GREEN'S EQUIVALENCES ON SEMIGROUPS OF TRANSFORMATIONS PRESERVING A ZIG-ZAG ORDER AND AN EQUIVALENCE RELATION

By

MR. Chaleamrach MALANGPOO

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Silpakorn University

Academic Year 2022

ความสัมพันธ์สมมูลของกรีนบนกึ่งกรุปของการแปลงแบบยืนยงอันดับของเฟนซ์และความสัมพันธ์สมมูล

โดย

นายเฉลิมราชย์ แมลงภู่

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ แผน ก แบบ ก 2 ระดับปริญญามหาบัณฑิต

ภาควิชาคณิตศาสตร์

มหาวิทยาลัยศิลปากร

ปีการศึกษา 2565

GREEN'S EQUIVALENCES ON SEMIGROUPS OF TRANSFORMATIONS PRESERVING A ZIG-ZAG ORDER AND AN EQUIVALENCE RELATION

By

MR. Chaleamrach MALANGPOO

A Thesis Submitted in Partial Fulfillment of the Requirements

for Master of Science (MATHEMATICS)

Department of MATHEMATICS

Silpakorn University

Academic Year 2022

Title	Green's Equivalences on Semigroups of Transformations Preserving a Zig-Zag Order and an Equivalence Relation
By	MR. Chaleamrach MALANGPOO
Field of Study	(MATHEMATICS)
Advisor	Assistant Professor Thanakorn Prinyasart, Ph.D.

Faculty of Science, Silpakorn University in Partial Fulfillment of the Requirements for the Master of Science

-- Dean of Faculty of Science

(Assistant Professor Narong Chimpalee, Ph.D.)

Approved by

-- Chair person

(Associate Professor Ratana Srithus, Ph.D.)

-- Advisor

(Assistant Professor Thanakorn Prinyasart, Ph.D.)

-- External Examiner

(Assistant Professor Watcharintorn Ruksasakchai, Ph.D.)

620720001 : MAJOR: MATHEMATICS

KEY WORDS : SEMIGROUPS, REGULAR ELEMENTS, GREEN'S EQUIVALENCES, ORDER PRESERVING TRANSFORMATIONS

CHALEAMRACH MALANGPOO : GREEN'S EQUIVALENCES ON SEMIGROUPS OF TRANSFORMATIONS PRESERVING A ZIG-ZAG ORDER AND AN EQUIVALENCE RELATION THESIS ADVISOR : ASST. PROF. THANAKORN PRINYASART, Ph.D.

Let E be an equivalence relation on a finite fence X such that every equivalence class in X/E is a subfence of X. Let $T(X)$ be the set of all transformations on X and

$$T_E(X) = \{f \in T(X) \mid \forall a, b \in E, (af, bf) \in E\}$$

be the set of all E-preserving transformations on X. The set of all order preserving transformations in $T_E(X)$ forms a subsemigroup of $T_E(X)$ denoted by

$$O_E(X) = \{f \in T_E(X) \mid \forall x, y \in X, x \leq y \implies xf \leq yf\}.$$

In this research, we study the semigroup $O_E(X)$. Some characterizations of Green's equivalences on $O_E(X)$ are presented as well.

Acknowledgements

I am greatly indebted to my thesis advisor, Assistant Professor Dr.Thanakorn Prinyasart, for his helpful comments and suggestions in preparing and writing this thesis. I appreciate every help from him.

I also wish to express my gratitude to my thesis committees, Associate Professor Dr. Ratana Srithus and Assistant Professor Dr. Watcharintorn Ruksasakchai, for their comments and suggestions. Moreover, I feel thankful to all of my teachers who have taught me for my knowledge and skills.

Finally, I would like to express my deep gratitude to my parents for their encouragement during the period of study.

Chaleamrach MALANGPOO

Contents

Chapter 1

Introduction

A semigroup $(S, \circ)$ is an algebraic structure which consists of a non-empty set S and an associative binary operation $\circ$ on S. A subsemigroup of $(S, \circ)$ is a non-empty subset T of S which is closed under $\circ$. For example, we know that $\emptyset \neq \mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{N}$ is closed under $+$. Then $(\mathbb{N}, +)$ is a subsemigroup of $(\mathbb{Z}, +)$.

For any non-empty set X, the set of all transformations on X is denoted by $T(X)$. Consider an ordered set $(X; \leq)$. We denote the set of all $\leq$ - preserving transformations on X by $OT(X)$, that is,

$$OT(X) = \{f \in T(X) : \forall x, y \in X, x \leq y \text{ implies } xf \leq yf\}.$$

An interesting example of an ordered set is a fence, whose order forms a path with alternating orientations. In fact, the relations

$$x_1 \leq x_2 \geq x_3, \ldots, x_{2m-1} \leq x_{2m} \geq x_{2m+1} \leq \ldots$$

or

$$x_1 \geq x_2 \leq x_3, \ldots, x_{2m-1} \geq x_{2m} \leq x_{2m+1} \geq \ldots$$

are the only comparability relations in a fence $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. It is easy to see that $T(X)$ is a semigroup under the composition of functions defined by

$$fg := \{(x,y) \in X \times X : (x,z) \in f \text{ and } (z,y) \in g \text{ for some } z \in X\},$$

and $OT(X)$ is a subsemigroup of $T(X)$.

Algebraic properties of $T(X)$ and its subsemigroups have been studied by many researchers. Jendana and Srithus [4] characterized a finite fence X having $OT(X)$ as a coregular semigroup and already described coregular elements of $OT(X)$. Tanyawong [5] described all regular semigroups $OT(X)$ where X is a finite fence. To date, we know the regularity of $OT(X)$ where X is a fence. This leads to a more complex semigroup, which is the semigroup of all transformations preserving a zig-zag order and an equivalence relation on X.

An equivalence relation is a binary relation that is reflexive, symmetric and transitive. Let E be an equivalence relation on X. We define

$$T_E(X) = \{f \in T(X) : \forall (a,b) \in E, (af, bf) \in E\}.$$

The set of all order-preserving transformations in $T_E(X)$ forms a subsemigroup of $T_E(X)$ denoted by

$$O_E(X) = \{f \in T_E(X) : \forall x, y \in X, x \leq y \text{ implies } xf \leq yf\}.$$

Green's relations are five equivalence ralations that characterize elements of a semigroup in terms of the ideals they generate. Let S be a semigroup and S^1 be the set S with an identity adjoined if S does not contain an identity. For $a, b \in S$, we define Green's relations $\mathfrak{L}, \mathfrak{R}, \mathfrak{J}, \mathfrak{H}$ and $\mathfrak{D}$ as follows:

1. $a\mathfrak{L}b$ if and only if $S^1a = S^1b$, that is, a, b generate the same left principal ideal;

2. $a\mathfrak{R}b$ if and only if $aS^1 = bS^1$, that is, a, b generate the same right principal ideal;

3. $a\mathfrak{J}b$ if and only if $S^1aS^1 = S^1bS^1$, that is, a, b generate the same two-sided principal ideal;

4. $a\mathfrak{H}b$ if and only if $a\mathfrak{L}b$ and $a\mathfrak{R}b$;

5. $a\mathfrak{D}b$ is the smallest equivalence relation that contains $\mathfrak{L}$ and $\mathfrak{R}$, namely, $\mathfrak{D} = \mathfrak{L} \circ \mathfrak{R}$.

