SKEW POLYNOMIALS AND SOME GENERALIZATIONS OF CIRCULANT MATRICES

By

Mr. Prarinya Morrakutjinda

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
Master of Science Program in Mathematics
Department of Mathematics
Graduate School, Silpakorn University
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พหุนามเสมือนและนัยทั่วไปบางประการของเมทริกซ์วัฏจักร

โดย
นายปริญญา มรกฎจินดา

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
ปีการศึกษา 2559

The Graduate School, Silpakorn University has approved and accredited the Thesis title of "Skew Polynomials and Some Generalizations of Circulant Matrices" submitted by Mr. Prarinya Morrakutjinda as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

……..

(Associate Professor Panjai Tantatsanawong, Ph.D.)

Dean of Graduate School

........../..................../..........

The Thesis Advisor

Somphong Jitman, Ph.D.

The Thesis Examination Committee

.... Chairman

(Ratana Srithus, Dr. rer. nat.)

............/......................../.............

.... Member

(Assistant Professor Sajee Pianskool, Ph.D.)

............/......................../.............

.... Member

(Somphong Jitman, Ph.D.)

............/......................../.............

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In this thesis, four generalizations of classical circulant matrices over the complex field $\mathbb{C}$ are introduced, namely, right conjugate-circulant, left conjugate-circulant, right conjugate-negacirculant and left conjugate-negacirculant matrices. Group structures of some subsets of such matrices are studied. Subsequently, some properties of skew polynomials over $\mathbb{C}$ are determined. The characterization of the set of all $n \times n$ right conjugate-circulant matrices over $\mathbb{C}$ (resp., the set of all $n \times n$ right conjugate-negacirculant matrices over $\mathbb{C}$) is given in terms of skew polynomials over $\mathbb{C}$.

Department of Mathematics Graduate School, Silpakorn University

Student's signature .... Academic Year 2016

Thesis Advisor's signature ....

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วิทยานิพนธ์นี้นำเสนอนัยทั่วไปของเมทริกซ์วัฏจักร 4 แบบ บนฟีลด์ $\mathbb{C}$ คือ เมทริกซ์วัฏจักรสังยุคขวา เมทริกซ์วัฏจักรสังยุคซ้าย เมทริกซ์วัฏจักรเชิงลบสังยุคขวา และเมทริกซ์วัฏจักรเชิงลบสังยุคซ้าย เราศึกษาโครงสร้างการเป็นกรุปของเซตย่อยบางเซตของเมทริกซ์เหล่านั้น พร้อมกันนี้ได้ศึกษาสมบัติบางประการของพหุนามเสมือนบนฟีลด์ $\mathbb{C}$ และใช้จำแนกโครงสร้างของเซตของเมทริกซ์วัฏจักรสังยุคขวาและเมทริกซ์วัฏจักรเชิงลบสังยุคขวามิติ $n \times n$ บนฟีลด์ $\mathbb{C}$ ในเทอมของพหุนามเสมือนบนฟีลด์ $\mathbb{C}$ ตามลำดับ

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
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Chapter 1

Introduction

The theory of circulant matrices over the complex field $\mathbb{C}$ has widely been studied and applied in many branches of Mathematics and Engineering. A circulant matrix is a matrix where each row is rotated one element to the right relative to the preceding row [4]. Precisely, an $n \times n$ circulant matrix is of the form

$$\begin{bmatrix} z_0 & z_1 & \cdots & z_{n-2} & z_{n-1} \\ z_{n-1} & z_0 & \cdots & z_{n-3} & z_{n-2} \\ z_{n-2} & z_{n-1} & \cdots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1 & z_2 & \cdots & z_{n-1} & z_0 \end{bmatrix}$$

for some $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$.

In applications, circulant matrices can be applied in solving linear systems using discrete Fourier tranform and they can be used in the MixColumns step of the Advanced Encryption Standard in cryptography. In [2], the eigenvalues of a circulant matrix have been studied. Determinants, norms, and the spread of circulant matrices with Tribohacci and generalized Lucas numbers have been studied in [6] and references therein. The probability that the determinant of an integer circulant $n \times n$ matrix is divisible by a prime p (where p does not divide n) have been studied in [9]. In [4] and [5], the inverse of a circulant matrix is studied.

It is well known [4] that the set of all circulant matrices is isomorphic to the ring $\mathbb{C}[x]/\langle x^n - 1\rangle$. In [8], some groups of circulant matrices have been studied.

Skew polynomials over the complex numbers have been studied in [3]. They have interesting properties and the set of all skew polynomials over $\mathbb{C}$ forms a non-commutative ring under the addition and multiplication defined in Chapter 2. In [1], some properties of skew polynomials have been studied and applied in coding theory.

In this thesis, four generalizations of circulant matrices are introduced, namely, right conjugate-circulant, left conjugate-circulant, right conjugate-negacirculant and left conjugate-negacirculant matrices. The algebraic characterizations and some properties of such matrices are studied in terms of skew polynomials over $\mathbb{C}$. These might motivate further study of properties of such matrices such as determinants, norms, diagonalizability etc. Moreover, applications of such matrices would be interesting for further studies.

The thesis is organized as follows. The formal definitions of right conjugate-circulant matrices, left conjugate-circulant matrices, right conjugate-negacirculant matrices, left conjugate-negacirculant matrices and skew polynomials over $\mathbb{C}$ are given in Chapter 2 as well as their basic properties. In Chapter 3, group structures of some subsets of such matrices are established. In Chapter 4 Section 4.1, the characterization of right conjugate-circulant matrices and their properties are established. The characterization of right conjugate-negacirculant matrices and their properties are given in Section 4.2. In Section 4.3, some relations among right conjugate-circulant matrices and right conjugate-negacirculant matrices are discussed.

Chapter 2

Preliminaries

In this chapter, some properties of skew polynomials over the complex field are discussed. The notions of complex right conjugate-circulant, left conjugate-circulant, right conjugate-negacirculant matrices and left conjugate-negacirculant matrices are mentioned.

2.1 Generalization of Circulant Matrices

For each $n \in \mathbb{N}$, let $M_n(\mathbb{C})$ denote the set of all $n \times n$ complex matrices and let $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\}$. Let $\xi : \mathbb{C} \to \mathbb{C}$ denote the complex conjugate, i.e., $\xi(z) = \overline{z}$. An $n \times n$ matrix A over $\mathbb{C}$ is said to be *right conjugate-circulant* (resp., *left conjugate-circulant*) if

$$A = \begin{bmatrix} z_0 & z_1 & \dots & z_{n-2} & z_{n-1} \\ \xi(z_{n-1}) & \xi(z_0) & \dots & \xi(z_{n-3}) & \xi(z_{n-2}) \\ \xi^2(z_{n-2}) & \xi^2(z_{n-1}) & \dots & \xi^2(z_{n-4}) & \xi^2(z_{n-3}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi^{n-1}(z_1) & \xi^{n-1}(z_2) & \dots & \xi^{n-1}(z_{n-1}) & \xi^{n-1}(z_0) \end{bmatrix}$$

$$(resp., A = \begin{bmatrix} z_0 & z_1 & \dots & z_{n-2} & z_{n-1} \\ \xi(z_1) & \xi(z_2) & \dots & \xi(z_{n-1}) & \xi(z_0) \\ \xi^2(z_2) & \xi^2(z_3) & \dots & \xi^2(z_0) & \xi^2(z_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi^{n-1}(z_{n-1}) & \xi^{n-1}(z_0) & \dots & \xi^{n-1}(z_{n-3}) & \xi^{n-1}(z_{n-2}) \end{bmatrix})$$

for some $(z_0, z_1, \dots, z_{n-1}) \in \mathbb{C}^n$. A right (resp., *left*) conjugate-circulant matrix of this form is denoted by $\mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \dots, z_{n-1}))$ (resp., $\mathrm{lcir}_{\mathrm{conj}}((z_0, z_1, \dots, z_{n-1}))$).

In similar fashion, an $n \times n$ matrix A over $\mathbb{C}$ is said to be *right conjugate-negacirculant* (resp., *left conjugate-negacirculant*) if

$$A = \begin{bmatrix} z_0 & z_1 & \dots & z_{n-2} & z_{n-1} \\ \xi(-z_{n-1}) & \xi(z_0) & \dots & \xi(z_{n-3}) & \xi(z_{n-2}) \\ \xi^2(-z_{n-2}) & \xi^2(-z_{n-1}) & \dots & \xi^2(z_{n-4}) & \xi^2(z_{n-3}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi^{n-1}(-z_1) & \xi^{n-1}(-z_2) & \dots & \xi^{n-1}(-z_{n-1}) & \xi^{n-1}(z_0) \end{bmatrix}$$

$$(resp., A = \begin{bmatrix} z_0 & z_1 & \dots & z_{n-2} & z_{n-1} \\ \xi(z_1) & \xi(z_2) & \dots & \xi(z_{n-1}) & \xi(-z_0) \\ \xi^2(z_2) & \xi^2(z_3) & \dots & \xi^2(-z_0) & \xi^2(-z_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi^{n-1}(z_{n-1}) & \xi^{n-1}(-z_0) & \dots & \xi^{n-1}(-z_{n-3}) & \xi^{n-1}(-z_{n-2}) \end{bmatrix})$$

for some $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$. It is denoted by $\mathrm{rncir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))$ (resp., $\mathrm{lncir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))$).

Such matrices become the classical circulant and negacirculant matrices if ξ is replaced by the identity map.

Since ξ^2 is the identity map, we have

$$\mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})) = \begin{cases} \begin{bmatrix} z_0 & z_1 & \cdots & z_{n-2} & z_{n-1} \\ \overline{z_{n-1}} & \overline{z_0} & \cdots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ z_{n-2} & z_{n-1} & \cdots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{z_2} & \overline{z_3} & \cdots & \overline{z_0} & \overline{z_1} \\ z_1 & z_2 & \cdots & z_{n-1} & z_0 \end{bmatrix} & \text{if } n \text{ is odd,} \\ \begin{bmatrix} z_0 & z_1 & \cdots & z_{n-2} & z_{n-1} \\ \overline{z_{n-1}} & \overline{z_0} & \cdots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ z_{n-2} & z_{n-1} & \cdots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_2 & z_3 & \cdots & z_0 & z_1 \\ \overline{z_1} & \overline{z_2} & \cdots & \overline{z_{n-1}} & \overline{z_0} \end{bmatrix} & \text{if } n \text{ is even,} \end{cases}$$

$$
\mathrm{lcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})) = \begin{cases} \begin{bmatrix} z_0 & z_1 & \cdots & z_{n-2} & z_{n-1} \\ \overline{z_1} & \overline{z_2} & \cdots & \overline{z_{n-1}} & \overline{z_0} \\ z_2 & z_3 & \cdots & z_0 & z_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{z_{n-2}} & \overline{z_{n-1}} & \cdots & \overline{z_{n-4}} & \overline{z_{n-3}} \\ z_{n-1} & z_0 & \cdots & z_{n-3} & z_{n-2} \end{bmatrix} & \text{if } n \text{ is odd,} \\[2em] \begin{bmatrix} z_0 & z_1 & \cdots & z_{n-2} & z_{n-1} \\ \overline{z_1} & \overline{z_2} & \cdots & \overline{z_{n-1}} & \overline{z_0} \\ z_2 & z_3 & \cdots & z_0 & z_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{n-2} & z_{n-1} & \cdots & z_{n-4} & z_{n-3} \\ \overline{z_{n-1}} & \overline{z_0} & \cdots & \overline{z_{n-3}} & \overline{z_{n-2}} \end{bmatrix} & \text{if } n \text{ is even,} \end{cases}
$$

