\varepsilon_k = 1$ and $e_k = 0$. Then $e_{k+1} = 1$. To conclude, we obtain $e_j = 0$ for $0 \leq j < m$, $e_m = 9$, $e_{m+1} = 8$, $e_j = 9$ for $m + 2 \leq j \leq k - 1$, $e_k = 0$, and $e_{k+1} = 1$. This completes the proof. □

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