It is well-known that in a finite semigroup, $\mathfrak{D} = \mathfrak{J}$ in [1].

Definition 1.1. [2] Let S be a semigroup and $x \in S$. We say that x is *regular* if there is $b \in S$ such that $x = xbx$. Moreover, S is *regular* if every element in S is regular.

Huisheng and Dingyu [3] described the nature of regular elements in $O_E(X)$ and characterized the Green's equivalences on $O_E(X)$ completely, where X is a finite chain. For $f \in T(X)$, we denote $\pi(f) = \{xf^{-1} : \; x \in Xf\}$. Notice that $f*$ is a function from $\pi(f)$ into Xf defined by $Af* = Af$ for each $A \in \pi(f)$. For each $f \in T(X)$, we let $E(f) = \{Af^{-1} : A \in X/E, Af^{-1} \neq \emptyset\}$. The characterization of Green's equivalences for $O_E(X)$, where X is a chain are already described as follows:

Theorem 1.2. *[3] Let $f, g \in O_E(X)$. Then the following statements are equivalent.*

1. *$(f, g) \in \mathfrak{R}$.*

2. *$\pi(f) = \pi(g)$ and $E(f) = E(g)$.*

3. *There exists an E^*- preserving order isomorphism $\phi : Xf \to Xg$ such that $g = f\phi$.*

Before we introduce the result of the relation $\mathfrak{L}$, we need to introduce some definition.

Definition 1.3. [3] Let $f \in O_E(X)$ and $\phi : \pi(f) \to \pi(g)$. If for every $A \in X/E$, there is $B \in X/E$ such that $\pi_A(f)\phi \subseteq \pi_B(g)$, then ϕ is called *E-admissible*. Moreover, if ϕ is bijective and ϕ, ϕ^{-1} are E- admissible, then ϕ is *E^*-admissible.*

Theorem 1.4. *[3] Let $f, g \in O_E(X)$. Then the following statements are equivalent.*

1. $(f,g) \in \mathfrak{L}$.

2. $Xf = Xg$ and for each $A \in X/E$, there exist $B, C \in X/E$ such that $Af \subseteq Bg$, $Ag \subseteq Cf$.

3. There exists an E^*- admissible order isomorphism $\phi : \pi(f) \to \pi(g)$ such that $f* = \phi g*$.

Theorem 1.5. *[3] Let $f, g \in O_E(X)$. Then the following statements are equivalent.*

1. $(f,g) \in \mathfrak{H}$.

2. $\pi(f) = \pi(g)$, $E(f) = E(g)$, $Xf = Xg$ and for each $A \in X/E$, there exist $B, C \in X/E$ such that $Af \subseteq Bg$, $Ag \subseteq Cf$.

3. There exists an E^*- preserving order isomorphism $\phi : Xf \to Xg$ and E^*- admissible order isomorphism $\psi : \pi(f) \to \phi(g)$ such that $g = f\phi$ and $f* = \psi g*$.

Theorem 1.6. *[3] Let $f, g \in O_E(X)$. Then the following statements are equivalent.*

1. $(f,g) \in \mathfrak{D}$.

2. There exist an E^*- preserving order isomorphism $\psi : Xf \to Xg$ and E^*- admissible order isomorphism $\phi : \pi(f) \to \phi(g)$ such that $\phi g* = f * \psi$.

Theorem 1.7. *[3] Let $f, g \in O_E(X)$ be regular elements. Then*

1. $f\mathfrak{L}g$ if and only if $\pi(f) = \pi(g)$;

2. $f\mathfrak{R}g$ if and only if $Xf = Xg$;

3. $f\mathfrak{D}g$ if and only if there exists a bijection $\phi : Xf \to Xg$ such that ϕ and ϕ^{-1} are order-preserving and E-preserving.

In this research, we aim to characterize the Green's equivalence relations on $O_E(X)$ where X is a finite fence and E is an equivalence relation on X such that every equivalence class in X/E is a subfence of X.

Chapter 2

Preliminaries

In this chapter, we provide definitions, theorems, lemmas and some examples related to this research.

2.1 Fences

Definition 2.1. A *relation* on a set X is a subset of $X \times X$.

Definition 2.2. For a relation $\leq$ on a set X and $x, y \in X$, the notation $x \leq y$ refers to $(x, y) \in \leq$, and the notation $x \geq y$ refers to $(y, x) \in \leq$.

Definition 2.3. Let A be a set. A relation $\leq$ on A is a *(partial) order* if

- for all $x \in A$, $x \leq x$, that is, $\leq$ satisfies *reflexivity*;
- for all $x, y, z \in A$, if $x \leq y$ and $y \leq z$, then $x \leq z$, that is, $\leq$ satisfies *transitivity*;
- for all $x, y \in A$, if $x \leq y$ and $y \leq x$, then $x = y$, that is, $\leq$ satisfies *antisymmetry*.

If $\leq$ is a partial order on A, the pair $(A, \leq)$ is called a *(partially) ordered set*. When there is no ambiguity, we denote the partially ordered set $(A, \leq)$ by A.

In this research, we will focus on the ordered sets called fences defined as follows.

Definition 2.4. An ordered set X is called a *fence* if the order forms a path with alternating orientation. Indeed, X is in which either

$$x_1 \leq x_2 \geq x_3, \ldots, x_{2m-1} \leq x_{2m} \geq x_{2m+1} \leq \ldots$$

or

$$x_1 \geq x_2 \leq x_3, \ldots, x_{2m-1} \geq x_{2m} \leq x_{2m+1} \geq \ldots$$

are the only comparability relations in the fence $X = \{x_1, x_2, \ldots, x_n, \ldots\}$.

2.2 Basic facts on functions

Definition 2.5. Let A and B be sets. A subset f of $A \times B$ is said to be a *function* from A into B if

$$A = \{a \in A : \exists! b \in B[(a, b) \in f]\}.$$

We denote the function f from A into B by $f : A \to B$. Moreover, for each $a \in A$, let af donote the unique $b \in B$ such that $(a, b) \in f$.