and

$$\operatorname{rncir}_{\mathrm{conj}}((z_0, z_1, \dots, z_{n-1})) = \begin{cases} \begin{bmatrix} z_0 & z_1 & \cdots & z_{n-2} & z_{n-1} \\ -\overline{z_{n-1}} & \overline{z_0} & \cdots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ -z_{n-2} & -z_{n-1} & \cdots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\overline{z_2} & -\overline{z_3} & \cdots & \overline{z_0} & \overline{z_1} \\ -z_1 & -z_2 & \cdots & -z_{n-1} & z_0 \end{bmatrix} & \text{if } n \text{ is odd,} \\ \begin{bmatrix} z_0 & z_1 & \cdots & z_{n-2} & z_{n-1} \\ -\overline{z_{n-1}} & \overline{z_0} & \cdots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ -z_{n-2} & -z_{n-1} & \cdots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -z_2 & -z_3 & \cdots & z_0 & z_1 \\ -\overline{z_1} & -\overline{z_2} & \cdots & -\overline{z_{n-1}} & \overline{z_0} \end{bmatrix} & \text{if } n \text{ is even,} \end{cases}$$

$$\operatorname{lncir}_{\text{conj}}((z_0, z_1, \ldots, z_{n-1})) = \begin{cases} \begin{bmatrix} z_0 & z_1 & \ldots & z_{n-2} & z_{n-1} \\ \overline{z_1} & \overline{z_2} & \ldots & \overline{z_{n-1}} & -\overline{z_0} \\ z_2 & z_3 & \ldots & -z_0 & -z_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{z_{n-2}} & \overline{z_{n-1}} & \ldots & -\overline{z_{n-4}} & -\overline{z_{n-3}} \\ z_{n-1} & -z_0 & \ldots & -z_{n-3} & -z_{n-2} \end{bmatrix} & \text{if } n \text{ is odd,} \\ \begin{bmatrix} z_0 & z_1 & \ldots & z_{n-2} & z_{n-1} \\ \overline{z_1} & \overline{z_2} & \ldots & \overline{z_{n-1}} & -\overline{z_0} \\ z_2 & z_3 & \ldots & -z_0 & -z_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{n-2} & z_{n-1} & \ldots & -z_{n-4} & -z_{n-3} \\ \overline{z_{n-1}} & -\overline{z_0} & \ldots & -\overline{z_{n-3}} & -\overline{z_{n-2}} \end{bmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Example 2.1. The matrices

$$\operatorname{rcir}_{\text{conj}}((1, 1-i, 2, 2+i)) = \begin{bmatrix} 1 & 1-i & 2 & 2+i \\ 2-i & 1 & 1+i & 2 \\ 2 & 2+i & 1 & 1-i \\ 1+i & 2 & 2-i & 1 \end{bmatrix}$$

and

$$\text{rncir}_{\text{conj}}((1, 1-i, 2, 2+i)) = \begin{bmatrix} 1 & 1-i & 2 & 2+i \\ -2+i & 1 & 1+i & 2 \\ -2 & -2-i & 1 & 1-i \\ -1-i & -2 & -2+i & 1 \end{bmatrix}$$

are right conjugate-circulant and right conjugate-negacirculant, respectively. Clearly, they are neither right circulant nor right negacirculant.

Denote by $\text{RCir}_{n,\text{rconj}}(\mathbb{C}) := \{\text{rcir}_{\text{conj}}(\boldsymbol{z}) \mid \boldsymbol{z} \in \mathbb{C}^n\}$ and $\text{RNCir}_{n,\text{rconj}}(\mathbb{C}) := \{\text{rncir}_{\text{conj}}(\boldsymbol{z}) \mid \boldsymbol{z} \in \mathbb{C}^n\}$ the set of complex $n \times n$ right conjugate-circulant matrices and the set of complex $n \times n$ right conjugate-negacirculant matrices, respectively.

Example 2.2. The matrices

$$\text{lcir}_{\text{conj}}((1, 1-i, 2, 2+i)) = \begin{bmatrix} 1 & 1-i & 2 & 2+i \\ 1+i & 2 & 2-i & 1 \\ 2 & 2+i & 1 & 1-i \\ 2-i & 1 & 1+i & 2 \end{bmatrix}$$

and

$$\text{lncir}_{\text{conj}}((1, 1-i, 2, 2+i)) = \begin{bmatrix} 1 & 1-i & 2 & 2+i \\ 1+i & 2 & 2-i & -1 \\ 2 & 2+i & -1 & -1+i \\ 2-i & -1 & -1-i & -2 \end{bmatrix}$$

are left conjugate-circulant and left conjugate-negacirculant, respectively.

Denote by $\text{LCir}_{n,\text{rconj}}(\mathbb{C}) := \{\text{lcir}_{\text{rconj}}(\boldsymbol{z}) \mid \boldsymbol{z} \in \mathbb{C}^n\}$ and $\text{LNCir}_{n,\text{rconj}}(\mathbb{C}) := \{\text{lncir}_{\text{conj}}(\boldsymbol{z}) \mid \boldsymbol{z} \in \mathbb{C}^n\}$ the set of complex $n \times n$ left conjugate-circulant matrices and the set of complex $n \times n$ left conjugate-negacirculant matrices, respectively.

The set $\mathrm{Cir}_{n,\mathrm{rconj}}(\mathbb{C}) := \mathrm{RCir}_{n,\mathrm{rconj}}(\mathbb{C}) \cup \mathrm{LCir}_{n,\mathrm{rconj}}(\mathbb{C})$ is called the *set of complex $n \times n$ conjugate-circulant matrices* and an element in $\mathrm{Cir}_{n,\mathrm{rconj}}(\mathbb{C})$ is called a *conjugate-circulant matrix* over $\mathbb{C}$. The set $\mathrm{NCir}_{n,\mathrm{rconj}}(\mathbb{C}) := \mathrm{RNCir}_{n,\mathrm{rconj}}(\mathbb{C}) \cup \mathrm{LNCir}_{n,\mathrm{rconj}}(\mathbb{C})$ is called the *set of complex $n \times n$ conjugate-negacirculant matrices* and an element in $\mathrm{NCir}_{n,\mathrm{rconj}}(\mathbb{C})$ is called a *conjugate-negacirculant matrix* over $\mathbb{C}$. For convenience, the indices of a matrix $[c_{ij}]_{n\times n} \in M_n(\mathbb{C})$ will be written as $0, 1, 2, \ldots, n-1$ and the computations will be done under modulo n.

2.2 Skew Polynomials

Skew polynomials over the complex field are recalled. Proofs of necessary properties are given. The readers may refer to [2, Chapter 2] for more details.

The set $\mathbb{C}[x : \mathrm{conj}] = \{z_0 + z_1x + \cdots + z_nx^n | z_i \in \mathbb{C} \text{ and } n \in \mathbb{N}_0\}$ of formal polynomials forms a ring under the usual addition of polynomials and where the multiplication is defined using the rule $xz = \bar{z}x$. The multiplication is extended to all elements in $\mathbb{C}[x : \mathrm{conj}]$ by associativity and distributivity. The ring $\mathbb{C}[x : \mathrm{conj}]$ is called the *skew polynomial ring* over $\mathbb{C}$ and an element in $\mathbb{C}[x : \mathrm{conj}]$ is called a *skew polynomial*. Clearly, the ring $\mathbb{C}[x : \mathrm{conj}]$ is non-commutative.

Given a ring R , an additive subgroup $I \subseteq R$ is called a *left* (resp., *right*) *ideal* of R if $ra \in I$ (resp., $ar \in I$) for all $r \in R$ and $a \in I$. It is said to be *two-sided ideal* if I is both a left ideal and a right ideal.

For each skew polynomial $f(x)$ in $\mathbb{C}[x : \mathrm{conj}]$, let $\langle f(x)\rangle := \{g(x)f(x) \mid g(x) \in \mathbb{C}[x : \mathrm{conj}]\}$ be the left ideal of $\mathbb{C}[x : \mathrm{conj}]$ generated by $f(x)$. Note that $\langle f(x)\rangle$ does not need to be two-sided. A polynomial $f(x)$ is said to be *central* if $f(x)g(x) = g(x)f(x)$ for all $g(x) \in \mathbb{C}[x : \mathrm{conj}]$.

Necessary and sufficient conditions for a left ideal $\langle x^n \pm 1\rangle$ to be two-sided are given as follows.

Proposition 2.3. *Let n be a positive integer. Then the following statements are equivalent:*

i) $x^n \pm 1$ is central in $\mathbb{C}[x : \mathrm{conj}]$

ii) $\langle x^n \pm 1\rangle$ *is two-sided.*

iii) n *is even.*

Proof. The statement i) implies ii) is clear.

To prove the statement ii) implies iii), assume that $\langle x^n \pm 1\rangle$ is two-sided. Suppose that n is odd. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then $zx^n \pm z = z\,(x^n \pm 1) = (x^n \pm 1)\,w = \overline{w}x^n \pm w$ for some $w \in \mathbb{C}$. Comparing the coefficients, we have $w = z = \overline{w}$, a contradiction.

Finally, we prove that the statement iii) implies i). Assume that n is even. Then

$$x(x^n \pm 1) = x^{n+1} \pm x = (x^n \pm 1)x \quad \text{and} \quad (x^n \pm 1)z = zx^n \pm z = z(x^n \pm 1)$$

for all $z \in \mathbb{C}$. Consequently, $x^n \pm 1$ commutes with every skew polynomial in $\mathbb{C}[x : \mathrm{conj}]$. $\square$

From Proposition 2.3, it follows that $\mathbb{C}[x : \mathrm{conj}]/\langle x^n \pm 1\rangle$ is well defined as a quotient ring if and only if n is even. In this case, the ring $\mathbb{C}[x : \mathrm{conj}]/\langle x^n \pm 1\rangle$ plays an important role in characterizing the right conjugate-circulant and right conjugate-negacirculant matrices.

Chapter 3

Group structures of Some Generalizations of Circulant Matrices

In this chapter, some properties of right conjugate-circulant matrices, left conjugate-circulant matrices, righ conjugate-negacirculant matrices and left conjugate-negacirculant matrices are discussed. The group structures of some subsets of such matrices are mentioned.

3.1 Conjugate-Circulant Matrices

In this section, we focus on the group structures of some subsets of conjugate-circulant matrices.

It is easy to see that $(\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C}), +)$ and $(\mathrm{LCir}_{n,\mathrm{conj}}(\mathbb{C}), +)$ are groups for all $n \in \mathbb{N}$. Since $\mathrm{Cir}_{1,\mathrm{conj}}(\mathbb{C}) \cong \mathbb{C}$, the structure $(\mathrm{Cir}_{1,\mathrm{conj}}(\mathbb{C}), +)$ is a group. Since $\mathrm{Cir}_{2,\mathrm{conj}}(\mathbb{C}) = \mathrm{LCir}_{2,\mathrm{conj}}(\mathbb{C})$ and $(\mathrm{LCir}_{2,\mathrm{conj}}(\mathbb{C}), +)$ is a group, the structure $(\mathrm{Cir}_{2,\mathrm{conj}}(\mathbb{C}), +)$ is a group. If $n \geq 3$, then

$$\mathrm{rcir}_{\mathrm{conj}}((1, 0, \ldots, 0)) + \mathrm{lcir}_{\mathrm{conj}}((0, \ldots, 0, 1)) \notin \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$$

and

$$\mathrm{rcir}_{\mathrm{conj}}((1, 0, \ldots, 0)) + \mathrm{lcir}_{\mathrm{conj}}((0, \ldots, 0, 1)) \notin \mathrm{LCir}_{n,\mathrm{conj}}(\mathbb{C}).$$

It follows that $(\mathrm{Cir}_{n,\mathrm{conj}}(\mathbb{C}),+)$ is not a group under the usual addition with $n \geq 3$. Next, we focus on invertible matrices in $\mathrm{RCir}_{n,\mathrm{rconj}}(\mathbb{C}), \mathrm{LCir}_{n,\mathrm{rconj}}(\mathbb{C})$ and $\mathrm{Cir}_{n,\mathrm{rconj}}(\mathbb{C})$.

The set $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}) := \{A \in \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C}) \mid \det(A) \neq 0\}$ is the set of invertible complex $n \times n$ right conjugate-circulant matrices. The set $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) := \{A \in \mathrm{LCir}_{n,\mathrm{conj}}(\mathbb{C}) \mid \det(A) \neq 0\}$ is the set of invertible complex $n \times n$ left conjugate-circulant matrices. The set $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C}) := \{A \in \mathrm{Cir}_{n,\mathrm{conj}}(\mathbb{C}) \mid \det(A) \neq 0\}$ is the set of invertible complex $n \times n$ conjugate-circulant matrices.

The following relations between left and right conjugate-circulant matrices can be obtained by the direct calculation.

Lemma 3.1. Let n be an even positive integer. Let $\boldsymbol{z} = (z_0, z_1, z_2, \ldots, z_{n-1}) \in \mathbb{C}^n$. Then $H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z}) = \mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{z})$, where $H = \begin{bmatrix} 1 & O_1 \\ O_1^T & \widetilde{I}_{n-1} \end{bmatrix}$, $\widetilde{I}_{n-1} = \mathrm{adiag}\,(1, 1, \ldots, 1)_{(n-1)\times(n-1)}$ is an antidiagonal matrix and $O_1 = \underbrace{(0, 0, \ldots, 0)}_{n-1 \text{ copies}}$.

Proof. We observe that

$$H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z}) = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} z_0 & z_1 & \ldots & z_{n-2} & z_{n-1} \\ \overline{z_{n-1}} & \overline{z_0} & \ldots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ z_{n-2} & z_{n-1} & \ldots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_2 & z_3 & \ldots & z_0 & z_1 \\ \overline{z_1} & \overline{z_2} & \ldots & \overline{z_{n-1}} & \overline{z_0} \end{bmatrix}$$

$$= \begin{bmatrix} z_0 & z_1 & \dots & z_{n-2} & z_{n-1} \\ \overline{z_1} & \overline{z_2} & \dots & \overline{z_{n-1}} & \overline{z_0} \\ z_2 & z_3 & \dots & z_0 & z_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{n-2} & z_{n-1} & \dots & z_{n-4} & z_{n-3} \\ \overline{z_{n-1}} & \overline{z_0} & \dots & \overline{z_{n-3}} & \overline{z_{n-2}} \end{bmatrix} = \mathrm{lcir}_{\mathsf{conj}}(\boldsymbol{z}).$$

Hence, $H\mathrm{rcir}_{\mathsf{conj}}(\boldsymbol{z}) = \mathrm{lcir}_{\mathsf{conj}}(\boldsymbol{z})$. □

Lemma 3.2. Let n be an even positive integer. Let $\boldsymbol{z} = (z_0, z_1, z_2, \dots, z_{n-1}) \in \mathbb{C}^n$. Then $\mathrm{rcir}_{\mathsf{conj}}(\boldsymbol{z})H = \mathrm{lcir}_{\mathsf{conj}}(\boldsymbol{\gamma})$ where $\boldsymbol{\gamma} = (z_0, z_{n-1}, z_{n-2}, \dots, z_2, z_1)$ and

$$H = \begin{bmatrix} 1 & O_1 \\ O_1^T & \widetilde{I}_{n-1} \end{bmatrix}.$$

Proof. We observe that

$$\mathrm{rcir}_{\mathsf{conj}}(\boldsymbol{z})H = \begin{bmatrix} z_0 & z_1 & \dots & z_{n-2} & z_{n-1} \\ \overline{z_{n-1}} & \overline{z_0} & \dots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ z_{n-2} & z_{n-1} & \dots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_2 & z_3 & \dots & z_0 & z_1 \\ \overline{z_1} & \overline{z_2} & \dots & \overline{z_{n-1}} & \overline{z_0} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z_0 & z_{n-1} & \cdots & z_2 & z_1 \\ \overline{z_{n-1}} & \overline{z_{n-2}} & \cdots & \overline{z_1} & \overline{z_0} \\ z_{n-2} & z_{n-3} & \cdots & z_0 & z_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_2 & z_1 & \cdots & z_4 & z_3 \\ \overline{z_1} & \overline{z_0} & \cdots & \overline{z_3} & \overline{z_2} \end{bmatrix} = \mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{\gamma}).$$

Hence, $\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z})H = \mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{\gamma})$. □

Next, we focus on the multiplicative group structures of $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$, $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ and $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$.

A necessary and sufficient condition for the set $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$ of $n \times n$ invertible complex right conjugate-circulant matrices to be a group under the usual matrix multiplication is given in the next theorem.

Theorem 3.3. *Let n be a positive integer. Then $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication if and only if $n = 1$ or n is even.*

Proof. Suppose $n \neq 1$ and n is odd. Let $\boldsymbol{a} = (2i, i, 1, 0, \ldots, 0)$. Then $\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})$ is invertible since $\det(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})) = (2^n + 1)i \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 .$$

Then

$$\begin{aligned} \overline{c}_{n-2,n-1} &= \overline{0 + \cdots + 0 + (-2) + 2} \\ &= \overline{0} \\ &= 0 \\ &\neq -4 \\ &= -2 + 0 + \cdots + 0 + (-2) \\ &= c_{n-1,0}. \end{aligned}$$

Hence, $(\mathrm{rcir}_{\mathrm{conj}}(\mathbf{a}))^2 \notin \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Conversely, assume that $n = 1$ or n is even. If $n = 1$, then $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}) \cong \mathbb{C} \setminus \{0\} \cong GL_n(\mathbb{C})$ is a group. Next, we consider the case where n is even.

Let $\mathrm{rcir}_{\mathrm{conj}}((a_0, a_1, \ldots, a_{n-1}))$ and $\mathrm{rcir}_{\mathrm{conj}}((b_0, b_1, \ldots, b_{n-1}))$ be elements in $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Let $[c_{ij}]_{n\times n} := \mathrm{rcir}_{\mathrm{conj}}((a_0, a_1, \ldots, a_{n-1}))\mathrm{rcir}_{\mathrm{conj}}((b_0, b_1, \ldots, b_{n-1}))$. Then, for each $0 \leq i, j \leq n-1$, we have

$$c_{ij} = \begin{cases} [a_{n-i}\ a_{n-i+1} \ldots a_{n-i-1}]\left[b_j\ \overline{b_{j-1}}\ bj-2 \ldots \overline{b_{j+1}}\right]^T & \text{if } i \text{ is even,} \\ [\overline{a_{n-i}}\ \overline{a_{n-i+1}} \ldots \overline{a_{n-i-1}}]\left[b_j\ \overline{b_{j-1}}\ bj-2 \ldots \overline{b_{j+1}}\right]^T & \text{if } i \text{ is odd.} \end{cases}$$

Precisely, for each $0 \leq i, j \leq n-1$, c_{ij} is of the form

$$\begin{aligned} \overline{c_{ij}} &= \overline{\sum_{k=0}^{\frac{n-2}{2}} a_{n-i+2k}b_{j-2k} + \sum_{k=0}^{\frac{n-2}{2}} a_{n-i+(2k+1)}\overline{b_{j-(2k+1)}}} \\ &= \sum_{k=0}^{\frac{n-2}{2}} \overline{a_{n-i+2k}b_{j-2k}} + \sum_{k=0}^{\frac{n-2}{2}} \overline{a_{n-i+(2k+1)}}b_{j-(2k+1)} \\ &= \sum_{k=0}^{\frac{n-2}{2}} \overline{a_{n-i+(2k+1)}}b_{j-(2k+1)} + \sum_{k=0}^{\frac{n-2}{2}} \overline{a_{n-i+2k}b_{j-2k}} \\ &= c_{i+1,j+1} \end{aligned}$$

if i is even, and

$$\begin{aligned} \overline{c_{ij}} &= \overline{\sum_{k=0}^{\frac{n-2}{2}} \overline{a_{n-i+2k}}b_{j-2k} + \sum_{k=0}^{\frac{n-2}{2}} \overline{a_{n-i+(2k+1)}}b_{j-(2k+1)}} \\ &= \sum_{k=0}^{\frac{n-2}{2}} a_{n-i+2k}\overline{b_{j-2k}} + \sum_{k=0}^{\frac{n-2}{2}} a_{n-i+(2k+1)}b_{j-(2k+1)} \\ &= \sum_{k=0}^{\frac{n-2}{2}} a_{n-i+(2k+1)}b_{j-(2k+1)} + \sum_{k=0}^{\frac{n-2}{2}} a_{n-i+2k}\overline{b_{j-2k}} \\ &= c_{i+1,j+1} \end{aligned}$$

if i is odd. It follows that

$$\mathrm{rcir}_{\mathrm{conj}}((a_0, a_1, \ldots, a_{n-1}))\mathrm{rcir}_{\mathrm{conj}}((b_0, b_1, \ldots, b_{n-1})) \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$ be an element in $\mathbb{C}^n$ such that

$$A := (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})) \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Then there exists a unique $B = [b_{ij}]_{n\times n}$ in $GL_n(\mathbb{C})$ such that $AB = I_n$. We will show that $B \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Note that

$$A \left[b_{0,0}\ b_{1,0} \ldots b_{n-1,0}\right]^T = [1\ 0 \ldots 0]^T.$$

From the equation above, we have the following system of equations.