Definition 2.6. Let A, B, C be sets, $f : A \to B$ and $g : B \to C$. The *composition* of f and g is the function $fg : A \to C$ defined by

$$fg = \{(a, c) \in A \times C : (a, b) \in f \text{ and } (b, c) \in g \text{ for some } b \in B\}.$$

Definition 2.7. Let A, B be sets and $f : A \to B$. For all subset X of A, the *restriction* of f to X is the function $f|_X : X \to B$ defined by

$$f|_X = \{(x, b) \in f : x \in X\}.$$

Definition 2.8. Let A, B be sets and $f : A \to B$. For all subset X of A, the *image* of X under f, which is denoted by Xf, is defined by

$$Xf = \{xf : x \in X\}.$$

For all subset Y of B, the *inverse image* of Y under f, which is denoted by Yf^{-1}, is defined by

$$Yf^{-1} = \{x \in X : xf \in Y\}.$$

If $Y = \{b\}$ for some $b \in B$, we denote Yf^{-1} by bf^{-1}.

Remark 2.9. Let A, B be sets and $f : A \to B$. Then, for all $X \subseteq A$ and $Y \subseteq B$,

$$X \subseteq (Xf)f^{-1} \text{ and } (Yf^{-1})f \subseteq Y.$$

Definition 2.10. Let A, B be sets and $f : A \to B$.

(i) f is *surjective* if $Af = B$.

(ii) f is *injective* if for all $a_1, a_2 \in A$, $a_1 = a_2$ whenever $a_1 f = a_2 f$.

(iii) f is *bijective* if f is surjective and injective.

2.3 Semigroups

Definition 2.11. A *binary operation* on a set X is a function from $X \times X$ into X. If $\circ$ is a binary operation on X, we denote $\circ((x, y))$ by $x \circ y$ for all $x, y \in X$.

Definition 2.12. Let $\circ$ be a binary operation on a set X. A subset Y of X is said to be *closed* under $\circ$ if $y_1 \circ y_2 \in Y$ for all $y_1, y_2 \in Y$, that is, $\circ|_{Y \times Y}$ is a binary operation on Y.

Definition 2.13. A binary operation $\circ$ on a set X is *associative* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in X$.

Definition 2.14. A pair $(S, \circ)$ is a *semigroup* if S is a non-empty set and $\circ$ is an associative binary operation on S.

Example 2.15. Let $\mathbb{Z}$ be the set of all integers, and $+$ be the usual addition on $\mathbb{Z}$. Then $(\mathbb{Z}, +)$ is a semigroup.

A semigroup $(S, \circ)$ is usually denoted as S, without mentioning the operator $\circ$, if there is no ambiguity.

Definition 2.16. Let $(S, \circ)$ be a semigroup and T be a non-empty subset of S. We call T a *subsemigroup* of S if $(T, \circ|_{T \times T})$ is a semigroup.

Remark 2.17. For any semigroup $(S, \circ)$ and any subset T of S, T is a subsemigroup of S if and only if T is closed under $\circ$.

Example 2.18. Consider the semigroup $(\mathbb{Z}, +)$, where $+$ is the usual addition on $\mathbb{Z}$. We know that $\emptyset \neq \mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{N}$ is closed under $+$. Then $\mathbb{N}$ is a subsemigroup of $\mathbb{Z}$.

Definition 2.19. Let X be a non-empty set. A *transformation* on X is a function from X to X. Let $T(X)$ denote the set of all transformations on X.

Remark 2.20. Let X be a non-empty set. Then $(T(X), \circ)$ is a semigroup, where $\circ$ is the function composition defined by

$$f \circ g = \{(x, y) \in X \times X : (x, z) \in f \text{ and } (z, y) \in g \text{ for some } z \in X\}.$$

From now on, for any transformations f, g on a non-empty set X, the composition $f \circ g$ is denoted simply as fg.

Definition 2.21. Let X, Y be two ordered sets and $f : X \to Y$ be a function. We say that f is *order-preserving* if for all $x, y \in X$, $x \leq y$ implies $xf \leq yf$. If f is an order-preserving bijection such that $x \leq y$ if and only if $xf \leq yf$, then we say that f is an *order isomorphism*.

Definition 2.22. Let $(X, \leq)$ be an ordered set. We denote the set of all $\leq$ - preserving transformations on X by $OT(X)$. Namely,

$$OT(X) = \{f \in T(X) : \forall x, y \in X, x \leq y \text{ implies } xf \leq yf\}.$$

Theorem 2.23. *Let $(X, \leq)$ be an ordered set. Then $OT(X)$ is a subsemigroup of $T(X)$.*

Proof. Obviously, $\emptyset \neq OT(X) \subseteq T(X)$. Let $f, g \in OT(X)$ and $x, y \in X$ such that $x \leq y$. Since $f \in OT(X)$, $xf \leq yf$. Since $g \in OT(X)$, $x(fg) = (xf)g \leq (yf)g = y(fg)$. Thus, $fg \in OT(X)$. It follows that $OT(X)$ is closed under the function composition. Hence, $OT(X)$ is a subsemigroup of $T(X)$. $\square$

Definition 2.24. Let S be a semigroup and $x \in S$, then x is a *regular element* if there is $b \in S$ such that $x = xbx$. Moreover, S is *regular* if every element in S is regular.

Theorem 2.25. *Let X be a non-empty set. Then $T(X)$ is regular.*

Proof. Let $\alpha \in T(X)$. For each $y \in X\alpha$, there exists x_y such that $x_y\alpha = y$. Since X is non-empty, there exists $x_0 \in X$. We define $\beta : X \to X$ by

$$y\beta = \begin{cases} x_y & \text{if } y \in X\alpha; \\ x_0 & \text{otherwise.} \end{cases}$$

We want to show that $\alpha\beta\alpha = \alpha$. We know that $\mathrm{dom}(\alpha\beta\alpha) = \mathrm{dom}(\alpha)$. Since $a\alpha\beta\alpha = (a\alpha)\beta\alpha = (x_{a\alpha})\alpha = a\alpha$ for all $a \in X$. We have that $\alpha\beta\alpha = \alpha$. Hence, α is regular.

□

2.4 Equivalence relations and partitions

Recall that, for a relation R on a set X,

- R is said to be reflexive, if $(x, x) \in R$ for every $x \in X$;
- R is said to be symmetric, if $(y, x) \in R$ whenever $(x, y) \in R$;
- R is said to be transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Definition 2.26. A relation R is said to be an *equivalence relation* if the relation R is reflexive, symmetric and transitive.