$$\begin{aligned}
a_0 b_{0,0} + a_1 b_{1,0} + a_2 b_{2,0} + \cdots + a_{n-1} b_{n-1,0} &= 1,\\
\overline{a}_{n-1} b_{0,0} + \overline{a}_0 b_{1,0} + \overline{a}_1 b_{2,0} + \cdots + \overline{a}_{n-2} b_{n-1,0} &= 0,\\
a_{n-2} b_{0,0} + a_{n-1} b_{1,0} + a_0 b_{2,0} + \cdots + a_{n-3} b_{n-1,0} &= 0,\\
&\ \vdots\\
\overline{a}_1 b_{0,0} + \overline{a}_2 b_{1,0} + \overline{a}_3 b_{2,0} + \cdots + \overline{a}_0 b_{n-1,0} &= 0.
\end{aligned}$$

Move the last equation to the top and apply the conjugation to all equations, we conclude that

$$\begin{aligned}
a_0 \overline{b}_{n-1,0} + a_1 \overline{b}_{0,0} + a_2 \overline{b}_{1,0} + \cdots + a_{n-1} \overline{b}_{n-2,0} &= 0,\\
\overline{a_{n-1} b_{n-1,0}} + \overline{a_0 b_{0,0}} + \overline{a_1 b_{1,0}} + \cdots + \overline{a_{n-2} b_{n-2,0}} &= 1,\\
a_{n-2} \overline{b}_{n-1,0} + a_{n-1} \overline{b}_{0,0} + a_0 \overline{b}_{1,0} + \cdots + a_{n-3} \overline{b}_{n-2,0} &= 0,\\
&\ \vdots\\
\overline{a_1 b_{n-1,0}} + \overline{a_2 b_{0,0}} + \overline{a_3 b_{1,0}} + \cdots + \overline{a_0 b_{n-2,0}} &= 0.
\end{aligned}$$

Hence,

$$A \left[\overline{b}_{n-1,0}\ \overline{b}_{0,0} \ldots \overline{b}_{n-2,0}\right]^T = [0\ 1\ 0 \ldots 0]^T.$$

Continue this process, we have

$$A^{-1} = B = \mathrm{rcir}_{\mathrm{conj}}((b_{0,0}, \overline{b}_{n-1,0}, \ldots, b_{2,0}, \overline{b}_{1,0})) \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Therefore, $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$ as desired. □

A necessary and sufficient condition for the set $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ of $n \times n$ invertible complex left conjugate-circulant matrices to be a group under the usual matrix multiplication is given in the next theorem.

Theorem 3.4. *Let n be a positive integer. Then $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication if and only if $n = 1$ or $n = 2$.*

Proof. Assume that $n \neq 1$ and $n \neq 2$. If n is odd, we consider the following 2 cases.

Case 1: $n = 3$. Let $\boldsymbol{a} = (i, i, 1)$. Then $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ since $\det(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})) = -2i \neq 0$. It follows that

$$(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 = \begin{bmatrix} 1 & -1+2i & 1+2i \\ 1-2i & 3 & 1-2i \\ 1+2i & -1+2i & 1 \end{bmatrix} \notin \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Case 2 : $n \geq 5$. Let $\boldsymbol{a} = (0, \ldots, 0, i, 2i)$. Then $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ since $\det(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})) = (2^n + 1)i \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 .$$

Then

$$\begin{aligned} \overline{c}_{n-2,0} &= \overline{0 + \cdots + 0} \\ &= \overline{0} \\ &= 0 \\ &\neq -5 \\ &= (-4) + 0 + \cdots + 0 + (-1) \\ &= c_{n-1,n-1}. \end{aligned}$$

Hence, $(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Next, we consider the case when n is even. Let $\boldsymbol{a} = (0, \ldots, 0, i, 2i)$. Then $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ because $\det(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})) = -2^{n+1} + 1 \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 .$$

Then

$$\begin{aligned}\overline{c}_{n-2,0} &= \overline{0+\cdots+0} \\ &= \overline{0} \\ &= 0 \\ &\neq 3 \\ &= 4+0+\cdots+0+(-1) \\ &= c_{n-1,n-1}.\end{aligned}$$

Hence, $(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Conversely, assume that $n = 1$ or $n = 2$. We consider 2 cases the following.
Case 1 : $n = 1$. Then $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) \cong \mathbb{C} \setminus \{0\} \cong GL_n(\mathbb{C})$ is a group.
Case 2 : $n = 2$. Then $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) = \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is a group by Theorem 3.3.
From Cases 1 and 2, the set $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication. □

A necessary and sufficient condition for the set $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ of $n \times n$ invertible complex conjugate-circulant matrices to be a group under the usual matrix multiplication is given in the next theorem.

Theorem 3.5. *Let n be a positive integer. Then the set $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication if and only if $n = 1$ or n is even.*

Proof. Suppose $n \neq 1$ and n is odd. Then we consider the following 2 cases.
Case 1 : $n = 3$. Let $\boldsymbol{a} = (i, i, 1)$. Then $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ because $\det(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})) = -2i \neq 0$. It follows that

$$(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 = \begin{bmatrix} 1 & -1+2i & 1+2i \\ 1-2i & 3 & 1-2i \\ 1+2i & -1+2i & 1 \end{bmatrix} \notin \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

and

$$(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 = \begin{bmatrix} 1 & -1+2i & 1+2i \\ 1-2i & 3 & 1-2i \\ 1+2i & -1+2i & 1 \end{bmatrix} \notin \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Hence, $(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}) \cup \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) = \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$.

Case 2 : $n \geq 5$. Let $\boldsymbol{a} = (0, \ldots, 0, i, 2i)$. Then $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ because $\det(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})) = (2^n + 1)i \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2.$$

Since

$$\begin{aligned} \overline{c}_{n-2,0} &= \overline{0 + \cdots + 0} \\ &= \overline{0} \\ &= 0 \\ &\neq -5 \\ &= (-4) + 0 + \cdots + 0 + (-1) \\ &= c_{n-1,n-1}, \end{aligned}$$

$(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

Since

$$\begin{aligned} \overline{c}_{n-2,n-1} &= \overline{2 + 0 + \cdots + 0} \\ &= \overline{2} \\ &= 2 \\ &\neq -2 \\ &= 0 + \cdots + 0 + (-2) \\ &= c_{n-1,0}, \end{aligned}$$

$(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $(\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$.

From Cases 1 and 2, $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Conversely, assume that $n = 1$ or n is even. If $n = 1$, then $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C}) \cong \mathbb{C} \setminus \{0\} \cong GL_n(\mathbb{C})$ is a group. Next, we consider the case where n is even.

Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1})$ be elements in $\mathbb{C}^n$. Then we consider the following 4 cases.

Case 1: $\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})$ and $\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b})$ are elements in $\widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

By Theorem 3.3, $\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b}) \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$.

Case 2: $\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{b}) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

By Lemma 3.2, $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{b}) = \mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})H$ for some $\boldsymbol{c} \in \mathbb{C}^n$. We have that

$$\begin{aligned}\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{b}) &= (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a}))(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})H) \\ &= (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c}))H \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C}).\end{aligned}$$

Case 3: $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b}) \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

By Lemma 3.1, $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) = H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})$. We have that

$$\begin{aligned}\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b}) &= (H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a}))(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b})) \\ &= H(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b})) \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C}).\end{aligned}$$

Case 4: $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})$ and $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{b})$ be elements in $\widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

By Lemmas 3.2 and 3.1, $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a}) = \mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})H$ for some $\boldsymbol{c} \in \mathbb{C}^n$ and $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{b}) = H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b})$. It follows that

$$\begin{aligned}\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{b}) &= (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})H)(H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b})) \\ &= \mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})H^2\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b}) \\ &= \mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})I_n\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b}) \\ &= \mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{b}).\end{aligned}$$

By Theorem 3.3, $\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{lcir}_{\mathrm{conj}}(\boldsymbol{b}) \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$.

From Cases 1–4, $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ is closed under multiplication.

Next, let A be an element in $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$. Then we consider the following 2 cases.

Case 1: $A \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Then by Theorem 3.3, $A^{-1} \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$.

Case 2: $A \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Then by Lemma 3.1, $A = H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c})$ for some $\boldsymbol{c} \in \mathbb{C}^n$.

We have that

$$
\begin{aligned}
A^{-1} &= (H\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1} \\
&= (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1}H^{-1} \\
&= (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1}H \quad (\text{ since } H^2 = I_n) \\
&\in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) \qquad (\text{ since } (\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1} \in \widehat{\mathrm{RCir}}_{n,\mathrm{conj}}(\mathbb{C})).
\end{aligned}
$$

Hence, $A^{-1} \in \widehat{\mathrm{LCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$.

From Cases 1 and 2, A^{-1} is an element in $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$. It follows that the set $\widehat{\mathrm{Cir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication. □

3.2 Conjugate-Negacirculant Matrices

In this section, we focus on the group structures of some subsets of conjugate-negacirculant matrices.

It is easy to see that $(\mathrm{RNCir}_{n,\mathrm{rconj}}(\mathbb{C}), +)$ and $(\mathrm{LNCir}_{n,\mathrm{rconj}}(\mathbb{C}), +)$ are groups for all $n \in \mathbb{N}$. Since $\mathrm{NCir}_{1,\mathrm{conj}}(\mathbb{C}) \cong \mathbb{C}$, the structure $(\mathrm{NCir}_{1,\mathrm{conj}}(\mathbb{C}), +)$ is a group. Since

$$\mathrm{rncir}_{\mathrm{conj}}((1,0)) + \mathrm{lncir}_{\mathrm{conj}}((0,1)) \notin \mathrm{RNCir}_{2,\mathrm{conj}}(\mathbb{C})$$

and

$$\mathrm{rncir}_{\mathrm{conj}}((1,0)) + \mathrm{lncir}_{\mathrm{conj}}((0,1)) \notin \mathrm{LNCir}_{2,\mathrm{conj}}(\mathbb{C}),$$

we have that $(\mathrm{NCir}_{2,\mathrm{conj}}(\mathbb{C}), +)$ is not a group under the usual addition. If $n \geq 3$, then

$$\mathrm{rncir}_{\mathrm{conj}}((1,0,\ldots,0)) + \mathrm{lncir}_{\mathrm{conj}}((0,\ldots,0,1)) \notin \mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$$

and

$$\mathrm{rncir}_{\mathrm{conj}}((1,0,\ldots,0)) + \mathrm{lncir}_{\mathrm{conj}}((0,\ldots,0,1)) \notin \mathrm{LNCir}_{n,\mathrm{conj}}(\mathbb{C})$$

It follows that $(\mathrm{NCir}_{n,\mathrm{conj}}(\mathbb{C}), +)$ is not a group under the usual addition with $n \geq 2$. Next, we focus on invertible matrices in $\mathrm{RNCir}_{n,\mathrm{rconj}}(\mathbb{C}), \mathrm{LNCir}_{n,\mathrm{rconj}}(\mathbb{C})$ and $\mathrm{NCir}_{n,\mathrm{rconj}}(\mathbb{C})$.

The set $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}) := \{A \in \mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C}) \mid \det(A) \neq 0\}$ is the set of invertible complex $n \times n$ right conjugate-negacirculant matrices. The set $\widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$

$= \{A \in \mathrm{LNCir}_{n,\mathrm{conj}}(\mathbb{C}) \mid \det(A) \neq 0\}$ is the set of invertible complex $n \times n$ left conjugate-negacirculant matrices. The set $\widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C}) = \{A \in \mathrm{NCir}_{n,\mathrm{conj}}(\mathbb{C}) \mid \det(A) \neq 0\}$ is the set of invertible complex $n\times n$ conjugate-negacirculant matrices.