Definition 2.27. Let E be an equivalence relation on X, and $x \in X$. We denote the set of all elements that is E-related to x by $[x]$, that is,

$$[x] = \{y \in X : (x, y) \in E\}.$$

The set $[x]$ is called the *equivalence class* of x. The set of all equivalence classes is denoted by X/E, that is,

$$X/E = \{[a] : a \in X\}.$$

Definition 2.28. Let X be a set. A nonempty collection $\mathcal{C}$ of subsets of X is a *partition* of X if $\bigcup \mathcal{C} = X$, and for all $A, B \in \mathcal{C}$, $A \cap B = \emptyset$ if and only if $A \neq B$.

Remark 2.29. If E is an equivalence relation on a set X, then X/E is a partition of X.

Proposition 2.30. *If f is a transformation on a non-empty set X, then $\pi(f)$ is a partition of X.*

Proof. Define a relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $x, y \in X$. It is easy to see that $\sim$ is an equivalence relation and $\pi(f) = X/\sim$. Hence, $\pi(f)$ is a partition of X. □

Definition 2.31. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of X. We say that $\mathcal{P}$ is a *refinement* of $\mathcal{Q}$ if for any $A \in \mathcal{P}$, there is $B \in \mathcal{Q}$ such that $A \subseteq B$.

Proposition 2.32. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of X. If $\mathcal{P}$ is a refinement of $\mathcal{Q}$ and $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then $\mathcal{P} = \mathcal{Q}$.*

Proof. Assume that $\mathcal{P}$ is a refinement of $\mathcal{Q}$ and $\mathcal{Q}$ is a refinement of $\mathcal{P}$. To show that $\mathcal{P} \subseteq \mathcal{Q}$, let $A \in \mathcal{P}$. Since $\mathcal{P}$ is a refinement of $\mathcal{Q}$, there is $B \in \mathcal{Q}$ such that $A \subseteq B$. Similarly, since $\mathcal{Q}$ is a refinement of $\mathcal{P}$, there is $C \in \mathcal{P}$ such that $B \subseteq C$. Since $A \subseteq B \subseteq C$, $A \cap C = A \neq \emptyset$. Since $\mathcal{P}$ is a partition and $A \cap C \neq \emptyset$, $A = C$, which implies that $A = B \in \mathcal{Q}$. Similarly, we can conclude that $\mathcal{Q} \subseteq \mathcal{P}$. Hence, $\mathcal{P} = \mathcal{Q}$. □

2.5 Transformations preserving an equivalence relation

Definition 2.33. Let E be an equivalence relation on X, and Y, Z be subsets of X. Let f be a function from Y to Z. We say that f is *E-preserving* if for any $a, b \in Y, (a, b) \in E$ implies $(af, bf) \in E$. Moreover, if for any $a, b \in Y, (a, b) \in E$ if and only if $(af, bf) \in E$, then we say that f is $E^* -$ *preserving.*

Definition 2.34. Let X be an ordered set and E be an equivalence relation on X. The set of all E-preserving transformation in $OT(X)$, denoted by $O_E(X)$, is defined by

$$O_E(X) = \{f \in OT(X) : (xf, yf) \in E \text{ for all } (x, y) \in E\}.$$

Lemma 2.35. *[3] Let f be E-preserving. Then, for each $B \in X/E$, there exists $B' \in X/E$ such that $Bf \subseteq B'$. Consequently, for any $A \in X/E, Af^{-1}$ is either $\emptyset$ or a union of some classes X/E.*

2.6 Green's equivalences

Definition 2.36. For any semigroup S, let S^1 be a semigroup with an identity adjoined if S has no identity, and let $S^1 = S$ if S contains an identity. For $a, b \in S$, we define the Green's relation $\mathfrak{L}, \mathfrak{R}, \mathfrak{J}, \mathfrak{H}$ and $\mathfrak{D}$ as follows:

1. $a\mathfrak{L}b$ if and only if $S^1a = S^1b$. Namely, $a\mathfrak{L}b$ if and only if $a = xb$ and $b = ya$ for some $x, y \in S^1$;

2. $a\mathfrak{R}b$ if and only if $aS^1 = bS^1$. Namely, $a\mathfrak{R}b$ if and only if $a = bx$ and $b = ay$ for some $x, y \in S^1$;

3. $a\mathfrak{J}b$ if and only if $S^1aS^1 = S^1bS^1$. Namely, $a\mathfrak{J}b$ if and only if $a = xby$ and $b = uav$ for some $x, y, u, v \in S^1$;

4. $a\mathfrak{H}b$ if and only if $a\mathfrak{L}b$ and $a\mathfrak{R}b$;

5. $a\mathfrak{D}b$ is the smallest equivalence relation that contains $\mathfrak{L}$ and $\mathfrak{R}$,
namely, $\mathfrak{D} = \mathfrak{L} \circ \mathfrak{R}$.

It is well-known that in a finite semigroup, $\mathfrak{D} = \mathfrak{J}$. Therefore, to characterize all Green's equivalences on $OT(X)$, where X is a finite fence, it is enough to consider only $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{H}$ and $\mathfrak{D}$.

Chapter 3

The characterization of Green's equivalences

For the rest of this research, let E be an equivalence relation on a finite fence X such that every equivalence class in X/E is a subfence in X.

We begin this section with the characterization of $\mathfrak{L}$ and $\mathfrak{R}$, which will be useful for the characterization of $\mathfrak{H}$ and $\mathfrak{D}$ later.

Theorem 3.1. *Let $f, g \in O_E(X)$. Then $(f, g) \in \mathfrak{R}$ if and only if $\pi(f) = \pi(g)$ and $E(f) = E(g)$.*

Proof. Assume that $(f, g) \in \mathfrak{R}$. Then there are $h, k \in O_E(X)$ such that $fh = g$ and $gk = f$. Let $P \in \pi(f)$. We have that Pfh is a singleton , which implies that $P \subseteq (Pg)g^{-1} = (Pfh)\,g^{-1} \in \pi(g)$. Thus, $\pi(f)$ is a refinement of $\pi(g)$. Similarly, we also have that $\pi(g)$ is a refinement of $\pi(f)$. Hence, $\pi(f) = \pi(g)$. Next, we will show that $E(f) = E(g)$. Let $U \in E(f)$. Then there exists $A \in X/E$ such that $A \cap Xf \neq \emptyset$ and $U = Af^{-1}$. Since h is E- preserving, there exists $B \in X/E$ such that $Ah \subseteq B$. Then $Ug = Ufh = Af^{-1}fh \subseteq Ah \subseteq B$. Hence, $U \subseteq Ugg^{-1} \subseteq Bg^{-1}$. Similarly, since $Bg^{-1} \in E(g)$, we can also have that $Bg^{-1} \subseteq V$ for some $V \in E(f)$. Thus, $U \subseteq V$ and $U, V \in E(f)$. Since $E(f)$ is a partition of X, we have $U = V$. So, $U = Bg^{-1} \in E(g)$. Hence, $E(f) \subseteq E(g)$. Similarly, we can also have that

$E(g) \subseteq E(f)$. Thus, $E(f) = E(g)$.

Conversely, assume that $\pi(f) = \pi(g)$ and $E(f) = E(g)$. Without loss of generality, we may assume that $X = \{x_1, x_2, \ldots, x_l\}, x_1 \leq x_2 \geq x_3 \leq \ldots \geq x_{l-1} \leq x_l$ and $\text{ran}(g) = \{x_i, x_{i+1}, \ldots, x_j\}$, where $1 \leq i \leq j \leq l$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} af & ; x = ag \text{ for some } a \in X; \\ y_i f & ; x = x_k \text{ for some } k \leq i \text{ and } y_i g = x_i \text{ for some } y_i \in X; \\ y_j f & ; x = x_k \text{ for some } k \geq j \text{ and } y_j g = x_j \text{ for some } y_j \in X. \end{cases}$$

Thus, γ is well-defined since $\pi(f) = \pi(g)$. Next, we will show that $\gamma \in O_E(X)$. First, we need to show that γ is order-preserving. Let $x_m, x_n \in X$ with $x_m \leq x_n$.

Case 1 : $x_m, x_n \in \text{ran}(g)$. Then there exist $a, b \in X$ such that $a \leq b, ag = x_m$ and $bg = x_n$. Thus, $x_m\gamma = af$ and $x_n\gamma = bf$. Since $a \leq b$ and f is order-preserving, we have $x_m\gamma = af \leq bf = x_n\gamma$.

Case 2 : $x_m \notin \text{ran}(g)$ or $x_n \notin \text{ran}(g)$. Since x_m and x_n are comparable, $m, n \leq i$ or $m, n \geq j$. If $m, n \leq i$, then $x_m\gamma = y_i f = x_n\gamma$. Similarly, if $m, n \geq j$, then $x_m\gamma = y_j f = x_n\gamma$.

Thus, $\gamma \in OT(X)$. Finally, we will show that γ is E-preserving. Let $(x, y) \in E$.

Case 1 : $x, y \in \text{ran}(g)$. Then there are $a_1, a_2 \in X$ such that $a_1 g = x$ and $a_2 g = y$. Thus, we have $x\gamma = a_1 f$ and $y\gamma = a_2 f$. Since $a_1, a_2 \in [x] g^{-1} \in E(g) = E(f)$, we have $(a_1 f, a_2 f) \in E$ implying $(x\gamma, y\gamma) \in E$.

Case 2 : $x \in \text{ran}(g)$ but $y \notin \text{ran}(g)$. Let F be a subfence of X such that x and y are both ends of F. Then $x_i \in F$ or $x_j \in F$. Without loss of generality, we may assume that $x_i \in F$. Then $y\gamma = x_i\gamma$. Since every equivalence class in X/E is a subfence of X, $F \subseteq A$ for some $A \in X/E$. So, $(x, x_i) \in E$. By the previous case, $(x\gamma, x_i\gamma) \in E$. Hence, $(x\gamma, y\gamma) \in E$.

Case 3 : $x, y \notin \text{ran}(g)$. Let F be a subfence of X such that x and y are both ends of F. Then $\text{ran}(g) \cap F = \emptyset$ or $\text{ran}(g) \subseteq F$. If $\text{ran}(g) \cap F = \emptyset$, then $x\gamma = y\gamma$, which implies that $(x\gamma, y\gamma) \in E$. Now, assume that $\text{ran}(g) \subseteq F$. Then $x\gamma, y\gamma \in \{x_i\gamma, x_j\gamma\}$. Moreover, since every equivalence class in X/E is a subfence of X, $F \subseteq A$ for some

$A \in X/E$. Then $(x_i, x_j) \in E$. By Case 1, we have that $(x_i\gamma, x_j\gamma) \in E$. Since $x\gamma, y\gamma \in \{x_i\gamma, x_j\gamma\}$ and $(x_i\gamma, x_j\gamma) \in E$, we can conclude that $(x\gamma, y\gamma) \in E$.

Thus, $\gamma \in O_E(X)$. Moreover, it is easy to see from the definition of γ that $g\gamma = f$. Similarly, we can also show that $g = f\delta$ for some $\delta \in O_E(X)$. Hence, $(f, g) \in \mathfrak{R}$.

□

For $f \in O_E(X)$, let

$$X_f := \{\operatorname{im} h : h \in O_E(X) \text{ and } hf = f\}.$$

The following lemma will be useful for the characterization of $\mathfrak{L}$.

Lemma 3.2. *Let $f \in O_E(X)$ and Y_f, Y_f' be minimal subfences in X_f. Then there exists a function in $O_E(X)$ that bijectively map Y_f onto Y_f'. Consequently, $|Y_f| = |Y_f'|$.*

Proof. Assume that $f \in O_E(X)$. Since $X_f = \{\operatorname{im} h : h \in O_E(X) \text{ and } hf = f\}$, and Y_f and Y_f' are minimal subfences in X_f, there exist $g, h \in O_E(X)$ such that $\operatorname{im} g = Y_f$, $gf = f$, $\operatorname{im} h = Y_f'$ and $hf = f$. We will show that there exists a bijective function $h|_{Y_f} : Y_f \to Y_f'$. Notice that $\operatorname{im} gh \subseteq \operatorname{im} h = Y_f'$ and $ghf = f$, and Y_f' is minimal in X_f. Then we have $\operatorname{im} h|_{Y_f} = Y_f h = \operatorname{im} gh = Y_f'$ implying $h|_{Y_f}$ is onto. Consequently, we have $|Y_f| \geq |Y_f'|$. On the other hand, we can find an onto function $h'|_{Y_f} : Y_f' \to Y_f$, then $|Y_f'| \geq |Y_f|$ which implies $|Y_f| = |Y_f'|$. Since $h|_{Y_f}$ is onto, $|Y_f| = |Y_f'|$ and Y_f, Y_f' are finite, we have that $h|_{Y_f}$ is injective and we can conclude that $h|_{Y_f}$ is a bijective function from Y_f to Y_f'. □

Theorem 3.3. *For $f, g \in O_E(X)$, let Y_f and Y_g be minimal subfences in X_f and X_g respectively. Then $(f, g) \in \mathfrak{L}$ if and only if $f|_{Y_f} = hg$ for some E^*-preserving order isomorphism $h : Y_f \to Y_g$.*