The following relations between left and right conjugate-negacirculant matrices can be obtained by the direct calculation.

Lemma 3.6. Let n be an even positive integer. Let $\boldsymbol{z} = (z_0, z_1, z_2, \ldots, z_{n-1}) \in \mathbb{C}^n$. Then $H\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{z}) = \mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{z})$ where $H = \begin{bmatrix} 1 & O_1 \\ O_1^T & -\widetilde{I}_{n-1} \end{bmatrix}$.

Proof. We observe that

$$H\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{z}) = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & \ldots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & \ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} z_0 & z_1 & \ldots & z_{n-2} & z_{n-1} \\ -\overline{z_{n-1}} & \overline{z_0} & \ldots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ -z_{n-2} & -z_{n-1} & \ldots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -z_2 & -z_3 & \ldots & z_0 & z_1 \\ -\overline{z_1} & -\overline{z_2} & \ldots & -\overline{z_{n-1}} & \overline{z_0} \end{bmatrix}$$

$$= \begin{bmatrix} z_0 & z_1 & \ldots & z_{n-2} & z_{n-1} \\ \overline{z_1} & \overline{z_2} & \ldots & \overline{z_{n-1}} & -\overline{z_0} \\ z_2 & z_3 & \ldots & -z_0 & -z_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{n-2} & z_{n-1} & \ldots & -z_{n-4} & -z_{n-3} \\ \overline{z_{n-1}} & -\overline{z_0} & \ldots & -\overline{z_{n-3}} & -\overline{z_{n-2}} \end{bmatrix} = \mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{z}).$$

Hence, $H\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{z}) = \mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{z})$. □

Lemma 3.7. Let n be an even positive integer. Let $\boldsymbol{z} = (z_0, z_1, z_2, \dots, z_{n-1}) \in \mathbb{C}^n$. Then $\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{z})H = \mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{\gamma})$ where $\boldsymbol{\gamma} = (z_0, -z_{n-1}, -z_{n-2}, \dots, -z_2, -z_1)$ and

$$H = \begin{bmatrix} 1 & O_1 \\ O_1^T & -\widetilde{I}_{n-1} \end{bmatrix}.$$

Proof. It can be seen that

$$\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{z})H = \begin{bmatrix} z_0 & z_1 & \dots & z_{n-2} & z_{n-1} \\ -\overline{z_{n-1}} & \overline{z_0} & \dots & \overline{z_{n-3}} & \overline{z_{n-2}} \\ -z_{n-2} & -z_{n-1} & \dots & z_{n-4} & z_{n-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -z_2 & -z_3 & \dots & z_0 & z_1 \\ -\overline{z_1} & -\overline{z_2} & \dots & -\overline{z_{n-1}} & \overline{z_0} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z_0 & -z_{n-1} & \dots & -z_2 & -z_1 \\ -\overline{z_{n-1}} & -\overline{z_{n-2}} & \dots & -\overline{z_1} & -\overline{z_0} \\ -z_{n-2} & -z_{n-3} & \dots & -z_0 & z_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -z_2 & -z_1 & \dots & z_4 & z_3 \\ -\overline{z_1} & -\overline{z_0} & \dots & \overline{z_3} & \overline{z_2} \end{bmatrix} = \mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{\gamma}).$$

Hence, $\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{z})H = \mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{\gamma})$. □

Next, we focus on the multiplicative group structures of $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$, $\widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ and $\widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$

A necessary and sufficient condition for the set $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ of $n \times n$ invertible complex right conjugate-negacirculant matrices to be a group under the usual matrix multiplication is given in the next theorem.

Theorem 3.8. *Let n be a positive integer. Then $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication if and only if $n = 1$ or n is even.*

Proof. Suppose $n \neq 1$ and n is odd. Let $\boldsymbol{a} = (2i, i, 0, \ldots, 0)$. Then $\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a})$ is invertible since $\det(\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a})) = (2^n - 1)i \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a}))^2 .$$

Then

$$\begin{aligned} -\overline{c}_{n-2,n-1} &= -\overline{(0 + \cdots + 0 + (-2) + 2)} \\ &= -\overline{0} \\ &= 0 \\ &\neq 4 \\ &= 2 + 0 + \cdots + 0 + 2 \\ &= c_{n-1,0}. \end{aligned}$$

Hence, $(\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Conversely, assume that $n = 1$ or n is even. If $n = 1$, then $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \cong \mathbb{C} \setminus \{0\} \cong GL_n(\mathbb{C})$ is a group. Next, we consider the case where n is even.

Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$, $\boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1}) \in \mathbb{C}^n$ be such that $\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a})$ and $\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b})$ are in $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Let $[c_{ij}]_{n\times n} := \mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{rncir}(\boldsymbol{b})$. Therefore, we need to show that

i) $\overline{c_{ij}} = c_{i+1,j+1}$ for all $0 \leq i \leq n-3$ and $i \leq j \leq n-3$,

ii) $-\overline{c_{i,n-1}} = c_{i+1,0}$ $0 \leq i \leq n-2$,

iii) $\overline{c_{ij}} = c_{i+1,j+1}$ for all $1 \leq i \leq n-2$ and $0 \leq j \leq i-1$.

Case 1 $0 \leq i \leq n-3$ and $i \leq j \leq n-3$.

Case 1.1 i is even. We have

$$\begin{aligned}
\overline{c_{ij}} &= \overline{\sum_{k=0,k \text{ is odd}}^{j-i} a_k\overline{b_{j-i-k}} + \sum_{k=0,k \text{ is even}}^{j-i} a_k b_{j-i-k} - \sum_{k=j-i+1,k \text{ is odd}}^{n-1} a_k\overline{b_{j-i-k}} - \sum_{k=j-i+1,k \text{ is even}}^{n-1} a_k b_{j-i-k}} \\
&= \sum_{k=0,k \text{ is odd}}^{j-i} \overline{a_k} b_{j-i-k} + \sum_{k=0,k \text{ is even}}^{j-i} \overline{a_k b_{j-i-k}} - \sum_{k=j-i+1,k \text{ is odd}}^{n-1} \overline{a_k} b_{j-i-k} - \sum_{k=j-i+1,k \text{ is even}}^{n-1} \overline{a_k b_{j-i-k}} \\
&= c_{i+1,j+1}.
\end{aligned}$$

Case 1.2 i is odd. We have

$$\begin{aligned}
\overline{c_{ij}} &= \overline{\sum_{k=0,k \text{ is odd}}^{j-i} \overline{a_k b_{j-i-k}} + \sum_{k=0,k \text{ is even}}^{j-i} \overline{a_k} b_{j-i-k} - \sum_{k=j-i+1,k \text{ is odd}}^{n-1} \overline{a_k b_{j-i-k}} - \sum_{k=j-i+1,k \text{ is even}}^{n-1} \overline{a_k} b_{j-i-k}} \\
&= \sum_{k=0,k \text{ is odd}}^{j-i} a_k b_{j-i-k} + \sum_{k=0,k \text{ is even}}^{j-i} a_k \overline{b_{j-i-k}} - \sum_{k=j-i+1,k \text{ is odd}}^{n-1} a_k b_{j-i-k} - \sum_{k=j-i+1,k \text{ is even}}^{n-1} a_k \overline{b_{j-i-k}} \\
&= c_{i+1,j+1}.
\end{aligned}$$

Case 2 $0 \leq i \leq n-2$ and $j = n-1$.

Case 2.1 i is even. It follows that

$$
\begin{aligned}
-\overline{c_{i,n-1}} &= -\overline{\left(\sum_{k=0,k \text{ is odd}}^{n-(i+1)} a_k\overline{b_{n-(i+1)-k}} + \sum_{k=0,k \text{ is even}}^{n-(i+1)} a_k b_{n-(i+1)-k} - \sum_{k=n-i,k \text{ is odd}}^{n-1} a_k\overline{b_{n-(i+1)-k}} - \sum_{k=n-i,k \text{ is even}}^{n-1} a_k b_{n-(i+1)-k} \right)} \\
&= -\sum_{k=0,k \text{ is odd}}^{n-(i+1)} \overline{a_k} b_{n-(i+1)-k} - \sum_{k=0,k \text{ is even}}^{n-(i+1)} \overline{a_k b_{n-(i+1)-k}} + \sum_{k=n-i,k \text{ is odd}}^{n-1} \overline{a_k} b_{n-(i+1)-k} + \sum_{k=n-i,k \text{ is even}}^{n-1} \overline{a_k b_{n-(i+1)-k}} \\
&= c_{i+1,0}.
\end{aligned}
$$

Case 2.1 i is odd. Then

$$
\begin{aligned}
-\overline{c_{i,n-1}} &= -\overline{\left(\sum_{k=0,k \text{ is odd}}^{n-(i+1)} \overline{a_k b_{n-(i+1)-k}} + \sum_{k=0,k \text{ is even}}^{n-(i+1)} \overline{a_k} b_{n-(i+1)-k} - \sum_{k=n-i,k \text{ is odd}}^{n-1} \overline{a_k b_{n-(i+1)-k}} - \sum_{k=n-i,k \text{ is even}}^{n-1} \overline{a_k} b_{n-(i+1)-k} \right)} \\
&= -\sum_{k=0,k \text{ is odd}}^{n-(i+1)} a_k b_{n-(i+1)-k} - \sum_{k=0,k \text{ is even}}^{n-(i+1)} a_k\overline{b_{n-(i+1)-k}} + \sum_{k=n-i,k \text{ is odd}}^{n-1} a_k b_{n-(i+1)-k} + \sum_{k=n-i,k \text{ is even}}^{n-1} a_k\overline{b_{n-(i+1)-k}} \\
&= c_{i+1,0}.
\end{aligned}
$$

Case 3 $1 \leq i \leq n-2$ and $0 \leq j \leq i-1$.