Proof. Assume that $(f, g) \in \mathfrak{L}$. Then there are $\alpha, \beta \in O_E(X)$ such that $\alpha f = g$ and $\beta g = f$. Since Y_f and Y_g are minimal elements in X_f and X_g, there are $\gamma_1, \gamma_2 \in O_E(X)$, such that $\operatorname{im} \gamma_1 = Y_f, \gamma_1 f = f$, and $\operatorname{im} \gamma_2 = Y_g, \gamma_2 g = g$. Let $h := \beta|_{Y_f}\gamma_2$. Since $\beta, \gamma_2 \in O_E(X)$, we have that h is order and E-preserving. Notice that

$\gamma_1\beta\gamma_2\alpha \in O_E(X)$ and $\operatorname{im}\gamma_1\beta\gamma_2\alpha = Y_f h\alpha$, and $\gamma_1\beta\gamma_2\alpha f = \gamma_1\beta\gamma_2 g = \gamma_1\beta g = \gamma_1 f = f$. Thus, $Y_f h\alpha \in X_f$. Next, we will show that $h := \beta|_{Y_f}\gamma_2$ is injective. Suppose that $\beta|_{Y_f}\gamma_2$ is not injective. Then $|Y_f\beta\gamma_2| < |Y_f|$ which contradicts to the fact that Y_f is a minimal subfence in X_f. Therefore, $h := \beta|_{Y_f}\gamma_2$ is an injective function. Since $h := \beta|_{Y_f}\gamma_2$ injectively maps Y_f into Y_g, we have $|Y_f| \leq |Y_g|$. Similarly, we can prove that $|Y_g| \leq |Y_f|$. Then $|Y_f| = |Y_g|$. Consequently, h is a bijection. Similarly, there exists an order and E-preserving bijection h_2 that maps Y_g onto Y_f. Since Y_f is finite, by Lagrange's theorem, $(hh_2)^n$ is the identity function on Y_f for some n, so $(hh_2)^n = h(h_2h)^{n-1}h_2$. Then $(h_2h)^{n-1}h_2$ is the inverse function of h. Since $(h_2h)^{n-1}h_2$ is order and E-preserving, h is an E^*-preserving order isomorphism. Notice that $hg = \beta|_{Y_f}\gamma_2 g = \beta|_{Y_f} g = f|_{Y_f}$. Hence, there is an E^*-preserving order isomorphism $h : Y_f \to Y_g$ such that $f|_{Y_f} = hg$. Conversely, assume that $f|_{Y_f} = hg$ for some E^*-preserving order isomorphism $h : Y_f \to Y_g$. Since Y_f is a element in X_f, there is $\gamma_1 \in O_E(X)$ such that $\operatorname{im}\gamma_1 = Y_f$ and $\gamma_1 f = f$. Let $\beta := \gamma_1 h \in O_E(X)$. Then $\beta g = \gamma_1 hg = \gamma_1 f = f$. Thus, we have $\beta := \gamma_1 h \in O_E(X)$ such that $f = \beta g$. Since $f|_{Y_f} = hg$ and h is an bijection, there exists an E^*-preserving order isomorphism $h^{-1} : Y_g \to Y_f$ such that $h^{-1}f = h^{-1}hg = g|_{Y_g}$. Similarly, there is $\beta' \in O_E(X)$ such that $g = \beta' f$. Therefore, $(f, g) \in \mathfrak{L}$. □

Theorem 3.4. *Let $f, g \in O_E(X)$, Y_f and Y_g be minimal subfences in X_f and X_g respectively. Then $(f, g) \in \mathfrak{H}$ if and only if $\pi(f) = \pi(g)$ and $E(f) = E(g)$, and $f|_{Y_f} = hg$ for some E^*-preserving order isomorphism $h : Y_f \to Y_g$.*

The following lemmas will be useful to proof the next theorem.

Lemma 3.5. *Let $f \in O_E(X)$, Y be a subfence of X and let $y_1, y_2 \in Yf$ with $y_1 < y_2$. Then there are $x_1, x_2 \in Y$ such that $x_1 f = y_1$, $x_2 f = y_2$ and $x_1 \leq x_2$.*

Proof. Assume that $y_1 < y_2$, $Y = \{x_1, x_2, \ldots, x_m\}$ and $Yf = \{z_1, z_2, \ldots, z_k\}$, where $x_1 < x_2 > x_3 < \cdots > x_m$ and $z_1 < z_2 > z_3 < \cdots > z_k$. We choose $i, j \in \{1, 2, \ldots, m\}$ such that $i < j$, $\{x_i f, x_j f\} = \{y_1, y_2\}$ and $j - i = \min\{|r - s| : \{x_r f, x_s f\} = \{y_1, y_2\}\}$.

Now, we are going to show that $j - i = 1$ by supposing that $j - i \geq 2$. Then there exists a subfence $\{x_{i+1}f, x_{i+2}f, \ldots, x_{j-1}f\}$ which is disjoint from $\{y_1, y_2\}$. Now, we let $\{y_1, y_2\} = \{z_p, z_{p+1}\}$ for some $p \in \{1, 2, \ldots, k-1\}$. Then we have

$$\{x_{i+1}f, x_{i+2}f, \ldots, x_{j-1}f\} \subseteq \{z_1, z_2, \ldots, z_{p-1}\} \text{ or}$$

$$\{x_{i+1}f, x_{i+2}f, \ldots, x_{j-1}f\} \subseteq \{z_{p+2}, z_{p+3}, \ldots, z_k\}.$$

Thus, $x_{i+1}f$ and $x_i f$ are not comparable or $x_{j-1}f$ and $x_j f$ are not comparable, which is a contradiction. Then we have $j - i = 1$ implying x_i, x_j are comparable. Since $y_1 < y_2$ and $f \in O_E(X)$, if $x_i f = y_1$ and $x_j f = y_2$, we have $x_i < x_j$, similarly for $x_j f = y_1$ and $x_i f = y_2$, we have $x_j < x_i$. □

Lemma 3.6. *Let $f, g \in O_E(X)$. If $\pi(f) = \pi(g)$ and $E(f) = E(g)$, then there exists an E^*- preserving order isomorphism $\psi : Xg \to Xf$ such that $f = g\psi$.*

Proof. Assume that $\pi(f) = \pi(g)$. Since for each $y \in Xg$, there is $x_y \in X$ such that $x_y g = y$, we can define $h : Xg \to X$ by $yh = x_y$. Then, for all $x \in X$, $xghg = x_{xg}g = xg$ implying $ghg = g$. Since $\pi(f) = \pi(g)$, we have $f = ghf$. Now, we are going to prove that hf is an E^*- preserving order isomorphism that $f = ghf$. First, let $ag, bg \in Xg$, where $a, b \in X$, such that $aghf = bghf$. Since $\pi(f) = \pi(g)$, we have $aghg = bghg$. Thus, $ag = bg$. Hence, hf is injective. Now, we let $y \in Xf$. Then there is $x \in X$ such that $xf = y$. Since $xghg = xg$ and $\pi(f) = \pi(g)$, we have that $xghf = xf$, then there is $xg \in Xg$ such that $xghf = xf = y$. Hence, hf is onto. Next, we will show that hf is order preserving. Let $ag, bg \in Xg$, where $a, b \in X$, with $ag \leq bg$. Since $ghg = g$, we have $aghg = ag \leq bg = bghg$. By lemma 3.5, there are $a', b' \in X$ such that $a' \leq b'$, $a'g = (agh)g$ and $b'g = (bgh)g$. Since $\pi(f) = \pi(g)$ and $f \in OT(X)$, $aghf = a'f \leq b'f = bghf$. Finally, we are going to show that hf is E^* - preserving. Let $x_1, x_2 \in Xg$. Then $x_1 h = x_{x_1}$ and $x_2 h = x_{x_2}$. So, $x_{x_1} \in [x_1]g^{-1}$ and $x_{x_2} \in [x_2]g^{-1}$. Since $(x_1, x_2) \in E$, $[x_1] = [x_2]$, so $x_{x_2} \in [x_1]g^{-1}$. Since $E(f) = E(g)$, $x_{x_1}, x_{x_2} \in [x_m]f^{-1}$ for some $x_m \in X/E$. Hence, $x_1 h, x_2 h \in [x_m]f^{-1}$. Thus, $x_1 hf, x_2 hf \in [x_m]$ implying $(x_1 hf, x_2 hf) \in E$. Conversely, we assume that

$(x_1hf, x_2hf) \in E$. Let $A_0 \in X/E$ such that $x_1hf, x_2hf \in A_0$. Then $x_1h, x_2h \in A_0f^{-1}$. Since $E(f) = E(g)$, we have $x_1h, x_2h \in A_mg^{-1}$ for some $A_m \in X/E$. So, $x_{x_1}, x_{x_2} \in A_mg^{-1}$. Then $x_{x_1}g, x_{x_2}g \in A_m$ implying $x_1, x_2 \in A_m$. Thus $(x_1, x_2) \in E$.

□

Lemma 3.7. *Let $f, g \in O_E(X)$. Assume that $\pi(f) = \pi(g)$. Then $X_f = X_g$. In particular, a subset of X is minimal in X_f if and only if it is minimal in X_g.*

Proof. To show that $X_f \subseteq X_g$, let $Y \in X_f$. Then $Y = Xh$ for some $h \in O_E(X)$ with $hf = f$. Thus, for any $x \in X$, since $\pi(f) = \pi(g)$ and $(xh)f = xf$, we have that $(xh)g = xg$. Consequently, $hg = g$, which implies that $Y = Xh \in X_g$. So, $X_f \subseteq X_g$. Similarly, we obtain that $X_g \subseteq X_f$. Hence, $X_f = X_g$. □

Theorem 3.8. *Let $f, g \in O_E(X)$ and let Y_f and Y_g be minimal subsets in X_f and X_g, respectively. Then $(f, g) \in \mathfrak{D}$ if and only if there exist E^*-preserving order isomorphism $h : Y_f \to Y_g$ and $\psi : Xf \to Xg$ such that $g|_{Y_g} = hf\psi$.*

Proof. Assume that $(f, g) \in \mathfrak{D}$. There is $\gamma \in O_E(X)$ such that $(f, \gamma) \in \mathfrak{L}$ and $(\gamma, g) \in \mathfrak{R}$. Since $(\gamma, g) \in \mathfrak{R}$, by Theorem 3.1, $\pi(\gamma) = \pi(g)$. By Lemma 3.7, Y_g is a minimal subfence in X_γ. Since $(f, \gamma) \in \mathfrak{L}$, by Theorem 3.3, there is an E^*-preserving order isomorphism $h : Y_g \to Y_f$ such that $\gamma|_{Y_g} = hf$. Since $(\gamma, g) \in \mathfrak{R}$, by Theorem 3.1 and Lemma 3.6, there is an E^*-preserving order isomorphism $\psi : X\gamma \to Xg$ such that $g = \gamma\psi$. Since $(f, \gamma) \in \mathfrak{L}$, it is easy to see that $X\gamma = Xf$, so the domain of ψ is Xf. Notice that $g|_{Y_g} = \gamma|_{Y_g}\psi = hf\psi$.

Conversely, assume that there exist E^*-preserving order isomorphisms $h : Y_g \to Y_f$ and $\psi : Xf \to Xg$ such that $g|_{Y_g} = hf\psi$. Let $\gamma := f\psi \in O_E(X)$. Since $\gamma = f\psi$ and $\gamma\psi^{-1} = f$, $(\gamma, f) \in \mathfrak{R}$. By Theorem 3.1 and Lemma 3.7, Y_f is a minimal subfence in X_γ. Since $g|_{Y_g} = hf\psi = h\gamma$, we have that $(g, \gamma) \in \mathfrak{L}$ by Theorem 3.3. Hence, $(f, g) \in \mathfrak{D}$.

□

Chapter 4

The characterization of Green's equivalences for regular elements

In chapter 4, we focus on the characterization of Green's equivalences on the semigroup $O_E(X)$ for regular elements. The definition of regular element has been introduced in chapter 2. We start this section with the relation $\mathfrak{L}$.

Theorem 4.1. *Let $f, g \in O_E(X)$ be regular. Then $(f, g) \in \mathfrak{L}$ if and only if $Xf = Xg$.*

Proof. Assume that $(f, g) \in \mathfrak{L}$. Then there exist $h, k \in O_E(X)$ such that $hf = g$ and $kg = f$. Thus, $Xf \subseteq Xhf = Xg$ and $Xg \subseteq Xkg = Xf$. Then $Xg = Xf$. Conversely, assume that $Xf = Xg$. Since g is regular, there exists $h \in O_E(X)$ such that $g = ghg$. We define $\gamma : X \to X$ by $x\gamma = y$, where $y \in \operatorname{ran}(gh)$ and $yg = xf$. First, we will show that γ is well-defined, that is, we will show that for each $x \in X$, there exists a unique $y \in \operatorname{ran}(gh)$ such that $yg = xf$. Since $Xf = Xg = (Xgh)g$, the existence is clear. Next, let $y_1, y_2 \in \operatorname{ran}(gh)$ be such that $y_1g = y_2g$. Since $y_1, y_2 \in \operatorname{ran}(gh)$, there exist $a_1, a_2 \in X$ such that $y_1 = a_1gh$ and $y_2 = a_2gh$. Then $y_1 = a_1gh = a_1ghgh = y_1gh = y_2gh = a_2ghgh = a_2gh = y_2$. Similarly, for any $x_1, x_2 \in X$ and $y_1, y_2 \in \operatorname{ran}(gh)$ such that $y_1g = x_1f$ and $y_2g = x_2f$, we also have that $x_1 \leq x_2$ implies $y_1 \leq y_2$, and $(x_1, x_2) \in E$ implies $(y_1, y_2) \in E$. Hence, $\gamma \in O_E(X)$. Moreover, it is easy to see that $\gamma g = f$. Similarly, we can also show that $g = \delta f$ for some $\delta \in O_E(X)$. Therefore, $(f, g) \in \mathfrak{L}$. □

The immediate consequence of the Theorem 3.1 and Theroem 4.1 is the characterization of $\mathfrak{H}$. Since $\mathfrak{H} = \mathfrak{L} \cap \mathfrak{R}$, we get the following theorem.