Case 3.1 i is even. we have

$$\begin{aligned}
\overline{c_{ij}} &= \overline{-\sum_{k=0,k \text{ is odd}}^{n+j-i} a_k\overline{b_{n+j-i-k}} - \sum_{k=0,k \text{ is even}}^{n+j-i} a_k b_{n+j-i-k} + \sum_{k=n+j-i+1,k \text{ is odd}}^{n-1} a_k\overline{b_{n+j-i-k}} + \sum_{k=n+j-i+1,k \text{ is even}}^{n-1} a_k b_{n+j-i-k}} \\
&= -\sum_{k=0,k \text{ is odd}}^{n+j-i} \overline{a_k} b_{n+j-i-k} - \sum_{k=0,k \text{ is even}}^{n+j-i} \overline{a_k b_{n+j-i-k}} + \sum_{k=n+j-i+1,k \text{ is odd}}^{n-1} \overline{a_k} b_{n+j-i-k} + \sum_{k=n+j-i+1,k \text{ is even}}^{n-1} \overline{a_k b_{n+j-i-k}} \\
&= c_{i+1,j+1}.
\end{aligned}$$

Case 3.2 i is odd. we have

$$\begin{aligned}
\overline{c_{ij}} &= \overline{-\sum_{k=0,k \text{ is odd}}^{n+j-i} \overline{a_k b_{n+j-i-k}} - \sum_{k=0,k \text{ is even}}^{n+j-i} \overline{a_k} b_{n+j-i-k} + \sum_{k=n+j-i+1,k \text{ is odd}}^{n-1} \overline{a_k b_{n+j-i-k}} + \sum_{k=n+j-i+1,k \text{ is even}}^{n-1} \overline{a_k} b_{n+j-i-k}} \\
&= -\sum_{k=0,k \text{ is odd}}^{n+j-i} a_k b_{n+j-i-k} - \sum_{k=0,k \text{ is even}}^{n+j-i} a_k\overline{b_{n+j-i-k}} + \sum_{k=n+j-i+1,k \text{ is odd}}^{n-1} a_k b_{n+j-i-k} + \sum_{k=n+j-i+1,k \text{ is even}}^{n-1} a_k\overline{b_{n+j-i-k}} \\
&= c_{i+1,j+1}.
\end{aligned}$$

From Cases 1 and 2, $\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b}) \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$ be an element in $\mathbb{C}^n$ be such that

$$A := (\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a})) \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Then there exists a unique $B = [b_{ij}]_{n\times n}$ in $GL_n(\mathbb{C})$ such that $AB = I_n$. We will show that $B \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Note that

$$A\,[b_{0,0}\ b_{1,0}\ldots b_{n-1,0}]^T = [1\ 0\ldots 0]^T.$$

From the equation above, we have the following system of equations.

$$\begin{aligned}
a_0b_{0,0} + a_1b_{1,0} + a_2b_{2,0} + \cdots + a_{n-1}b_{n-1,0} &= 1,\\
-\bar{a}_{n-1}b_{0,0} + \bar{a}_0b_{1,0} + \bar{a}_1b_{2,0} + \cdots + \bar{a}_{n-2}b_{n-1,0} &= 0,\\
-a_{n-2}b_{0,0} - a_{n-1}b_{1,0} + a_0b_{2,0} + \cdots + a_{n-3}b_{n-1,0} &= 0,\\
&\vdots\\
-\bar{a}_1b_{0,0} - \bar{a}_2b_{1,0} - \bar{a}_3b_{2,0} - \cdots + \bar{a}_0b_{n-1,0} &= 0.
\end{aligned}$$

Multiply the last eqution with -1 and move it to the top and take the conjugate to every equation, we have

$$\begin{aligned}
-a_0\bar{b}_{n-1,0} + a_1\bar{b}_{0,0} + a_2\bar{b}_{1,0} + \cdots + a_{n-1}\bar{b}_{n-2,0} &= 0,\\
\overline{a_{n-1}b_{n-1,0}} + \overline{a_0b_{0,0}} + \overline{a_1b_{1,0}} + \cdots + \overline{a_{n-2}b_{n-2,0}} &= 1,\\
a_{n-2}\bar{b}_{n-1,0} - a_{n-1}\bar{b}_{0,0} + a_0\bar{b}_{1,0} + \cdots + a_{n-3}\bar{b}_{n-2,0} &= 0,\\
&\vdots\\
\overline{a_1b_{n-1,0}} - \overline{a_2b_{0,0}} - \overline{a_3b_{1,0}} - \cdots + \overline{a_0b_{n-2,0}} &= 0.
\end{aligned}$$

Hence,

$$A\left[-\bar{b}_{n-1,0}\ \bar{b}_{0,0}\ldots\bar{b}_{n-2,0}\right]^T = [0\ 1\ 0\ldots 0]^T.$$

Continue this process, we have

$$A^{-1} = B = \mathrm{rncir}_{\mathrm{conj}}((b_{0,0}, -\bar{b}_{n-1,0}, -b_{n-2,0}, \ldots, -b_{2,0}, -\bar{b}_{1,0})) \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Therefore, $\widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$ as desired. □

A necessary and sufficient condition for the set $\widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ of $n\times n$ invertible complex left conjugate-negacirculant matrices to be a group under the usual matrix multiplication is given in the next theorem.

Theorem 3.9. *Let n be a positive integer. Then $\widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication if and only if $n = 1$.*

Proof. Assume that $n \neq 1$. If n is odd, we consider 2 cases.
Case 1: $n = 3$. Let $\boldsymbol{a} = (i, i, 1)$. Then $\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ since $\det(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a})) = -2 \neq 0$. It follows that

$$(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}))^2 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \notin \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Case 2 : $n \geq 5$. Let $\boldsymbol{a} = (0, \ldots, 0, i, 2i)$. Then $\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ since $\det(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a})) = (2^n - 1)i \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}))^2.$$

Then

$$\begin{aligned} -\overline{c}_{n-2,0} &= -\overline{(0 + \cdots + 0)} \\ &= -\overline{0} \\ &= 0 \\ &\neq -5 \\ &= (-4) + 0 + \cdots + (-1) \\ &= c_{n-1,n-1}. \end{aligned}$$

Hence, $(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $\widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Next, we consider the case when n is even. Then we have 2 cases to consider. Case 1: $n = 2$. Let $\boldsymbol{a} = (1-i, 1)$. Then $\text{lncir}_{\text{conj}}(\boldsymbol{a}) \in \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$ because $\det(\text{lncir}_{\text{conj}}(\boldsymbol{a})) = -3 \neq 0$. It follows that

$$(\text{lncir}_{\text{conj}}(\boldsymbol{a}))^2 = \begin{bmatrix} -2i & -2i \\ -2i & 2i \end{bmatrix} \notin \widehat{\text{LNCir}}_{\text{n,conj}}(\mathbb{C}).$$

Case 2: $n \geq 4$. Let $\boldsymbol{a} = (0, \ldots, 0, i, 2i)$. Then $\text{lncir}_{\text{conj}}(\boldsymbol{a}) \in \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$ because $\det(\text{lncir}_{\text{conj}}(\boldsymbol{a})) = -(2^n+1)i \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\text{lncir}_{\text{conj}}(\boldsymbol{a}))^2.$$

Then

$$\begin{aligned} -\overline{c}_{n-2,0} &= -\overline{(0+\cdots+0)} \\ &= -\overline{0} \\ &= 0 \\ &\neq 3 \\ &= 4+0+\cdots+0+(-1) \\ &= c_{n-1,n-1}. \end{aligned}$$

Hence, $(\text{lncir}_{\text{conj}}(\boldsymbol{a}))^2 \notin \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$. Therefore, $\widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Conversely, assume that $n = 1$. Then

$$\widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C}) \cong \mathbb{C} \setminus \{0\} \cong GL_n(\mathbb{C})$$

is a group under the usual matrix multiplication. □

A necessary and sufficient condition for the set $\widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C})$ of $n \times n$ invertible complex conjugate-negacirculant matrices to be a group under the usual matrix multiplication is given in the next theorem.

Theorem 3.10. *Let n be a positive integer. Then the set $\widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication if and only if $n = 1$ or n is even.*

Proof. Suppose $n \neq 1$ and n is odd. Then we consider the following 2 cases.
Case 1 : $n = 3$. Let $\boldsymbol{a} = (i, i, 1)$. Then $\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$ because $\det(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a})) = -2 \neq 0$. It follows that

$$(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}))^2 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \notin \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

and

$$(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}))^2 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \notin \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Hence, $(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}))^2 \notin \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \cup \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C}) = \widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$.
Case 2 : $n \geq 5$. Let $\boldsymbol{a} = (0, \ldots, 0, i, 2i)$. Then $\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}) \in \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$ because $\det(\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a})) = (2^n - 1)i \neq 0$. Let

$$[c_{ij}]_{n\times n} := (\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a}))^2.$$

Since

$$\begin{aligned} -\overline{c}_{n-2,0} &= -(\overline{0 + \cdots + 0}) \\ &= -\overline{0} \\ &= 0 \\ &\neq -5 \\ &= (-4) + 0 + \cdots + 0 + (-1) \\ &= c_{n-1,n-1}, \end{aligned}$$

$(\text{lncir}_{\text{conj}}(\boldsymbol{a}))^2 \notin \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$.

Since

$$\begin{aligned} -\overline{c}_{n-2,n-1} &= -(\overline{2+0+\cdots+0}) \\ &= -\overline{2} \\ &= -2 \\ &\neq 2 \\ &= 0+\cdots+0+2 \\ &= c_{n-1,0}, \end{aligned}$$

$(\text{lncir}_{\text{conj}}(\boldsymbol{a}))^2 \notin \widehat{\text{RNCir}}_{n,\text{conj}}(\mathbb{C})$. Therefore, $(\text{lncir}_{\text{conj}}(\boldsymbol{a}))^2 \notin \widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C})$.

From Cases 1 and 2, $\widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C})$ is not a subgroup of $GL_n(\mathbb{C})$. It follows that $\widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices.

Conversely, assume that $n = 1$ or n is even. If $n = 1$, then $\widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C}) \cong \mathbb{C} \setminus \{0\} \cong GL_n(\mathbb{C})$ is a group. Next, we consider the case where n is even. Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1})$ be elements in $\mathbb{C}^n$. Then we consider the following 4 cases.

Case 1: $\text{rncir}_{\text{conj}}(\boldsymbol{a})$ and $\text{rncir}_{\text{conj}}(\boldsymbol{b})$ are elements in $\widehat{\text{RNCir}}_{n,\text{conj}}(\mathbb{C})$.

By Theorem 3.8, $\text{rncir}_{\text{conj}}(\boldsymbol{a})\text{rncir}_{\text{conj}}(\boldsymbol{b}) \in \widehat{\text{RNCir}}_{n,\text{conj}}(\mathbb{C}) \subseteq \widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C})$.

Case 2: $\text{rncir}_{\text{conj}}(\boldsymbol{a}) \in \widehat{\text{RNCir}}_{n,\text{conj}}(\mathbb{C})$ and $\text{lncir}_{\text{conj}}(\boldsymbol{b}) \in \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$.

By Lemma 3.7, $\text{lncir}_{\text{conj}}(\boldsymbol{b}) = \text{rncir}_{\text{conj}}(\boldsymbol{c})H$ for some $\boldsymbol{c} \in \mathbb{C}^n$. We have that

$$\begin{aligned} \text{rncir}_{\text{conj}}(\boldsymbol{a})\text{lncir}_{\text{conj}}(\boldsymbol{b}) &= (\text{rncir}_{\text{conj}}(\boldsymbol{a}))(\text{rncir}_{\text{conj}}(\boldsymbol{c})H) \\ &= (\text{rncir}_{\text{conj}}(\boldsymbol{a})\text{rncir}_{\text{conj}}(\boldsymbol{c}))H \in \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C}) \subseteq \widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C}). \end{aligned}$$

Case 3: $\text{lncir}_{\text{conj}}(\boldsymbol{a}) \in \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$ and $\text{rncir}_{\text{conj}}(\boldsymbol{b}) \in \widehat{\text{RNCir}}_{n,\text{conj}}(\mathbb{C})$.