Theorem 4.2. *Let $f, g \in O_E(X)$ be regular. Then $(f, g) \in \mathfrak{H}$ if and only if $Xf = Xg, \pi(f) = \pi(g)$ and $E(f) = E(g)$.*

Finally, we present the characterization of $\mathfrak{D}$ - equivalence for two regular elements in the following theorem.

Theorem 4.3. *Let $f, g \in O_E(X)$ be regular. Then $(f, g) \in \mathfrak{D}$ if and only if there exists $\phi : Xg \to Xf$ such that, for all $x, y \in Xg$, the following conditions hold :*

1. *$x \le y$ if and only if $x\phi \le y\phi$;*
2. *$(x, y) \in E$ if and only if $(x\phi, y\phi) \in E$.*

Proof. Assume that $(f, g) \in \mathfrak{D}$. Then there exists $h \in O_E(X)$ such that $(f, h) \in \mathfrak{L}$ and $(h, g) \in \mathfrak{R}$. Since $(h, g) \in \mathfrak{R}$, there is $\gamma \in O_E(X)$ such that $h = g\gamma$. Since $(f, h) \in \mathfrak{L}$, we have $Xf = Xh$, so $\operatorname{ran}\left(\gamma|_{\operatorname{ran}(g)}\right) \subseteq Xh = Xf$. We will show that $\gamma|_{\operatorname{ran}(g)}$ is a bijection satisfying conditions (1) and (2). Since $h = g\gamma$, we have $|Xh| = |Xg\gamma| \le |Xg|$. Since $Xf = Xh$, we get that $|Xf| \le |Xg|$. Similarly, since $(g, f) \in \mathfrak{D}$, we have that $|Xg| \le |Xf|$. Hence, $|Xg| = |Xf|$. Since $Xf = Xh = Xg\gamma|_{\operatorname{ran}(g)}$, we have that $\gamma|_{\operatorname{ran}(g)}$ is surjective, which also implies that $\gamma|_{\operatorname{ran}(g)}$ is bijective. Next, we will show that $\gamma|_{\operatorname{ran}(g)}$ satisfies condition (1). For a subset S of $X \times X$, let $\le \cap S = \{(a, b) \in S | a \le b\}$. Notice that since $\gamma|_{\operatorname{ran}(g)}$ is an order-preserving bijection, we can define an injective map $\gamma_1 : \le \cap (Xg \times Xg) \to \le \cap (Xf \times Xf)$ by $\gamma_1(x, y) = x\gamma, y\gamma$ for all $x, y \in \le \cap (Xg \times Xg)$. Since Xg and Xf are subfences of X and $|Xg| = |Xf|$, we have that $|\le \cap (Xg \times Xg)| = |\le \cap (Xf \times Xf)|$. Therefore, γ_1 is bijective, which implies that γ satisfies condition (1). Finally, we will show that $\gamma|_{\operatorname{ran}(g)}$ satisfies condition (2). We know that $\gamma|_{\operatorname{ran}(g)}$ is E- preserving. Now, let $x_1, x_2 \in \operatorname{ran}(g)$ be such that $(x_1\gamma, x_2\gamma) \in E$. Since $x_1 = a_1 g$ and $x_2 = a_2 g$ for some $a_1, a_2 \in X$, we have

$(a_1 g\gamma, a_2 g\gamma) \in E$. Then $(a_1 h, a_2 h) \in E$. Note that $E(h) = Ah^{-1}|A \in X/E, Ah^{-1} \neq \emptyset$. Let $a_1 h, a_2 h \in A_0$ for some $A_0 \in X/E$. Then $a_1, a_2 \in A_0 h^{-1} \in E(h)$. Since $E(h) = E(g)$, There exists $A_1 \in X/E$ such that $a_1, a_2 \in A_1 g^{-1}$. Then $a_1 g, a_2 g \in A_1$. Hence, $(a_1 g, a_2 g) \in E$ implying $(x_1, x_2) \in E$. Therefore, $\gamma|_{\mathrm{ran}(g)}$ satisfies condition (2).

Conversely, let $\phi : Xg \to Xf$ be a bijective function satisfying conditions (1) and (2). Let $h = g\phi$. Since ϕ satisfies condition (1), h is order-preserving. Moreover, since ϕ satisfies condition (2), $h \in O_E(X)$ and $E(h) = E(g)$. Since ϕ is a bijection, we have that $\pi(h) = \pi(g)$ and $Xh = Xf$. Hence, by Theorem 3.1 and Theorem 4.1, we have that $(h, g) \in \mathfrak{R}$ and $(h, f) \in \mathfrak{L}$, which implies that $(f, g) \in \mathfrak{D}$.

□

References

[1] Gracinda M. S. Gomes, Jean-eric Pin, Pedro V. Silva. (2002). Semigroups, Algorithms, Automata and Languages. World Scientific, 94.

[2] Green, J. A. (1951). On the structure of semigroups. Annals of Mathematics, 54(1), 163-172.

[3] Huisheng, P., Dingyu, Z. (2005). Green's equivalences on semigroups of transformations preserving order and an equivalence relation. Semigroup Forum, 71(2), 241–251.

[4] Jendana, K., Srithus, R. (2015). Coregularity of order-preserving self-mapping semigroups of fences. Communications of the Korean Mathematical Society, 30(4), 349–361.

[5] Tanyawong, R., Srithus, R., Chinram, R. (2016). Regular subsemigroups of the semigroups of transformations preserving a fence. Asian-European Journal of Mathematics, 9(1), 1650003.

Biography

Name MISTER Chaleamrach Malangpoo

Address 288/1 No. 5, Nongchang Sub-district,
Nongchang District, Uthai Thani, 61110.

Date of Birth 5 December 1996

Education

2018 Bachelor of Science in Mathematics,
Srinakharinwirot University.

2021 Master of Science in Mathematics,
Slipakorn University.