By Lemma 3.6, $\text{lncir}_{\text{conj}}(\boldsymbol{a}) = H\text{rncir}_{\text{conj}}(\boldsymbol{a})$. We have that

$$\begin{aligned} \text{lncir}_{\text{conj}}(\boldsymbol{a})\text{rncir}_{\text{conj}}(\boldsymbol{b}) &= (H\text{rncir}_{\text{conj}}(\boldsymbol{a}))(\text{rncir}_{\text{conj}}(\boldsymbol{b})) \\ &= H(\text{rncir}_{\text{conj}}(\boldsymbol{a})\text{rncir}_{\text{conj}}(\boldsymbol{b})) \in \widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C}) \subseteq \widehat{\text{NCir}}_{n,\text{conj}}(\mathbb{C}). \end{aligned}$$

Case 4: $\text{lncir}_{\text{conj}}(\boldsymbol{a})$ and $\text{lncir}_{\text{conj}}(\boldsymbol{b})$ be elements in $\widehat{\text{LNCir}}_{n,\text{conj}}(\mathbb{C})$.

By Lemmas 3.7 and 3.6, $\text{lncir}_{\text{conj}}(\boldsymbol{a}) = \text{rncir}_{\text{conj}}(\boldsymbol{c})H$ for some $\boldsymbol{c} \in \mathbb{C}^n$ and

$\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{b}) = H\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b})$. It follows that

$$\begin{aligned}\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{b}) &= (\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c})H)(H\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b}))\\ &= \mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c})H^2\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b})\\ &= \mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c})I_n\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b})\\ &= \mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c})\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b}).\end{aligned}$$

By Theorem 3.8, $\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{lncir}_{\mathrm{conj}}(\boldsymbol{b}) \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$. From Cases 1–4, $\widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$ is closed under multiplication.

Next, let A be an element in $\widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Then we consider the following 2 cases.

Case 1: $A \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Then by Theorem 3.8, $A^{-1} \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

Case 2: $A \in \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C})$. Then by Lemma 3.6, $A = H\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c})$ for some $\boldsymbol{c} \in \mathbb{C}^n$. We have that

$$\begin{aligned}A^{-1} &= (H\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1}\\ &= (\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1}H^{-1}\\ &= (\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1}H \quad (\text{ since } H^2 = I_n)\\ &\in \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \qquad (\text{ since } (\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{c}))^{-1} \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C})).\end{aligned}$$

Hence, $A^{-1} \in \widehat{\mathrm{LNCir}}_{n,\mathrm{conj}}(\mathbb{C}) \subseteq \widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$.

From Cases 1 and 2, A^{-1} is an element in $\widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$. It follows that the set $\widehat{\mathrm{NCir}}_{n,\mathrm{conj}}(\mathbb{C})$ forms a group under the usual matrix multiplication. □

Chapter 4

Characterizations

4.1 Characterization of Right Conjugate-Circulant Matrices

From Section 3.1, $\mathrm{LCir}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices and $\mathrm{Cir}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual addition of matrices with $n \geq 3$. It follows that $\mathrm{LCir}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{Cir}_{n,\mathrm{conj}}(\mathbb{C})$ can not be rings. To study the ring structures of such matrices, it is therefore sufficient to consider $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$.

In this section, the algebraic structure of $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ is studied. The characterization of $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ is given in terms of skew polynomials.

Proposition 4.1. *Let n be a positive integer. Then $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ is a vector space over $\mathbb{C}$ under the usual addition and the scalar multiplication defined by*

$$a \cdot \mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})) := \mathrm{rcir}_{\mathrm{conj}}((a, 0, \ldots, 0))\, \mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))$$

for all $a \in \mathbb{C}$ and $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$.

Proof. Clearly, the sum of two right conjugate-circulant matrices is a right conjugate-circulant. Since

$$\mathrm{rcir}_{\mathrm{conj}}((a, 0, \ldots, 0))\, \mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})) = \mathrm{rcir}_{\mathrm{conj}}((az_0, az_1, \ldots, az_{n-1}))$$

for all $a \in \mathbb{C}$ and $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$, the proposition follows. □

Corollary 4.2. *Let n be a positive integer. Then* $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ *is a vector space over* $\mathbb{R}$ *under the usual addition and the scalar multiplication.*

Proof. Note that $\mathrm{rcir}_{\mathrm{conj}}((a,0,\ldots,0)) = aI_n$ for all $a \in \mathbb{R}$. By Proposition 4.1, the result follows. □

A necessary and sufficient condition for $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ to be a ring is given in the next theorem.

Theorem 4.3. *Let n be a positive integer. Then* $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ *is a subring of* $M_n(\mathbb{C})$ *if and only if $n = 1$ or n is even.*

Proof. Suppose $n \neq 1$ and n is odd. Let $a \in \mathbb{C}$ be such that $a \neq \overline{a}$ and let

$$[c_{ij}]_{n\times n} := \mathrm{rcir}_{\mathrm{conj}}((a,a,\ldots,a))\,\mathrm{rcir}_{\mathrm{conj}}((a,a,\ldots,a)).$$

Then

$$\begin{aligned}
\overline{c_{0,0}} &= \overline{\sum_{i=1}^{\frac{n+1}{2}} a^2 + \sum_{i=1}^{\frac{n-1}{2}} a\cdot\overline{a}} \\
&= \left(\frac{n+1}{2}\right)\overline{a^2} + \left(\frac{n-1}{2}\right) a\cdot\overline{a} \\
&\neq \left(\frac{n+1}{2}\right) a\cdot\overline{a} + \left(\frac{n-1}{2}\right)\overline{a^2} \\
&= \sum_{i=1}^{\frac{n+1}{2}} a\cdot\overline{a} + \sum_{i=1}^{\frac{n-1}{2}} \overline{a^2} \\
&= c_{1,1}.
\end{aligned}$$

Hence, $\mathrm{rcir}_{\mathrm{conj}}((a,a,\ldots,a))\mathrm{rcir}_{\mathrm{conj}}((a,a,\ldots,a)) \notin \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subring of $M_n(\mathbb{C})$.

Conversely, assume that $n = 1$ or n is even. If $n = 1$, then $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C}) = \mathbb{C} = M_n(\mathbb{C})$ is a ring. Next, we consider the case where n is even. Let $\mathrm{rcir}_{\mathrm{conj}}((a_0,a_1,\ldots,a_{n-1}))$ and $\mathrm{rcir}_{\mathrm{conj}}((b_0,b_1,\ldots,b_{n-1}))$ be elements in $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$. Then

$$\mathrm{rcir}_{\mathrm{conj}}((a_0,a_1,\ldots,a_{n-1})) - \mathrm{rcir}_{\mathrm{conj}}((b_0,b_1,\ldots,b_{n-1}))$$

$$= \mathrm{rcir}_{\mathrm{conj}}((a_0 - b_0, a_1 - b_1, \ldots, a_{n-1} - b_{n-1})) \in \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C}).$$

Using the arguments similar to those in the proof of Theorem 3.3,

$$\mathrm{rcir}_{\mathrm{conj}}((a_0, a_1, \ldots, a_{n-1}))\mathrm{rcir}_{\mathrm{conj}}((b_0, b_1, \ldots, b_{n-1})) \in \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C}).$$

Therefore, $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ is a subring of $M_n(\mathbb{C})$ as desired. □

In the case where n is even, there is a direct link between the ring $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ and the quotient ring of skew polynomials $\mathbb{C}[x, \mathrm{conj}]/\langle x^n - 1\rangle$.

Theorem 4.4. *Let n be an even positive integer. Then $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x, \mathrm{conj}]/\langle x^n - 1\rangle$ as rings.*

Proof. Let $T : \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C}) \to \mathbb{C}[x, \mathrm{conj}]/\langle x^n - 1\rangle$ be defined by

$$T\left(\mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))\right) = \sum_{i=0}^{n-1} z_i x^i + \langle x^n - 1\rangle.$$

Let $\boldsymbol{z} = (z_0, z_1, \ldots, z_{n-1})$ and $\boldsymbol{w} = (w_0, w_1, \ldots, w_{n-1})$ be vectors in $\mathbb{C}^n$. Then

$$\begin{aligned}
T\left(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z}) + \mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{w})\right) &= T\left(\mathrm{rcir}_{\mathrm{conj}}((z_0 + w_0, z_1 + w_1, \ldots, z_{n-1} + w_{n-1}))\right) \\
&= \sum_{i=0}^{n-1}(z_i + w_i)x^i + \langle x^n - 1\rangle \\
&= \sum_{i=0}^{n-1}(z_i x^i + w_i x^i) + \langle x^n - 1\rangle \\
&= \left(\sum_{i=0}^{n-1} z_i x^i + \langle x^n - 1\rangle\right) + \left(\sum_{i=0}^{n-1} w_i x^i + \langle x^n - 1\rangle\right) \\
&= T\left(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z})\right) + T\left(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{w})\right).
\end{aligned}$$

Let $[c_{ij}]_{n\times n} = \mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z})\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{w})$. By Theorem 4.3, we have $[c_{ij}]_{n\times n} \in \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ and hence,

$$\begin{aligned}
T\left(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z})\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{w})\right) &= T\left([c_{ij}]_{n\times n}\right) \\
&= T\left(\mathrm{rcir}_{\mathrm{conj}}(c_{0,0}, c_{0,1}, \ldots, c_{0,n-1})\right) \\
&= \sum_{i=0}^{n-1} c_{0,i} x^i + \langle x^n - 1\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \left(\sum_{i=0}^{\frac{n-2}{2}} z_{2k} w_{i-2k} + \sum_{i=0}^{\frac{n-2}{2}} z_{2k+1} \overline{w_{i-(2k+1)}} \right) x^i + \langle x^n - 1 \rangle \\
&= \sum_{i=0}^{n-1} \left(\sum_{i=2j+k} z_{2j} w_k + \sum_{i=(2j+1)+k} z_{2j+1} \overline{w_k} \right) x^i + \langle x^n - 1 \rangle \\
&= \sum_{i=0}^{n-1} \sum_{i=j+k(mod\ n)} \left(z_j x^j w_k x^k \right) + \langle x^n - 1 \rangle \\
&= \left(\sum_{k=0}^{n-1} z_k x^k + \langle x^n - 1 \rangle \right) \left(\sum_{k=0}^{n-1} w_k x^k + \langle x^n - 1 \rangle \right) \\
&= T\left(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{z})\right) T\left(\mathrm{rcir}_{\mathrm{conj}}(\boldsymbol{w})\right).
\end{aligned}
$$

Then T is a ring homomorphism.

To show that T is injective, let $\mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})) \in ker(T)$. Then

$$
\sum_{i=0}^{n-1} z_i x^i + \langle x^n - 1 \rangle = T\left(\mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))\right) = \langle x^n - 1 \rangle .
$$

It follows that $\sum_{i=0}^{n-1} z_i x^i \in \langle x^n - 1 \rangle$. Since $deg \left(\sum_{i=0}^{n-1} z_i x^i \right) \leq n - 1$, we have $z_i = 0$ for all $i = 0, 1, \ldots, n-1$. Hence, $ker(T) = \langle x^n - 1 \rangle$ and T is injective.

For each $f(x) + \langle x^n - 1 \rangle \in \mathbb{C}[x, \mathrm{conj}] / \langle x^n - 1 \rangle$, there exists $\sum_{i=0}^{n-1} z_i x^i \in \mathbb{C}[x, \mathrm{conj}]$ such that

$$
f(x) + \langle x^n - 1 \rangle = \sum_{i=0}^{n-1} z_i x^i + \langle x^n - 1 \rangle
$$

by the division algorithm. Then

$$
T\left(\mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))\right) = \sum_{i=0}^{n-1} z_i x^i + \langle x^n - 1 \rangle = f(x) + \langle x^n - 1 \rangle .
$$

Hence, T is surjective. Therefore, T is a ring isomorphism and $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x, \mathrm{conj}] / \langle x^n - 1 \rangle$ as rings. □

4.2 Characterization of Right Conjugate-Negacirculant Matrices

From Section 3.2, $\mathrm{LNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual multiplication of matrices and $\mathrm{NCir}_{n,\mathrm{conj}}(\mathbb{C})$ is not a group under the usual addition of matrices with $n \geq 2$. It follows that $\mathrm{LNCir}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{NCir}_{n,\mathrm{conj}}(\mathbb{C})$ can not be rings. To study the ring structures of such matrices, it is therefore sufficient to consider $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$.

In this section, the algebraic structure of $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is studied. The characterization of $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is given in terms of skew polynomials.

Proposition 4.5. *Let n be a positive integer. Then $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is a vector space over $\mathbb{C}$ under the usual addition and the scalar multiplication defined by*

$$a \cdot \mathrm{rncir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})) := \mathrm{rncir}_{\mathrm{conj}}((a, 0, \ldots, 0))\, \mathrm{rncir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))$$

for all $a \in \mathbb{C}$ and $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$.

Proof. Clearly, the sum of two right conjugate-negacirculant matrices is a right conjugate-negacirculant. Since $\mathrm{rncir}_{\mathrm{conj}}((a, 0, \ldots, 0))\, \mathrm{rncir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})) = \mathrm{rncir}_{\mathrm{conj}}((az_0, az_1, \ldots, az_{n-1}))$ for all $a \in \mathbb{C}$ and $(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n$, the proposition follows. □

Corollary 4.6. *Let n be a positive integer. Then $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is a real vector space under the usual addition and the scalar multiplication.*

Proof. Note that $\mathrm{rncir}_{\mathrm{conj}}((a, 0, \ldots, 0)) = aI_n$ for all $a \in \mathbb{R}$. By Proposition 4.5, the result follows. □

A necessary and sufficient condition for $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ to be a ring is given in the next theorem.

Theorem 4.7. *Let n be a positive integer. Then $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is a subring of $M_n(\mathbb{C})$ if and only if $n = 1$ or n is even.*

Proof. Suppose $n \neq 1$ and n is odd. Let $a \in \mathbb{C}$ be such that $a \neq \overline{a}$ and let

$$[c_{ij}]_{n \times n} := \mathrm{rncir}_{\mathrm{conj}}((a, a, \ldots, a))\, \mathrm{rncir}_{\mathrm{conj}}((a, a, \ldots, a)).$$

Then

$$\begin{aligned}
\overline{c_{0,0}} &= \overline{a^2 - \sum_{i=1}^{\frac{n-1}{2}} a \cdot \overline{a} - \sum_{i=1}^{\frac{n-1}{2}} a^2} \\
&= \overline{a^2} - \sum_{i=1}^{\frac{n-1}{2}} \overline{a} \cdot a - \sum_{i=1}^{\frac{n-1}{2}} \overline{a^2} \\
&\neq \overline{a^2} - \sum_{i=1}^{\frac{n+1}{2}} \overline{a} \cdot a - \sum_{i=1}^{\frac{n-3}{2}} \overline{a^2} \\
&= c_{1,1}.
\end{aligned}$$

Hence, $\mathrm{rncir}_{\mathrm{conj}}((a, a, \ldots, a))\mathrm{rncir}_{\mathrm{conj}}((a, a, \ldots, a)) \notin \mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$. Therefore, $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is not a subring of $M_n(\mathbb{C})$.

Conversely, assume that $n = 1$ or n is even. If $n = 1$, then $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C}) \cong \mathbb{C} \cong M_n(\mathbb{C})$ is a ring. Next, we focus on the case where n is even. Let $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$, $\boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1}) \in \mathbb{C}^n$. It is not difficult to see that

$$\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a}) - \mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b}) = \mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a} - \boldsymbol{b}) \in \mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$$

Using the arguments similar to those in the proof of Theorem 3.8,

$$\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{a})\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{b}) \in \widehat{\mathrm{RNCir}}_{n,\mathrm{conj}}(\mathbb{C}).$$

Therefore, $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is a subring of $M_n(\mathbb{C})$. □

In the case where n is even, the ring $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ can be characterized using the quotient skew polynomial ring $\mathbb{C}[x, \mathrm{conj}]/\langle x^n + 1\rangle$.

Theorem 4.8. *Let n be an even positive integer. Then $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x, \mathrm{conj}]/\langle x^n + 1\rangle$ as rings.*

Proof. Let $S : \mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C}) \to \mathbb{C}[x, \mathrm{conj}]/\langle x^n + 1\rangle$ be defined by

$$S\left(\mathrm{rncir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))\right) = \sum_{i=0}^{n-1} z_i x^i + \langle x^n + 1\rangle.$$

Let $\boldsymbol{z} = (z_0, z_1, \ldots, z_{n-1})$ and $\boldsymbol{w} = (w_0, w_1, \ldots, w_{n-1})$ be vectors in $\mathbb{C}^n$. Similar to Theorem 4.4, S is an additive group isomorphism. Let

$$[c_{ij}]_{n\times n} := \mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{z})\mathrm{rncir}_{\mathrm{conj}}(\boldsymbol{w}).$$

$$
\begin{aligned}
S\left(\operatorname{rncir}_{\operatorname{conj}}(\boldsymbol{z})\operatorname{ncir}_{\operatorname{conj}}(\boldsymbol{w})\right) &= S\left([c_{ij}]_{n\times n}\right)\\
&= S\left(\operatorname{rncir}_{\operatorname{conj}}(c_{0,0}, c_{0,1}, \dots, c_{0,n-1})\right)\\
&= \sum_{i=0}^{n-1} c_{0,i}x^i + \langle x^n+1\rangle\\
&= \sum_{i=0}^{n-1}\left(\sum_{i=2j+k} z_{2j}w_k\right)x^i + \sum_{i=0}^{n-1}\left(\sum_{i=(2j+1)+k} z_{2j}\overline{w_k}\right)x^i\\
&\quad - \sum_{i=0}^{n-1}\left(\sum_{i=2j+k(mod\ n)} z_{2j}w_k\right)x^i - \sum_{i=0}^{n-1}\left(\sum_{i=(2j+1)+k(mod\ n)} z_{2j+1}\overline{w_k}\right)x^i\\
&\quad + \langle x^n+1\rangle\\
&= \sum_{i=0}^{n-1}\left(\sum_{i=2j+k} z_{2j}w_k\right)x^i + \sum_{i=0}^{n-1}\left(\sum_{i=(2j+1)+k} z_{2j+1}\overline{w_k}\right)x^i\\
&\quad + \sum_{i=n}^{2n-2}\left(\sum_{i=2j+k(mod\ n)} z_{2j}w_k\right)x^i + \sum_{i=n}^{2n-2}\left(\sum_{i=(2j+1)+k(mod\ n)} z_{2j+1}\overline{w_k}\right)x^i\\
&\quad + \langle x^n+1\rangle\\
&= \sum_{k=0}^{n-1}\sum_{i=j+k}\left(z_jx^jw_kx^k\right) + \sum_{k=0}^{2n-2}\sum_{i=j+k}\left(z_jx^jw_kx^k\right) + \langle x^n+1\rangle\\
&= \left(\sum_{k=0}^{n-1} z_kx^k + \langle x^n+1\rangle\right)\left(\sum_{k=0}^{n-1} w_kx^k + \langle x^n+1\rangle\right)\\
&= S\left(\operatorname{rncir}_{\operatorname{conj}}(\boldsymbol{z})\right)S\left(\operatorname{rncir}_{\operatorname{conj}}(\boldsymbol{w})\right).
\end{aligned}
$$

Using the statement similar to those in the proof of Theorem 4.4, S is a bijection. Hence, S is a ring isomorphism. Therefore, $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ is isomorphic to $\mathbb{C}[x, \mathrm{conj}]/\langle x^n+1\rangle\ \rangle$ as rings. □

4.3 Isomorphisms

From the previous two sections, the algebraic structures of $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ are studied. They are complex vector spaces. In addition, if n is even, they are also rings. In this section, some relations among them are discussed.

The vector spaces $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ are isomorphic.

Theorem 4.9. *Let n be a positive integer. Then $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$*

are isomorphic as complex vector spaces, where the scalar multiplication defined in Propositions 4.1 and 4.5.

Proof It is not difficult to verify that a map $\psi : \mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C}) \to \mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ defined by

$$\psi\left(\mathrm{rcir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1}))\right) = \mathrm{rncir}_{\mathrm{conj}}((z_0, z_1, \ldots, z_{n-1})).$$

is a linear isomorphism. □

From [7, Theorem 3.6], the set $\mathrm{Cir}_n(\mathbb{C})$ of $n \times n$ complex circulant matrices and the set $\mathrm{NCir}_n(\mathbb{C})$ of $n \times n$ complex negacirculant matrices are isomorphic as rings. Hence, it would be possible that $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ and $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ are isomorphic. However, the idea proof of [7, Theorem 3.6] can not be applied. Therefore, we propose this problem as a conjecture.

Conjecture 4.10. *Let n be an even positive integer. Then the rings* $\mathrm{RCir}_{n,\mathrm{conj}}(\mathbb{C})$ *and* $\mathrm{RNCir}_{n,\mathrm{conj}}(\mathbb{C})$ *are isomorphic.*

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Presentations and Publications

Presentations

- P. Morrakutjinda and S. Jitman, *Skew Polynomials and Some Generalizations of Circulant Matrices*, The 11^{th} IMT–GT International Conference on Mathematics, Statistics and Its Applications, Ambassador City Jomtien Hotel, Pattaya, Thailand, 24 November, 2015.

Publications

- P. Morrakutjinda and S. Jitman, *Skew Polynomials and Some Generalizations of Circulant Matrices*, Proceedings of the 11^{th} IMT–GT International Conference on Mathematics, Statistics and Its Applications, Department of Mathematics Faculty of Science King Mongkut's Institute of Technology Ladkrabang, Thailand, 2015, pp. 14-24.

Biography

Name	Mr. Prarinya Morrakutjinda
Address	56 Village No.6, Khlong Mai Sub-district,
	Sam phran District, Nakhon Pathom,
	73110.
Date of Birth	27 August 1991
Education	
2013	Bachelor of Science in Mathematics,
	(First Class Honors), Silpakorn University.
2016	Master of Science in Mathematics,
	Silpakorn University.
Scholarship	Science Achievement Scholarship of Thailand (SAST).